Endochronic Modelling of Static Triaxial Response of Sand

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ABSTRACT

The response of sand to static loading is complex due to its nonlinear stress-strain and pore pressure generation behaviour, necessitating modeling sand as a two phase system. In the present study a new pseudo coupled constitutive model based on the endochronic theory is developed. The endochronic model is based on the observation of the state of stress in the neighborhood of a point in a plastic material and depends on the set of all previous states of deformation of that neighborhood. The rapidity of deformation states is achieved by introducing a time scale $\xi$. The pore pressure generation is related to the drained volumetric response to capture the undrained response under static loading. The model is incorporated into a finite difference code FLAC$^{3D}$ to perform drained and undrained static triaxial tests on sand. The developed constitutive law is validated with the published experimental triaxial responses of sand.

1. INTRODUCTION

The response of sand to static loading is quite complex due to its heterogeneous and nonlinear volumetric behaviour in comparison to other engineering materials. In order to model different aspects of the sand response to static loading the following observed behaviors are to be addressed: (1) The nonlinear volumetric response; (2) densification due to change in deviatoric state; (3) the deviatoric response and its dependence on normal stress; and (4) the effects of unloading and reloading, during isotropic compression and shearing (Wu et al., 1985). A constitutive relationship should therefore be capable of modeling the strength characteristics, compression and dilatancy of sand. In the present study the static response of sand is described based on the endochronic theory.

2. ENDOCHRONIC THEORY

In plasticity theories the discontinuities observed in material behavior upon loading beyond the "yield point" is described conceptually using a yield surface in stress space. However, the phenomenon of yield is usually a gradual transition from a linear to a non-linear stress-strain response and is difficult to establish precisely the point of occurrence of yielding, the conceptual difficulties that are encountered by the introduction of the yield surface are completely circumvented by the theory of plasticity which is developed on the basis that the state of stress in the neighborhood of a point in a plastic material which depends on the set of all previous states of deformation of that neighborhood. In this theory, the stress is a function of the strain history, defined with respect to the intrinsic time scale $\xi$, which is a property of the material (Valanis, 1970). Valanis (1970, 1971) developed the endochronic theory of plasticity based on the concept of intrinsic time.

3. UNDER DRAINED STATIC RESPONSE OF SAND

In the present study the new improved endochronic theory (Valanis, 1996, Wu et al., 1985, Valanis, 1998, Valanis, 1981, Valanis and Read, 1988) based on the Gibbs formulation (Wu and Wang, 1983), is used to simulate the triaxial conditions. The behaviour of drained sand is described in combination of three effects, i.e., the hydrostatic behaviour that governs the volume change due to the change in the hydrostatic stress, the deviatoric behaviour relating the deviatoric stress to the deviatoric strain, and densification (including dilation) that is the volume change due to shearing. In the present study the hydrostatic behaviour is described in the original form (Wu et al., 1985) and the hydrostatic part of volumetric strain is given as

$$d\varepsilon_{\text{V}} = \frac{d\sigma}{K_v} + \frac{\sigma_d}{K_v} \alpha \frac{\lambda}{K_v} \gamma_{\text{d}}$$

for $d\sigma > 0$

$$d\varepsilon_{\text{V}} = \frac{d\sigma}{K_v}$$

for $d\sigma <= 0$ (1)
In which $K_o$, $\lambda$, $\sigma_i$ are constants. The deviatoric behaviour is derived as

$$s = S \frac{d\theta}{dz} + \theta \rho(z_i - \bar{z}) \frac{d\theta}{dz} + Ro \theta$$  \hspace{0.5cm} \text{(2)}$$

The deviatoric intrinsic time $dz_i = \frac{d\zeta}{f(z_i, \sigma)}$ \hspace{0.5cm} \text{(3)}

The intrinsic time is expressed as

$$d\zeta = |d\varepsilon - k \frac{d\sigma}{E_o}|$$  \hspace{0.5cm} \text{with } 0 < k, \varepsilon < 1 \hspace{0.5cm} \text{(4)}$$

In which $S$ is related to yield stress; $Z_i$ and $\zeta_i$ are deviatoric intrinsic times; $s$ = deviatoric stress; $e$ = deviatoric strain; $\theta$ = $s/2\mu$; $2\mu$ = shear modulus; $f$ = isotropic hardening function. This new definition of intrinsic time removes the drawbacks for the simple endochronic theory (Valanis, 1981, Rivlin, 1981) and, in addition, the case of $k_0 = 1$ leads to the existence of yield surface. In case of a true triaxial test, the three axes are principal axes and are denoted by $x$, $y$, and $z$, with $z$ vertical. The components of the stress $\sigma$ and strain $\varepsilon$ are $\sigma_x$, $\sigma_y$, and $\sigma_z$ and $\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_z$. In case of triaxial test $\sigma_1 = \sigma_y$ and $\sigma_0 = \sigma_z$. The components of the deviatoric strain are $\varepsilon_x = \varepsilon_y = -1/3(\varepsilon_z - \varepsilon_0)$ and $\varepsilon_z = -2/3(\varepsilon_z - \varepsilon_0)$. Thus the stress component $Q_z = Q_y = -0.5 Q_x$. Thus,

$$s_i = \sqrt{\frac{2\mu_i(1 + \beta\xi_i)}{\beta(n_i - k)}[1 - (1 + \beta\xi_s)^{-(n_i-k)}]} + \sqrt{\frac{2\mu_i(1 + \beta\xi_s)}{\beta(n_i - k)}[1 - (1 + \beta\xi_s)^{-(n_i-k)}]} \hspace{0.5cm} \text{(5)}$$

In present study, the differential form of Eq. 5 is used in the finite difference code to study the response of sand. The lateral stresses are obtained as $S_x = S_y = -S_z/2$, under conventional triaxial testing conditions. In Eq.5; $n = 1 + a/\beta$; and $n_0 = 1 + a_0/\beta$. Where $a$ and $\beta$ are material constants.

Dilatancy i.e., volume changes associated with distortion of granular media, is an important phenomenon in simulating the response of sand. In the present study the effect of dilatancy is simulated by coupling deviatoric and hydrostatic behaviour using rate equations for the internal variables (Valanis and Read, 1988). The deviatoric volumetric strain is

$$\varepsilon_k^v = I'\gamma' (\sigma_0 - \sigma_o)Z_0 + \frac{1}{\gamma'}[D_0 - I'\gamma' (\sigma_0 - \sigma_o)][1 - e^{-\gamma'Z_0}] \hspace{0.5cm} \text{(6)}$$

Where, $Z_0 = \text{deviatoric intrinsic time}$; $D_0 = \text{slope}$ of the densification curve; $I'\gamma' = \text{constant}$ determined by curve fitting; $\sigma_0$ and $\sigma_o$ are the initial confining stress and critical deviatoric stress. The parameter $Z_0$ is scaled as

$$dZ_0 = \frac{d\xi_{\sigma} - d\sigma}{1 + \xi_{\sigma}^2}; \text{ in which } d\xi_{\sigma} = d\varepsilon - k\frac{d\sigma}{E_o} \hspace{0.5cm} \text{(7)}$$

The material parameters are obtained from the experimental stress – strain response curve as per the procedure recommended by Wu and Wang (1983) and Wu et al. (1985).

4. STATIC RESPONSE OF SAND UNDER UNDRAINED CONDITIONS

In the present study, the response of sand under undrained condition is simulated by coupling the pore pressure mechanism to the endochronic constitutive relationship. To simulate the behaviour of undrained condition it is important to simulate the pore pressure generation and dissipation. Hence a two phase system has to be effectively modeled to capture the response of sand under undrained condition. The present study is the first of its kind to effectively model the undrained response of sand using the endochronic theory. In order to model the undrained response effectively and to reduce the complexity of the model a coupled model proposed by Martin et al (1975) is incorporated in to the endochronic model to simulate undrained behaviour of sand. This method utilizes an incremental mechanism to compute the progressive increase in the pore pressure and is given as

$$\Delta u = E_i \times \Delta \varepsilon_{od} \hspace{0.5cm} \text{where, } E_i = k_0 (\sigma_i^v)^{0.5} \left(\frac{\sigma_i^v}{\sigma_o^v}\right)^{0.5} \hspace{0.5cm} \text{(9)}$$

The parameters $m$ and $k_0$ are obtained as discussed in Martin et al (1975).The excess pore pressure increment evaluated from Eq. 8 is deducted from Eq. 5 the total stress to yield the effective stress.

5. NUMERICAL SIMULATION

In the present study the endochronic model is implemented into the 3D finite difference code FLAC$^{\text{3D}}$ using the user defined module CPPUDM. The visual C++ code written incorporating the constitutive equation is compiled to generate the dynamic link library (dll) containing the constitutive equation which is invoked in $FLAC^{\text{3D}}$ when the model is loaded. In the drained triaxial test, the soil sample is modeled using 3D cylindrical wedge elements of size 1mm which is reflected to form a cylindrical grid scaled to represent the actual triaxial specimen size of 38mm radius and 76mm height. The discretized model is presented in Figure 1. The user defined endochronic constitutive law is applied to the soil model using the CPPUDM module. Each zone in the soil model is updated and behaves according to the proposed constitutive law. The top and the bottom ends of the sample are fixed simulating a rough condition between the porous plate and the soil sample. Confining pressure corresponding to the test conditions is applied along the periphery of the soil sample. The deviatoric stress
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is applied as a velocity at the top and bottom ends of the soil sample. The displacement and the stresses developed in the middle of the sample are used to obtain the stress–strain relationship of the soil sample.

![Discretized Finite Difference Model](image)

**Fig. 1:** Discretized Finite Difference Model

6. VALIDATION

The proposed endochronic constitutive models for drained and undrained conditions are validated with the published triaxial response of Toyoura Sand (Verdugo and Ishihara, 1996). The materials parameters for numerical simulation are obtained by curve fitting the experimental stress-strain plots using Eq. 5, 6 and 8 and is presented in Table 1. The drained static behaviour of Toyoura sand is compared with the simulated response in Figures 2 and 3. It can be observed from the figure that the simulated response matches well with the experimental findings. However, the slight discrepancies observed in the simulated volumetric strain from that of the experimental curves are due to the fact that all strains considered in the endochronic theory are plastic strains.

![Comparison of Static Triaxial Response of Toyoura Sand](image)

**Fig. 2:** Comparison of Static Triaxial Response of Toyoura Sand (Dense Condition, $e = 0.82$; Confining Stress of 270 kPa)

![Comparison of Static Triaxial Response of Toyoura Sand](image)

**Fig. 3:** Comparison of Static Triaxial Response of Toyoura Sand (Med-Dense Condition, $e = 0.76$; Confining Stress of 270 kPa)

It can also be observed from Figures 2a and 3a that the endochronic theory is capable to predict well the drained behaviour of Toyoura Sand. The yield point of the soil is

<table>
<thead>
<tr>
<th>Material Parameter</th>
<th>Toyoura Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear Modulus (MPa)</td>
<td>35</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.33</td>
</tr>
<tr>
<td>$n$</td>
<td>60</td>
</tr>
<tr>
<td>$S_s$(kPa)</td>
<td>200</td>
</tr>
<tr>
<td>$\sigma_0$(kPa)</td>
<td>450</td>
</tr>
<tr>
<td>$I^1y^1$(kPa$^{-1}$)</td>
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</tr>
<tr>
<td>$D_o$</td>
<td>0.7</td>
</tr>
<tr>
<td>$y^1$</td>
<td>2.3</td>
</tr>
<tr>
<td>$\xi$</td>
<td>90</td>
</tr>
<tr>
<td>$m$</td>
<td>12</td>
</tr>
<tr>
<td>$k_2$</td>
<td>0.375</td>
</tr>
<tr>
<td>Minimum Void Ratio</td>
<td>0.656</td>
</tr>
<tr>
<td>Maximum Void Ratio</td>
<td>0.873</td>
</tr>
<tr>
<td>Slope of Steady state line</td>
<td>0.039</td>
</tr>
</tbody>
</table>
predicted well by the endochronic model, however there is over prediction of stress after yielding in dense sand conditions. The undrained compression behaviour of medium dense sand is predicted well as can be seen from Figure 3a. It is observed from Figures 2b and 3b that the simulated volumetric response matches reasonably well with the experimental findings. It is found that the endochronic model is capable of simulating the stress-strain response with an accuracy of over 90%. The validation of the developed endochronic model with the experimental undrained response of the Toyoura Sand (Verdugo and Ishihara, 1996) is presented in Figure 4.

Fig. 4: Deviatoric Stress q’ Versus Strain Plot for Toyoura Sand

It can be observed from Figure 4 that the deviatoric stress q’ versus strain curve is simulated well by the developed model as a whole. Although the stress-strain response is predicted well till the maximum yield point, the maximum yield stress is slightly under predicted by about 5 to 10%. The ultimate stress state simulated by the endochronic model matches well the experimental behaviour. Figure 5 presents the comparison of the experimental and the simulated pore pressure response for different confining pressures. It is observed that the simulated pore pressure response matches well with the experimental response for different confining pressures.

Fig. 5: Plot of r, Versus Strain for Toyoura Sand, Comparison of Experimental

7. SUMMARY

In this study, the improved endochronic constitutive law is coupled to a pore pressure generation mechanism to simulate the static response of sand under drained and undrained conditions. The proposed constitutive law is incorporated into a 3D finite difference code FLAC3D. The developed model is validated with the published experimental drained and undrained response of Toyoura sand. It is observed that the endochronic constitutive law is capable of predicting the static drained and undrained response of sand with over 90% accuracy.

REFERENCES


