Reliability Evaluation of Earth Slopes by First Order Reliability Method

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ABSTRACT

This paper pertains to a study on the probabilistic approach of stability analysis of earth slopes. The study is concerned with the evaluation of reliability index and the corresponding probability of failure associated with a given slip circle. The critical deterministic slip circles determined at the initial phase of the study have been considered for further analysis. The present analysis is based on the FORM coupled with the Ordinary Method of Slices for evaluation of factor of safety. The statistical properties of the geotechnical parameters have been assumed to follow different distributions, namely, normal distribution and lognormal distribution. To elucidate the present analysis as well as bring out the difference clearly, a few numerical example problems have been solved using both FORM and the popularly used Mean-Value First-Order Second-Moment method (MVFOSM) based on the Taylor series expansion of the factor of safety and the results compared to enable to arrive at significant observations.

1. INTRODUCTION

Stability analysis of earth slopes are conventionally done following a deterministic approach based on limit equilibrium methods of slices in which the critical slip surface and the associated minimum factor of safety are determined. An important shortcoming of this approach is that uncertainties in engineering properties of soils, pore water pressures, and loads are not reflected in the factor of safety. A systematic consideration of these uncertainties warrants a probabilistic approach of analysis wherein the stability status is found out in terms of the probability of failure instead of, or complementary to, the conventional factor of safety. Reliability analysis is carried out to take into account uncertainties in the random variables and the reliability index is often used to indicate the degree of uncertainty in the calculated factor of safety.

Until recently, most of the slope reliability studies reported in the literature were based on the relatively simpler MVFOSM method (Wu and Kraft, 1970; Alonso, 1976; Tang et al., 1976; Vanmarcke, 1977; Chowdhury, 1988; Wolff, 1985; Christian et al., 1994; Duncan, 2000) and the relatively more sophisticated Advanced First-Order-Second-Moment method (AFOSM) (Hasofer and Lind, 1974). But the above methods suffer from some deficiencies. For example, MFOSM method does not use the distribution information about the random variables when it is available. Further, the performance function is linearised at the mean values of the basic variables. When the performance function is nonlinear, significant error may be introduced by neglecting higher order terms in the Taylor’s series expansion. More importantly, the reliability index fails to be constant under different but mechanically equivalent formulations of the same performance function. The use of MVFOSM introduces error of an unknown magnitude; the degree of error depends on the degree of nonlinearity of the performance function and the coefficients of variation of the random variables.

AFOSM method, on the other hand, is applicable for normal random variables only. In contrast, the First Order Reliability Method (FORM) (Haldar and Mahadevan, 2000) is free from the above shortcomings. However, the application of the FORM is few in geotechnical engineering in general and in slope stability analysis, in particular. An excellent review on application of various reliability analysis methods in geotechnical engineering problems is available in the literature (Baecher and Christian, 2003, Chowdhury et al., 2010).

In view of the above, in this paper an attempt has been made to develop a computational scheme for evaluation of
earth slopes based on the FORM. The developed methodology coded in FORTRAN has been used to elucidate the application of FORM to a simple slope in homogeneous soil when coupled with the Ordinary Method of Slices for circular slip surfaces.

2. FORMULATION

Steps in the First Order Reliability Method

In the formulation of the computational scheme for reliability analysis using FORM, the generalized algorithm suggested by Haldar and Mahadevan (2000) has been implemented. For the sake of completeness the same is given below.

Step 1
To define the appropriate limit state equation.

In the case of limit equilibrium slope stability analysis, the limit state equation is given by

\[ g(\mathbf{x}) = F - 1 \]  

where, \( F \) is the factor of safety.

Step 2
To assume an initial value of the reliability index \( \beta \).

Step 3
To assume the initial values of the of the design point \( x_i^* \), \( i=1,2,\ldots,n \). In the absence of any other information, the initial design point can be assumed to be at the mean values of the random variables.

Step 4
To compute the mean and standard deviation at the design point of the equivalent normal distribution for those variables that are nonnormal. Following Rackwitz and Fiessler (1976), these are given by:

\[ \mu_{X_i}^N = x_i^* - \Phi^{-1}[F_{X_i}(x_i^*)] \sigma_{X_i}^N \]  

\[ \sigma_{X_i}^N = \sum_{j=1}^{n} \frac{\partial g}{\partial X_i} \left[ \partial \sigma_{X_i}^N \right] \phi^{-1}[F_{X_i}(x_i^*)] \]  

Step 5
To compute partial derivatives \( \left( \frac{\partial g}{\partial X_i} \right)^* \) evaluated at the design point.

Step 6
To compute the direction cosines \( \alpha_{X_i} \) at the design point as

\[ \alpha_{X_i} = \frac{\left( \frac{\partial g}{\partial X_i} \right)^* \sigma_{X_i}^N}{\left( \sum_{j=1}^{n} \left( \frac{\partial g}{\partial X_i} \right)^* \sigma_{X_i}^N \right)^{1/2}} \]  

Step 7
To compute the new values for checking point as \( x_i^* \)

\[ x_i^* = \mu_{X_i}^N - \alpha_i \beta \sigma_{X_i}^N \]  

Steps 4 through 7 may be repeated until the estimates of \( \beta \) converge with a predetermined tolerance (say, 0.005). Once the direction cosines converge, the new checking point can be estimated, keeping \( \beta \) as the unknown parameter.

Step 8
To compute an updated value for \( \beta \) using the condition that the limit state equation must be satisfied at the new checking point.

Step 9
To repeat Steps 3 through 8 until \( \beta \) converges to a predetermined tolerance level. A tolerance level of 0.001 can be used.

Probability of Failure
Once the value of the reliability index, \( \beta \), is determined by any of the methods discussed above, the probability of failure, \( p_F \), is then obtained as:

\[ p_F = \Phi(-\beta) \]  

where \( \Phi(.) \) is the standard normal cumulative probability density function values of which are tabulated in standard texts.

Developed Computer Program
The above algorithm for FORM in conjunction with the Ordinary Method of Slices (OMS) has been coded in FORTRAN in the program RELFORM. All computations involved in this study have been carried out using the developed computer program.

3. ILLUSTRATIVE EXAMPLE

To elucidate the application of the developed computational scheme a simple slope of height 10m and inclination 45° in a homogeneous soil has been considered. The effective stress strength parameters \( c' \) and \( \tan \phi' \) are treated as random variables having their statistical properties as given in Table 1. The bulk unit weight, \( \gamma \) and the pore water pressure ratio, \( r_u \), have been assumed as 18.0 kN/m\(^3\) and 0.2 respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>Coefficient of Variation</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c' )</td>
<td>18.0 kPa</td>
<td>0.2</td>
<td>0.36 kPa</td>
</tr>
<tr>
<td>( \tan \phi' )</td>
<td>0.577</td>
<td>0.1</td>
<td>0.0577</td>
</tr>
</tbody>
</table>

4. RESULTS AND DISCUSSION

Reliability Index for an Arbitrary Slip Circle

Using the program RELFORM, computations have been carried out for an arbitrarily selected slip circle \( (x_0=4.0, y_0=16.0, r=16.492) \) with respect to an axes system passing...
through the toe of the slope such that the sloping portion of the ground surface has a positive slope (1:1).

Mean Factor of Safety
Considering the mean values of the soil parameter, OMS has yielded a factor of safety of 1.59.

Probability Distributions for the Basic Random Variables:
Reliability indexes have been determined for four different cases (Table 2) of combinations of the assumed probability distributions for the basic random variables, i.e., the strength parameters $c'$ and $\tan \phi$.

Table 3 presents the results of analysis. It is seen that the $\beta$-value varies significantly with the type of distribution considered for $c'$ and $\tan \phi$. Further, the normal-normal assumption yields the lowest value of $\beta$, while the lognormal-lognormal assumption yields the highest value of $\beta$. Again, the lognormal assumption for $c'$ always yields higher values. In other words, it is the type of distribution of $c'$ (rather than that of $\tan \phi$) which controls the value of $\beta$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Random Variable</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$c'$</td>
<td>Lognormal</td>
</tr>
<tr>
<td>1</td>
<td>$\tan \phi'$</td>
<td>Normal</td>
</tr>
<tr>
<td>2</td>
<td>$c'$</td>
<td>Normal</td>
</tr>
<tr>
<td>2</td>
<td>$\tan \phi'$</td>
<td>Lognormal</td>
</tr>
<tr>
<td>3</td>
<td>$c'$</td>
<td>Lognormal</td>
</tr>
<tr>
<td>3</td>
<td>$\tan \phi'$</td>
<td>Lognormal</td>
</tr>
<tr>
<td>4</td>
<td>$c'$</td>
<td>Normal</td>
</tr>
<tr>
<td>4</td>
<td>$\tan \phi'$</td>
<td>Normal</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case</th>
<th>$c'$</th>
<th>$\tan \phi'$</th>
<th>Values of $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Normal</td>
<td>Normal</td>
<td>4.03</td>
</tr>
<tr>
<td>2</td>
<td>Normal</td>
<td>Lognormal</td>
<td>4.13</td>
</tr>
<tr>
<td>3</td>
<td>Lognormal</td>
<td>Normal</td>
<td>4.65</td>
</tr>
<tr>
<td>4</td>
<td>Lognormal</td>
<td>Lognormal</td>
<td>5.14</td>
</tr>
</tbody>
</table>

Reliability Index for the Critical Slip Circle

Critical Deterministic Slip Circles and the Associated Minimum Factor of Safety ($F_{\min}$)

For the purpose of this study, computer program has also been developed for determination of the critical slip circle and the associated minimum factor of safety ($F_{\min}$). This developed computer program DETCRIT is based on the Ordinary Method of Slices (OMS) for the evaluation of Factor of safety and the Sequential Unconstrained Minimization Technique (SUMT) of nonlinear programming to search for the surface of minimum Factor of safety. This optimization technique for constrained minimization makes use of the powerful unconstrained minimization algorithms such as the Powell Method of Multidimensional minimization and the Quadratic Interpolation method of unidimensional minimization. One advantage of using the Quadratic Interpolation method is that it requires only function evaluation and no gradient evaluation.

For the same example problem, using the developed computer program DETCRIT for deterministic analysis, the deterministic critical slip circle ($x_0=1.37$, $y_0=10.44$, $r=10.519$) has been determined. The corresponding minimum factor of safety has been obtained as $F_{\min}=1.27$.

Reliability Index for the Deterministic Critical Slip Circle:

For the above problem and the deterministic critical slip circle, Table 4 shows the values of the reliability index $\beta$ for different assumed probability distributions for the basic random variables. As in Table 3, again it is seen that the $\beta$-value varies significantly with the type of distribution considered for $c'$ and $\tan \phi$, but in this case it is more evident that it is the type of distribution of $c'$ (rather than that of $\tan \phi$) which controls the value of $\beta$.

Table 4:

<table>
<thead>
<tr>
<th>Case</th>
<th>$c'$</th>
<th>$\tan \phi'$</th>
<th>Values of $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Normal</td>
<td>Normal</td>
<td>1.78</td>
</tr>
<tr>
<td>2</td>
<td>Normal</td>
<td>Lognormal</td>
<td>1.77</td>
</tr>
<tr>
<td>3</td>
<td>Lognormal</td>
<td>Normal</td>
<td>2.02</td>
</tr>
<tr>
<td>4</td>
<td>Lognormal</td>
<td>Lognormal</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Comparison between Values of Reliability Index obtained by using FORM and MVFOSM.

It is of interest to compare the values of the reliability index $\beta$ obtained above based on the First Order Reliability Method (FORM) with those obtained by using the widely used and relatively simpler Mean-Value First-Order Second-Moment Method (MVFOSM). In obtaining the probability of failure using Equation (6), $\beta$ obtained from the MVFOSM are sometimes modified to suit the assumption that the factor of safety is lognormally distributed, rather than normally distributed. As pointed out by Hassan and Wolff (1999), some investigators e.g., US. Navy use this assumption.

For the illustrative example described above, reliability analysis has been carried out based on the MVFOSM for both the arbitrary slip circle and the deterministic critical slip circle. The values of the reliability index $\beta$-have been computed using another developed computer program RELMVFOSM assuming (i) F is normally distributed and, also, (ii) F is lognormally distributed.

Table 5 presents a comparison of the $\beta$-values obtained from the FORM and the MVFOSM. It is seen that for both
the slip circles analysed, MVFOSM yields much less values of the reliability index when compared to those yielded by FORM. The MVFOSM results are thus conservative. Further, $\beta$-values from MVFOSM corresponding to the lognormal assumption for Factor of Safety are seen to be close to those from FORM corresponding to Normal-Normal assumption for the distribution of the two basic variates, $c'$ and $\tan \phi'$.

Table 5: Comparison Between Values of Reliability Index $\beta$ Obtained from the FORM and MVFOSM Methods

<table>
<thead>
<tr>
<th>Method used for Reliability Analysis</th>
<th>Slip Circle Analysed</th>
<th>Arbitrary</th>
<th>Deterministic Critical</th>
</tr>
</thead>
<tbody>
<tr>
<td>FORM</td>
<td>$c'$, $\tan \phi'$</td>
<td>4.03</td>
<td>1.78</td>
</tr>
<tr>
<td>Normal</td>
<td>Normal</td>
<td>4.13</td>
<td>1.77</td>
</tr>
<tr>
<td>Lognormal</td>
<td>Normal</td>
<td>4.65</td>
<td>2.02</td>
</tr>
<tr>
<td>Lognormal</td>
<td>Lognormal</td>
<td>5.14</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Comparison of Values of Probability of Failure

Once the values of the reliability indexes have been obtained, it is of greater interest to study how the order of magnitude of the failure probability varies in the above cases. Corresponding to the value of $\beta$ in a given case, Equation (6) is used to obtain the values of the probability of failure ($p_F$). Table 6 presents these values for the most common cases when only normal distribution is used.

Table 6: Probability of Failure, $p_F$ for the Cases in Which Normal Distribution has been Assumed

<table>
<thead>
<tr>
<th>Method used for Reliability Analysis</th>
<th>Slip Circle Analysed</th>
<th>Arbitrary</th>
<th>Critical</th>
</tr>
</thead>
<tbody>
<tr>
<td>FORM</td>
<td>$c'$: and $\tan \phi'$ both normal</td>
<td>2.789x10^{-4}</td>
<td>3.754x10^{-2}</td>
</tr>
<tr>
<td>MVFOSM</td>
<td>$F$ normally distributed</td>
<td>7.622x10^{-3}</td>
<td>5.821x10^{-2}</td>
</tr>
</tbody>
</table>

It can be seen from Table 6 that even when only normal distribution is assumed for the basic variates, MVFOSM with normal assumption for $F$ can indicate a value of $p_F$ which is an order of magnitude higher than that obtained from the more accurate FORM method of reliability analysis.

5. CONCLUDING REMARKS

On the basis of the observations made from the study undertaken in this paper, the following concluding remarks can be made:

1. The developed computational scheme based on the First-Order Reliability Method (FORM) in conjunction with the Ordinary Method of Slices (OMS) for the determination of Factor of safety appears to be sound tool for reliability evaluation of earth slopes.

2. The results obtained for the illustrative example of a simple slope in homogeneous soil have revealed that the types of probability distributions considered for the basic random variables have marked influence on the values of reliability index and hence, probability of failure.

3. Based on the above observations it can be stated that the MVFOSM in which there is no provision for incorporating the probability distribution of the basic variates, is likely to lead to results that could be in error to an uncertain degree. However, the results obtained for a simple slope situation indicates that MVFOSM is a conservative method when compare to the FORM. On the other hand the limitation of the Hasofer-Lind AFOSM Method that it is applicable only when the basic variates are normally distributed must also be taken note of.

4. The developed procedure based on FORM should, however, be applied to a wide variety of slope stability problems before arriving at a definite conclusion regarding its accuracy and applicability.

REFERENCES


