INVERSE ANALYSIS: A TOOL FOR MODEL PARAMETER ESTIMATION

Arindam Dey Assistant Professor, Department of Civil Engineering, IIT Guwahati, arindam.dey@iitg.ernet.in
Prabir K. Basudhar Professor, Department of Civil Engineering, IIT Kanpur, pkbd@iitk.ac.in

ABSTRACT: This paper reports the development of a generalized inverse analysis formulation for the parameter estimation of four-parameter Burger model that is used to represent the time-dependent deformation of a viscoelastic soil medium such as a consolidating clay stratum. Aided by a suitable optimization technique (Sequential Quadratic Programming, SQP), a mathematical programming has been developed in terms of identification of the design vector, objective function and design constraints. In order to comprehend the efficacy and establish the proper functionality of the developed technique, a synthetic case study accounting only the loading cycle of the Burger model has been considered. Prime issues related to the back-estimation of the parameters namely identification of variable bounds, global optimality and optimal number of data-points required are explored and reported herein. The efficacy of the developed technique is also illustrated with a case-study.

INTRODUCTION
Modeling and model parameter estimation is of paramount importance for a correct prediction of foundation behavior. Lumped parameter modeling, represented by discrete or interconnected springs, dashpots and/or other mechanical elements, is a convenient technique to simulate such behavior. Although four-parameter Burger model is able to provide a realistic representation of the time-dependent behavior of a consolidating viscoelastic stratum [1], its use is heavily restricted in geotechnical engineering primarily due to the difficulty in determining the model parameters. For known values of model parameters, a forward analysis can easily predict the response of the system. On the contrary, determination of model parameters from the observed response of the system is quite difficult. Very often, such inverse analysis problems are ill-posed characterized by the presence of several solutions giving identical responses. Hence, the determined model parameters become artifacts providing the response of the system, as observed, although in many cases, might not have physical significance. An iterative inverse analysis technique combined with optimization algorithm may provide satisfactory solution to the problem.

This paper reports the development of such a generalized methodology to determine the model parameters of a four-element Burger model from the observational behavior of a viscoelastic soil medium. The developed technique is studied in detail to comprehend its capability with the aid of a synthetic case-study considering only the loading cycle of the model.

PARAMETER ESTIMATION: INVERSE ANALYSIS
Parameter estimation is a convenient approach to determine the model parameters using observational data gathered from either laboratory [2] or in-situ investigations [3]. Plenty of research pertaining to inverse analysis of parameters in the field of geophysics and ground water flow has been reported. Considerable number of applications has also been made in geotechnical engineering in different vistas as consolidation, rock mechanics, slope stability, retaining structures, load-settlement behavior and bearing capacity of subgrade, pile-settlement analysis and tunneling works. Although well-established in geotechnical engineering, the ad hoc use of this technique is not free from controversial issues [4]. In the modern era, application of optimization techniques and neural networks has opened the gateways for more rational approaches.

Solution techniques for inverse analysis problems may be based on closed form solution [5], numerical iterative procedures [6], probabilistic and statistical approaches [7] or adoption of special optimization techniques [6]. These methods estimate the parameters of the model by determining those values that provide an optimal match of the output of the model to that of the real system, governed by a scalar measure-of-fit to ensure the quality of the match. The parameter adjustment algorithm finally selects the optimal parameter values by minimizing the measure-of-fit in a systematic procedure (Fig. 1). Thus, the selected optimization algorithm plays a key role in the inverse analysis. This paper reports the use of fmincon: an in-built constrained optimization solver module available in Matlab, to solve the formulated constrained problem.

Fig. 1 Inverse analysis algorithm to determine model parameters
PROBLEM STATEMENT
Fig. 2 depicts the rheological sketch of the four-parameter viscoelastic Burger model subjected to constant stress ($\sigma$) for a particular time-period ($t_a$) and the corresponding stress-strain behavior with time showing the different phases of response. The problem is to estimate the model parameters ($k_1$, $k_2$, $\eta_1$, $\eta_2$) from the measured deformation-time behavior of a viscoelastic medium. In order to comprehend the applicability and efficacy of the developed methodology, only the loading cycle of the response is considered for the back-estimation of parameters.

ANALYSIS
The following section describes the mathematical programming formulation after identifying the design vector, objective function and design constraints.

Design Variables and Design Vector
The design variables for the present study comprises of the elastic and viscoelastic coefficients of the Maxwell and Kelvin-Voigt sub-units. Thus, the design vector becomes:

$$\vec{V} = [k_1, k_2, \eta_1, \eta_2]^T$$

Objective Function
The choice of the objective function should reflect the difference between the measured and predicted values. Hence, considering the measured and predicted values at any instant of time, using the method of least squares, the normalized form of the objective function is shown as:

$$F(\vec{V}) = \frac{1}{n_p} \sum_{p=1}^{n_p} \left( \frac{\epsilon_{expr}(t_{np}) - \epsilon_{BU}(t_{np})}{\epsilon_{expr}(t_{np})} \right)^2$$

where, $n_p$ is the number of data-points considered for inverse analysis, $t_{np}$ is the elapsed time corresponding to the considered data-point, $\epsilon_{expr}(t_{np})$ is the measured magnitude of deformation at the instant of time corresponding to the considered data-point and $\epsilon_{BU}(t_{np})$ is the predicted magnitude of deformation at the instant of time corresponding to the considered data-point and is defined as:

$$\epsilon_{BU}(t_{np}) = \sigma_0 \left[ \frac{1}{k_1} + \frac{t_{np}}{\eta_1} + \frac{1}{k_2} \left( 1 - \exp \left( \frac{-k_2 t_{np}}{\eta_2} \right) \right) \right]$$

Thus, the objective function is multivariable and nonlinear.

Design Constraints
In order to ensure the feasibility and acceptability of the estimated magnitudes and to reduce the search domain, variable bounds are imposed on the model parameters, and are expressed in a standard form as follows:

$$|v_i - v_{i}^{\text{min}}| \leq 0.0, \quad |v_i - v_{i}^{\text{max}}| \leq 0.0 \quad \forall i = 1, 2, 3, 4$$

where, $v_{i}^{\text{max}}$ and $v_{i}^{\text{min}}$ are the maximum and minimum bounds on the design parameter $v_i$. It is observed that the constraints are linear. The standard format of expressing the above in a mathematical programming formulation is as follows:

$$g_i(\vec{V}) \leq 0.0, \quad i = 1, 2, 3, \ldots, 8$$

Mathematical Programming Formulation
Minimization of the objective function with respect to the design variables results in the determination of the optimal value. Hence, the optimization problem for the present study is:

Find optimal design variable vector $\vec{V}_{\text{opt}}$, such that,

$$F = F(\vec{V}_{\text{opt}})$$

is the minimum of $F(\vec{V})$

subject to

$$g_i(\vec{V}) \leq 0.0, \quad i = 1, 2, \ldots, 8$$

In the above formulation, the objective function and the design constraints, respectively being nonlinear and linear functions of the design variables, the problem is one of nonlinear programming.

SOLVING NONLINEAR CONSTRAINED OPTIMIZATION PROBLEM
The present study uses fmincon to solve the nonlinear constrained multivariable optimization problem and estimate the parameters of Burger model from the measured time-dependent behavior of viscoelastic medium. The optimization scheme adopted in MatLab incorporates Sequential Quadratic Programming (SQP). At all the major iteration, an approximation is made of the Hessian of the Lagrangian function using a quasi-Newton method. This generates a Quadratic Programming (QP) sub-problem whose solution forms a search direction using a line search technique in order to proceed with the iterative process. A detailed discussion on the optimization technique is provided by [8].

RESULTS AND DISCUSSIONS
Surface of Objective Function
The surface of the developed technique depends largely on the proper understanding of the developed technique. In this regard, this article reports a sensitivity study in order to examine the variation in the objective function with the basic model parameters. Since there are four design variables governing the objective function, it is not possible to depict the simultaneous effect of all parameters in a
Inverse Analysis: A Tool for Parameter Estimation

single plot. Hence, a series of plots are necessary considering two typical design parameters at a time while keeping the other two constants. Fig. 3 depicts a typical variation of the objective function with the corresponding design variables; for the sake of brevity, the rest of the variations are not presented here. It is observed that the contours of the objective function, obtained by varying any two of the design variables, are very steep initially and thereafter very flat over a wide region.

These figures do not provide any precise idea about the nature of the objective function. However, it is understandable that the simultaneous variation of all the four design parameters creates a complex hyper-surface in the design space that is difficult to visualize and present graphically. Thus, the difficulty in estimating the parameter increases manifold. This is mainly because the function is very flat over a wide region and several combinations of the design variables may result in nearly identical objective function value on reaching the plateau while satisfying the convergence.

Synthetic Study: Variable Bounds, Global Optimality and Minimum Number of Data-points

In order to gain experience for successful implementation of the developed methodology, a synthetic study is developed considering the following parameters:

- \( k_1 = 300 \) kPa
- \( k_2 = 30 \) kPa
- \( \eta_1 = 3 \times 10^5 \) kPa-days
- \( \eta_2 = 3 \times 10^4 \) kPa-days
- \( \tau_0 = 5000 \) days
- \( \sigma = 200 \) kPa

For most of the geotechnical engineering modeling problems, no \textit{a-priori} information is generally available for the contributory model parameters. Under such circumstances, it is not possible to impose both upper and lower bounds on the design variables. Hence, to gain further insight, a study is carried out wherein only the lower bound is maintained and a total slack is provided to the upper bound. For an identical initial design vector, three different sets of lower bounds are chosen to determine their effect on the convergence of the results. Table 1 shows that for each of the case, the obtained results agree to the actual design vector. It was comprehended that the imposition of a nominal lower bound is sufficient to obtain meaningful results. Hence, for further studies, the nominal lower bound is chosen as \([1 1 1 1]^T\).

Table 1 Effect of lower bound on the obtained solution

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimated Magnitudes</th>
<th>Actual Design Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_1 )</td>
<td>300</td>
<td>300</td>
</tr>
<tr>
<td>( k_2 )</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>( \eta_1 )</td>
<td>3\times10^5</td>
<td>3\times10^{5/2}</td>
</tr>
<tr>
<td>( \eta_2 )</td>
<td>3\times10^4</td>
<td>3\times10^4</td>
</tr>
</tbody>
</table>

In general, for determining model parameters, it is necessary and sufficient to have number of observational data-points equal to the number of unknown parameters. This is possible only when the data-points are precisely determined. However, it is agreeable that such precise solution of observational data-points is not generally possible from load-deformation-time behavior from real-time investigations. Hence, inverse analysis is carried out considering 2-5 uniformly or randomly selected data-points, in order to find out the minimum number of data-points required to obtain reasonable results with the minimum computational effort. In terms of the RMSE between the estimated and model parameters, Fig. 4 reveals that a minimum of 4 data-points (uniformly or randomly distributed in time-deformation response) are necessary to obtain a reasonable solution from inverse analysis for this problem.

Case Studies

Feda (1992) reported the result of long-term settlement characteristics of a 1-D oedometric compression test carried out on Sedlec Kaolin clay under a constant stress of 0.05
Arindam Dey & Prabir K. Basudhar

MPa. The compression curve consisted of segments mutually

intersecting in bifurcation points \((t = 2 \text{ yrs})\) which are believed to mark the structural collapses due to the load imposed. This is stated to occur due to the clustering of Kaolinite flakes into aggregates, and subsequently get broken and totally remolded and transformed to form a compression fabric. Table 3 enumerates the results of the inverse analysis technique to determine the model parameters using the actual settlement-time data. It is observed that a minimum of 5 data-points along the actual settlement-time curve is required for proper back-estimation of the model parameters. A very high value of the parameter \(\eta\) suggests that the behavior of the subgrade resembled a Poynting-Thompson model, a degenerated case of a Burger model [1]. Figure 5 reveals the match of the actual and the predicted settlement-time results.

**Table 3** Model parameters estimated by inverse analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Number of observation points</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_1)</td>
<td>2 111 141.2 145.1</td>
</tr>
<tr>
<td>(k_2)</td>
<td>3115.7 3058.1 3082.9 1322.6</td>
</tr>
<tr>
<td>(\eta_1)</td>
<td>1.4x10^4 1.42x10^4 1.41x10^4 8.5x10^4</td>
</tr>
<tr>
<td>(\eta_2)</td>
<td>103.12 83.23 80.81 122.68</td>
</tr>
<tr>
<td>RMSE</td>
<td>3.2x10^{-3} 3.1x10^{-3} 1.5x10^{-3} 3.1x10^{-3}</td>
</tr>
</tbody>
</table>

**CONCLUSIONS**

This paper reports the development of a generalized inverse analysis formulation, aided by a suitable optimization technique, for the parameter estimation of four-parameter Burger model. Based on the above studies, it is observed that the developed methodology is sufficiently effective in determining the model parameters. The synthetic study revealed that for obtaining reasonable results, only a nominal lower bound supplemented by a total slack in the upper bound is necessary. The developed scheme ensured the global optimality of the solution. A minimum 4-5 uniformly or randomly selected data-points are required to achieve the optimal solution. As a future endeavor, the developed technique can be efficiently used to determine the lumped parameter models for different classes and categories of soil.

**REFERENCES**


