APPENDIX B

Methods for Solution of Simultaneous Linear Equations

Introduction

Many problems in engineering and mathematical physics require the solution of a system of simultaneous linear algebraic equations. Stress analysis, heat transfer, and vibration analysis are engineering problems for which the finite element formulation for solution typically involves the solving of simultaneous linear equations. This appendix introduces methods applicable to both longhand and computer solutions of simultaneous linear equations. Many methods are available for the solution of equations; for brevity's sake, we will discuss only some of the more common methods.

B.1 General Form of the Equations

In general, the set of equations will have the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\
\vdots \qquad \vdots \qquad \quad \vdots \quad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n
\end{aligned} \tag{B.1.1}
$$

where the a_{ij}'s are the coefficients of the unknown x_j's, and the c_i's are the known right-side terms. In the structural analysis problem, the a_{ij}'s are the stiffness coefficients k_{ij}'s, the x_j's are the unknown nodal displacements d_i's, and the c_i's are the known nodal forces F_i's.

If the c's are not all zero, the set of equations is *nonhomogeneous*, and all equations must be independent to yield a unique solution. Stress analysis problems typically involve solving sets of nonhomogeneous equations.

If the c's are all zero, the set of equations in *homogeneous*, and nontrivial solutions exist only if all equations are not independent. Buckling and vibration problems typically involve homogeneous sets of equations.

B.2 Uniqueness, Nonuniqueness, and Nonexistence of Solution

To solve a system of simultaneous linear equations means to determine a unique set of values (if they exist) for the unknowns that satisfy every equation of the set simultaneously. A unique solution exists if and only if the determinant of the square coefficient matrix is not equal to zero. (All of the engineering problems considered in this text result in square coefficient matrices.) The problems in this text usually result in a system of equations that has a unique solution. Here we will briefly illustrate the concepts of uniqueness, nonuniqueness, and nonexistence of solution for systems of equations.

Uniqueness of Solution

$$\begin{aligned} 2x_1 + 1x_2 &= 6 \\ 1x_1 + 4x_2 &= 17 \end{aligned} \tag{B.2.1}$$

For Eqs. (B.2.1), the determinant of the coefficient matrix is not zero, and a unique solution exists, as shown by the single common point of intersection of the two Eqs. (B.2.1) in Figure B–1.

Nonuniqueness of Solution

$$\begin{aligned} 2x_1 + 1x_2 &= 6 \\ 4x_1 + 2x_2 &= 12 \end{aligned} \tag{B.2.2}$$

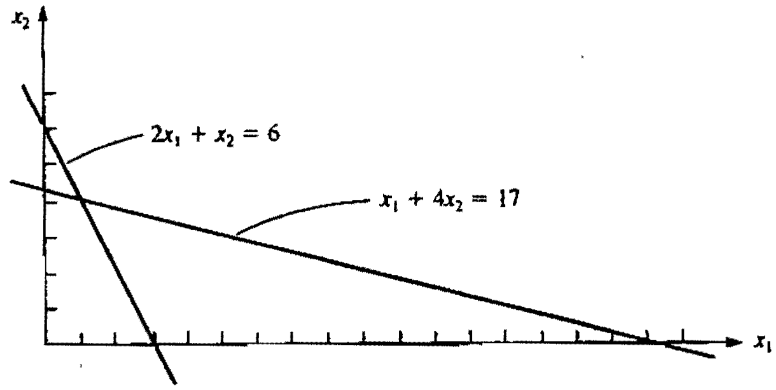

Figure B–1 Uniqueness of solution

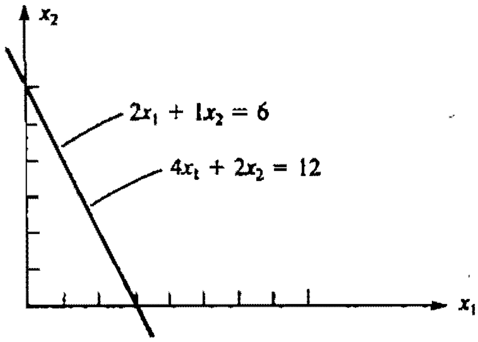

Figure B–2 Nonuniqueness off solution

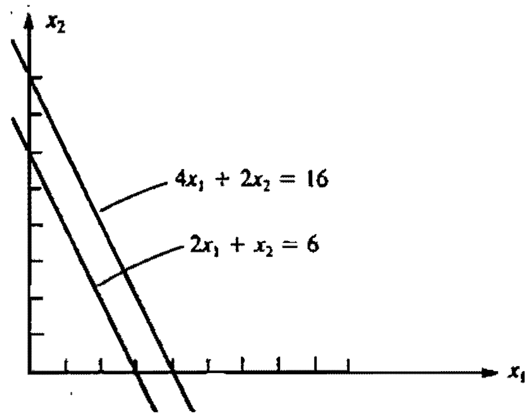

Figure B–3 Nonexistence of solution

For Eqs. (B.2.2), the determinant of the coefficient matrix is zero; that is,

$$\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$$

Hence the equations are called *singular*, and either the solution is not unique or it does not exist. In this case, the solution is not unique, as shown in Figure B–2.

Nonexistence of Solution

$$\begin{aligned} 2x_1 + x_2 &= 6 \\ 4x_1 + 2x_2 &= 16 \end{aligned} \qquad \text{(B.2.3)}$$

Again, the determinant of the coefficient matrix is zero. In this case, no solution exists because we have parallel lines (no common point of intersection), as shown in Figure B–3.

B.3 Methods for Solving Linear Algebraic Equations

We will now present some common methods for solving systems of linear algebraic equations that have unique solutions. Some of these methods work best for small sets of equations solved longhand, whereas others are well suited for computer application.

Cramer's Rule

We begin by introducing a method known as *Cramer's rule*, which is useful for the longhand solution of small numbers of simultaneous equations. Consider the set of equations

$$\underline{a}\underline{x} = \underline{c} \qquad \text{(B.3.1)}$$

or, in index notation,

$$\sum_{i=1}^{n} a_{ij}x_j = c_i \tag{B.3.2}$$

We first let $\underline{d}^{(i)}$ be the matrix $\underline{a}$ with column i replaced by the column matrix $\underline{c}$. Then the unknown x_i's are determined by

$$x_i = \frac{|\underline{d}^{(i)}|}{|\underline{a}|} \tag{B.3.3}$$

As an example of Cramer's rule, consider the following equations:

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 &= 2 \\ 2x_1 - 4x_2 + 2x_3 &= 1 \\ 4x_2 + x_3 &= 3 \end{aligned} \tag{B.3.4}$$

In matrix form, Eqs. (B.3.4) become

$$\begin{bmatrix} -1 & 3 & -2 \\ 2 & -4 & 2 \\ 0 & 4 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \\ 3 \end{Bmatrix} \tag{B.3.5}$$

By Eq. (B.3.3), we can solve for the unknown x_i's as

$$x_1 = \frac{|\underline{d}^{(1)}|}{|\underline{a}|} = \frac{\begin{vmatrix} 2 & 3 & -2 \\ 1 & -4 & 2 \\ 3 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} -1 & 3 & -2 \\ 2 & -4 & 2 \\ 0 & 4 & 1 \end{vmatrix}} = \frac{-41}{-10} = 4.1$$

$$x_2 = \frac{|\underline{d}^{(2)}|}{|\underline{a}|} = \frac{\begin{vmatrix} -1 & 2 & -2 \\ 2 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix}}{-10} = 1.1 \tag{B.3.6}$$

$$x_3 = \frac{|\underline{d}^{(3)}|}{|\underline{a}|} = \frac{\begin{vmatrix} -1 & 3 & 2 \\ 2 & -4 & 1 \\ 0 & 4 & 3 \end{vmatrix}}{-10} = -1.4$$

In general, to find the determinant of an $n \times n$ matrix, we must evaluate the determinants of n matrices of order $(n-1) \times (n-1)$. It has been shown that the solution of n simultaneous equations by Cramer's rule, evaluating determinants by expansion by minors, requires $(n-1)(n+1)!$ multiplications. Hence, this method takes large amounts of computer time and therefore is not used in solving large systems of simultaneous equations either longhand or by computer.

Inversion of the Coefficient Matrix

The set of equations $\underline{a}\underline{x} = \underline{c}$ can be solved for $\underline{x}$ by inverting the coefficient matrix $\underline{a}$ and premultiplying both sides of the original set of equations by $\underline{a}^{-1}$, such that

$$\begin{aligned} \underline{a}^{-1}\underline{a}\underline{x} &= \underline{a}^{-1}\underline{c} \\ \underline{I}\underline{x} &= \underline{a}^{-1}\underline{c} \\ \underline{x} &= \underline{a}^{-1}\underline{c} \end{aligned} \tag{B.3.7}$$

Two methods for determining the inverse of a matrix (the cofactor method and row reduction) were discussed in Appendix A.

The inverse method is much more time-consuming (because much time is required to determine the inverse of $\underline{a}$) than either the elimination method or the iteration method, which are discussed subsequently. Therefore, inversion is practical only for small systems of equations.

However, the concept of inversion is often used during the formulation of the finite element equations, even though elimination or iteration is used in achieving the final solution for the unknowns (such as nodal displacements).

Besides the tedious calculations necessary to obtain the inverse, the method usually involves determining the inverse of sparse, banded matrices (stiffness matrices in structural analysis usually contain many zeros with the nonzero coefficients located in a band around the main diagonal). This sparsity and banded nature can be used to advantage in terms of storage requirements and solution algorithms on the computer. The inverse results in a dense, full matrix with loss of the advantages resulting from the sparse, banded nature of the original coefficient matrix.

To illustrate the solution of a system of equations by the inverse method, consider the same equations that we solved previously by Cramer's rule. For convenience's sake, we repeat the equations here.

$$\begin{bmatrix} -1 & 3 & -2 \\ 2 & -4 & 2 \\ 0 & 4 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 1 \\ 3 \end{Bmatrix} \tag{B.3.8}$$

The inverse of this coefficient matrix was found in Eq. (A.3.11) of Appendix A. The unknowns are then determined as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = -\frac{1}{10} \begin{bmatrix} -12 & -11 & -2 \\ -2 & -1 & -2 \\ 8 & 4 & -2 \end{bmatrix} \begin{Bmatrix} 2 \\ 1 \\ 3 \end{Bmatrix} = \begin{Bmatrix} 4.1 \\ 1.1 \\ -1.4 \end{Bmatrix} \tag{B.3.9}$$

Gaussian Elimination

We will now consider a commonly used method called *Gaussian elimination* that is easily adapted to the computer for solving systems of simultaneous equations. It is based on triangularization of the coefficient matrix and evaluation of the unknowns by back-substitution starting from the last equation.

The general system of n equations with n unknowns given by

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix} \tag{B.3.10}$$

will be used to explain the Gaussian elimination method.

1. Eliminate the coefficient of x_1 in every equation except the first one. To do this, select a_{11} as the pivot, and
   a. Add the multiple $-a_{21}/a_{11}$ of the first row to the second row.
   b. Add the multiple $-a_{31}/a_{11}$ of the first row to the third row.
   c. Continue this procedure through the nth row.

   The system of equations will then be reduced to the following form:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a'_{22} & \dots & a'_{2n} \\ \vdots & & & \vdots \\ 0 & a'_{n2} & \dots & a'_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c'_2 \\ \vdots \\ c'_n \end{Bmatrix} \tag{B.3.11}$$

2. Eliminate the coefficient of x_2 in every equation below the second equation. To do this, select a'_{22} as the pivot, and
   a. Add the multiple $-a'_{32}/a'_{22}$ of the second row to the third row.
   b. Add the multiple $-a'_{42}/a'_{22}$ of the second row to the fourth row.
   c. Continue this procedure through the nth row.

   The system of equations will then be reduced to the following form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a'_{22} & a'_{23} & \dots & a'_{2n} \\ 0 & 0 & a''_{33} & \dots & a''_{3n} \\ \vdots & & & & \vdots \\ 0 & 0 & a''_{n3} & \dots & a''_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c'_2 \\ c''_3 \\ \vdots \\ c''_n \end{Bmatrix} \tag{B.3.12}$$

   We repeat this process for the remaining rows until we have the system of equations (called *triangularized*) as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ 0 & a'_{22} & a'_{23} & a'_{24} & \dots & a'_{2n} \\ 0 & 0 & a''_{33} & a''_{34} & \dots & a''_{3n} \\ 0 & 0 & 0 & a'''_{44} & \dots & a'''_{4n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & a^{n-1}_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c'_2 \\ c''_3 \\ c'''_4 \\ \vdots \\ c^{n-1}_n \end{Bmatrix} \tag{B.3.13}$$

3. Determine x_n from the last equation as

$$x_n = \frac{c^{n-1}_n}{a^{n-1}_{nn}} \tag{B.3.14}$$

and determine the other unknowns by back-substitution. These steps are summarized in general form by

$$a_{ij} = a_{ij} - a_{kj}\frac{a_{ik}}{a_{kk}} \qquad \begin{aligned} k &= 1, 2, \ldots, n-1 \\ i &= k+1, \ldots, n \\ j &= k, \ldots, n+1 \end{aligned} \tag{B.3.15}$$

$$x_i = \frac{1}{a_{ii}}\left(a_{i,n+1} - \sum_{r=i+1}^{n} a_{ir}x_r\right)$$

where $a_{i,n+1}$ represent the latest right side c's given by Eq. (B.3.13).

We will solve the following example to illustrate the Gaussian elimination method.

Example B.1

Solve the following set of simultaneous equations using Gauss elimination method.

$$\begin{aligned} 2x_1 + 2x_2 + 1x_3 &= 9 \\ 2x_1 + 1x_2 \qquad &= 4 \\ 1x_1 + 1x_2 + 1x_3 &= 6 \end{aligned} \tag{B.3.16}$$

Step 1

Eliminate the coefficient of x_1 in every equation except the first one. Select $a_{11} = 2$ as the pivot, and

a. Add the multiple $-a_{21}/a_{11} = -2/2$ of the first row to the second row
b. Add the multiple $-a_{31}/a_{11} = -1/2$ of the first row to the third row.
   We then obtain

$$\begin{aligned} 2x_1 + 2x_2 + 1x_3 &= 9 \\ 0x_1 - 1x_2 - 1x_3 &= 4 - 9 = -5 \\ 0x_1 + 0x_2 + \tfrac{1}{2}x_3 &= 6 - \tfrac{9}{2} = \tfrac{3}{2} \end{aligned} \tag{B.3.17}$$

Step 2

Eliminate the coefficient of x_2 in every equation below the second equation. In this case, we accomplished this in step 1.

Step 3

Solve for x_3 in the third of Eqs. (B.3.17) as

$$x_3 = \frac{\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)} = 3$$

Solve for x_2 in the second of Eqs. (B.3.17) as

$$x_2 = \frac{-5+3}{-1} = 2$$

Solve for x_1 in the first of Eqs. (B.3.17) as

$$x_1 = \frac{9-2(2)-3}{2} = 1$$

To illustrate the use of the index Eqs. (B.3.15), we re-solve the same example as follows. The ranges of the indexes in Eqs. (B.3.15) are $k = 1, 2$; $i = 2, 3$; and $j = 1, 2, 3, 4$.

Step 1

For $k = 1$, $i = 2$, and j indexing from 1 to 4,

$$\begin{aligned}
a_{21} &= a_{21} - a_{11}\frac{a_{21}}{a_{11}} = 2 - 2\left(\frac{2}{2}\right) = 0 \\
a_{22} &= a_{22} - a_{12}\frac{a_{21}}{a_{11}} = 1 - 2\left(\frac{2}{2}\right) = -1 \\
a_{23} &= a_{23} - a_{13}\frac{a_{21}}{a_{11}} = 0 - 1\left(\frac{2}{2}\right) = -1 \\
a_{24} &= a_{24} - a_{14}\frac{a_{21}}{a_{11}} = 4 - 9\left(\frac{2}{2}\right) = -5
\end{aligned} \tag{B.3.18}$$

Note that these new coefficients correspond to those of the second of Eqs. (B.3.17), where the right-side a's of Eqs. (B.3.18) are those from the previous step [here from Eqs. (B.3.16)], the right-side a_{24} is really $c_2 = 4$, and the left-side a_{24} is the new $c_2 = -5$.

For $k = 1$, $i = 3$, and j indexing from 1 to 4,

$$\begin{aligned}
a_{31} &= a_{31} - a_{11}\frac{a_{31}}{a_{11}} = 1 - 2\left(\frac{1}{2}\right) = 0 \\
a_{32} &= a_{32} - a_{12}\frac{a_{31}}{a_{11}} = 1 - 2\left(\frac{1}{2}\right) = 0 \\
a_{33} &= a_{33} - a_{13}\frac{a_{31}}{a_{11}} = 1 - 1\left(\frac{1}{2}\right) = \frac{1}{2} \\
a_{34} &= a_{34} - a_{14}\frac{a_{31}}{a_{11}} = 6 - 9\left(\frac{1}{2}\right) = \frac{3}{2}
\end{aligned} \tag{B.3.19}$$

where these new coefficients correspond to those of the third of Eqs. (B.3.17) as previously explained.

Step 2

For $k = 2$, $i = 3$, and $j\ (= k)$ indexing from 2 to 4,

$$a_{32} = a_{32} - a_{22}\left(\frac{a_{32}}{a_{22}}\right) = 0 - (-1)\left(\frac{0}{-1}\right) = 0$$

$$a_{33} = a_{33} - a_{23}\left(\frac{a_{32}}{a_{22}}\right) = \frac{1}{2} - (-1)\left(\frac{0}{-1}\right) = \frac{1}{2} \qquad \text{(B.3.20)}$$

$$a_{34} = a_{34} - a_{24}\left(\frac{a_{32}}{a_{22}}\right) = \frac{3}{2} - (-5)\left(\frac{0}{-1}\right) = \frac{3}{2}$$

where the new coefficients again correspond to those of the third of Eqs. (B.3.17), because step 1 already eliminated the coefficients of x_2 as observed in the third of Eqs. (B.3.17), and the a's on the right side of Eqs. (B.3.20) are taken from Eqs. (B.3.18) and (B.3.19).

Step 3

By Eqs. (B.3.15), for x_3, we have

$$x_3 = \frac{1}{a_{33}}(a_{34} - 0)$$

or, using a_{33} and a_{34} from Eqs. (B.3.20),

$$x_3 = \frac{1}{(\frac{1}{2})}\left(\frac{3}{2}\right) = 3$$

where the summation is interpreted as zero in the second of Eqs. (B.3.15) when $r > n$ (for x_3, $r = 4$, and $n = 3$). For x_2, we have

$$x_2 = \frac{1}{a_{22}}(a_{24} - a_{23}x_3)$$

or, using the appropriate a's from Eqs. (B.3.18),

$$x_2 = \frac{1}{-1}[-5 - (-1)(3)] = 2$$

and for x_1, we have

$$x_1 = \frac{1}{a_{11}}(a_{14} - a_{12}x_2 - a_{13}x_3)$$

or, using the a's from the first of Eqs. (B.3.16),

$$x_1 = \tfrac{1}{2}[9 - 2(2) - 1(3)] = 1$$

In summary, the latest a's from the previous steps have been used in Eqs. (B.3.15) to obtain the x's. ■

Note that the pivot element was the diagonal element in each step. However, the diagonal element must be nonzero because we divide by it in each step. An original matrix with all nonzero diagonal elements does not ensure that the pivots in each step will remain nonzero, because we are adding numbers to equations below the pivot in each following step. Therefore, a test is necessary to determine whether the pivot a_{kk} at each step is zero. If it is zero, the current row (equation) must be interchanged with one of the following rows—usually with the next row unless that row has a zero at the position that would next become the pivot. Remember that the right-side corresponding element in $\underline{c}$ must also be interchanged. After making this test and, if necessary, interchanging the equations, continue the procedure in the usual manner.

An example will now illustrate the method for treating the occurrence of a zero pivot element.

Example B.2

Solve the following set of simultaneous equations.

$$\begin{aligned} 2x_1 + 2x_2 + 1x_3 &= 9 \\ 1x_1 + 1x_2 + 1x_3 &= 6 \\ 2x_1 + 1x_2 \qquad &= 4 \end{aligned} \tag{B.3.21}$$

It will often be convenient to set up the solution procedure by considering the coefficient matrix $\underline{a}$ plus the right-side matrix $\underline{c}$ in one matrix without writing down the unknown matrix $\underline{x}$. This new matrix is called the *augmented matrix*. For the set of Eqs. (B.3.21), we have the augmented matrix written as

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 0 & 4 \end{array}\right] \tag{B.3.22}$$

We use the steps previously outlined as follows:

Step 1

We select $a_{11} = 2$ as the pivot and

a. Add the multiple $-a_{21}/a_{11} = -1/2$ of the first row to the second row of Eq. (B.3.22).
b. Add the multiple $-a_{31}/a_{11} = -2/2$ of the first row to the third row of Eq. (B.3.22) to obtain

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & -1 & -1 & -5 \end{array}\right] \tag{B.3.23}$$

At the end of step 1, we would normally choose a_{22} as the next pivot. However, a_{22} is now equal to zero. If we interchange the second and third rows of Eq. (B.3.23), the new a_{22} will be nonzero and can be used as a pivot. Interchanging rows 2 and 3 results in

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{array}\right] \tag{B.3.24}$$

For this special set of only three equations, the interchange has resulted in an upper-triangular coefficient matrix and concludes the elimination procedure. The back-substitution process of step 3 now yields

$$x_3 = 3 \qquad x_2 = 2 \qquad x_1 = 1$$ ■

A second problem when selecting the pivots in sequential manner without testing for the best possible pivot is that loss of accuracy due to rounding in the results can occur. In general, the pivots should be selected as the largest (in absolute value) of the elements in any column. For example, consider the set of equations given by

$$\begin{aligned} 0.002x_1 + 2.00x_2 &= 2.00 \\ 3.00x_1 + 1.50x_2 &= 4.50 \end{aligned} \tag{B.3.25}$$

whose actual solution is given by

$$x_1 = 1.0005 \qquad x_2 = 0.999 \tag{B.3.26}$$

The solution by Gaussian elimination without testing for the largest absolute value of the element in any column is

$$\begin{aligned} 0.002x_1 + 2.00x_2 &= 2.00 \\ -2998.5x_2 &= -995.5 \\ x_2 &= 0.3320 \\ x_1 &= 668 \end{aligned} \tag{B.3.27}$$

This solution does not satisfy the second of Eqs. (B.3.25). The solution by interchanging equations is

$$\begin{aligned} 3.00x_1 + 1.50x_2 &= 4.50 \\ 0.002x_1 + 2.00x_2 &= 2.00 \end{aligned}$$

or

$$\begin{aligned} 3.00x_1 + 1.50x_2 &= 4.50 \\ 1.999x_2 &= 1.997 \\ x_2 &= 0.999 \\ x_1 &= 1.0005 \end{aligned} \tag{B.3.28}$$

Equations (B.3.28) agree with the actual solution [Eqs. (B.3.26)].

Hence, in general, the pivots should be selected as the largest (in absolute value) of the elements in any column. This process is called *partial pivoting*. Even better results can be obtained by choosing the pivot as the largest element in the whole matrix of the remaining equations and performing appropriate interchanging of rows. This is called *complete pivoting*. Complete pivoting requires a large amount of testing, so it is not recommended in general.

The finite element equations generally involve coefficients with different orders of magnitude, so Gaussian elimination with partial pivoting is a useful method for solving the equations.

Finally, it has been shown that for n simultaneous equations, the number of arithmetic operations required in Gaussian elimination is n divisions, $\frac{1}{3}n^3 + n^2$ multiplications, and $\frac{1}{3}n^3 + n$ additions. If partial pivoting is included, the number of comparisons needed to select pivots is $n(n+1)/2$.

Other elimination methods, including the Gauss–Jordan and Cholesky methods, have some advantages over Gaussian elimination and are sometimes used to solve large systems of equations. For descriptions of other methods, see References [1–3].

Gauss–Seidel Iteration

Another general class of methods (other than the elimination methods) used to solve systems of linear algebraic equations is the *iterative methods*. Iterative methods work well when the system of equations is large and sparse (many zero coefficients). The Gauss–Seidel method starts with the original set of equations $\underline{a}\underline{x} = \underline{c}$ written in the form

$$
\begin{aligned}
x_1 &= \frac{1}{a_{11}}(c_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\
x_2 &= \frac{1}{a_{22}}(c_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\
&\vdots \\
x_n &= \frac{1}{a_{nn}}(c_n - a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{n,n-1}x_{n-1})
\end{aligned}
\qquad \text{(B.3.29)}
$$

The following steps are then applied.

1. Assume a set of initial values for the unknowns $x_1, x_2, \ldots, x_n$, and substitute them into the right side of the first of Eqs. (B.3.29) to solve for the new x_1.
2. Use the latest value for x_1 obtained from step 1 and the initial values for $x_3, x_4, \ldots, x_n$ in the right side of the second of Eqs. (B.3.29) to solve for the new x_2.
3. Continue using the latest values of the x's obtained in the left side of Eqs. (B.3.29) as the next trial values in the right side for each succeeding step.
4. Iterate until convergence is satisfactory.

A good initial set of values (guesses) is often $x_i = c_i/a_{ii}$. An example will serve to illustrate the method.

Example B.3

Consider the set of linear simultaneous equations given by

$$\begin{aligned} 4x_1 - x_2 \qquad\qquad &= 2 \\ -x_1 + 4x_2 - x_3 \qquad &= 5 \\ -x_2 + 4x_3 - x_4 &= 6 \\ -x_3 + 2x_4 &= -2 \end{aligned} \tag{B.3.30}$$

Using the initial guesses given by $x_i = c_i/a_{ii}$, we have

$$x_1 = \tfrac{2}{4} = \tfrac{1}{2} \qquad x_2 = \tfrac{5}{4} \approx 1 \qquad x_3 = \tfrac{6}{4} \approx 1 \qquad x_4 = -1$$

Solving the first of Eqs. (B.3.30) for x_1 yields

$$x_1 = \tfrac{1}{4}(2 + x_2) = \tfrac{1}{4}(2 + 1) = \tfrac{3}{4}$$

Solving the second of Eqs. (B.3.30) for x_2, we have

$$x_2 = \tfrac{1}{4}(5 + x_1 + x_3) = \tfrac{1}{4}(5 + \tfrac{3}{4} + 1) = 1.68$$

Solving the third of Eqs. (B.3.30) for x_3, we have

$$x_3 = \tfrac{1}{4}(6 + x_2 + x_4) = \tfrac{1}{4}[6 + 1.68 + (-1)] = 1.672$$

Solving the fourth of Eqs. (B.3.30) for x_4, we obtain

$$x_4 = \tfrac{1}{2}(-2 + x_3) = \tfrac{1}{2}(-2 + 1.67) = -0.16$$

The first iteration has now been completed. The second iteration yields

$$\begin{aligned} x_1 &= \tfrac{1}{4}(2 + 1.68) = 0.922 \\ x_2 &= \tfrac{1}{4}(5 + 0.922 + 1.672) = 1.899 \\ x_3 &= \tfrac{1}{4}[6 + 1.899 + (-0.16)] = 1.944 \\ x_4 &= \tfrac{1}{2}(-2 + 1.944) = -0.028 \end{aligned}$$

Table B–1 lists the results of four iterations of the Gauss–Seidel method and the exact solution. From Table B–1, we observe that convergence to the exact solution has proceeded rapidly by the fourth iteration, and the accuracy of the solution is dependent on the number of iterations. ■

In general, iteration methods are self-correcting, such that an error made in calculations at one iteration will be corrected by later iterations. However, there are certain systems of equations for which iterative methods are not convergent.

Table B–1 Results of four iterations of the Gauss–Seidel method for Eqs. (B.3.30)

Iteration	x_1	x_2	x_3	x_4
0	0.5	1.0	1.0	−1.0
1	0.75	1.68	1.672	−0.16
2	0.922	1.899	1.944	−0.028
3	0.975	1.979	1.988	−0.006
4	0.9985	1.9945	1.9983	−0.0008
Exact	1.0	2.0	2.00	0

When the equations can be arranged such that the diagonal terms are greater than the off-diagonal terms, the possibility of convergence is usually enhanced.

Finally, it has been shown that for n simultaneous equations, the number of arithmetic operations required by Gauss–Seidel iteration is n divisions, n^2 multiplications, and $n^2 - n$ additions for each iteration.

▲ B.4 Banded-Symmetric Matrices, Bandwidth, Skyline, and Wavefront Methods ▲

The coefficient matrix (stiffness matrix) for the linear equations that occur in structural analysis is always symmetric and banded. Because a meaningful analysis generally requires the use of a large number of variables, the implementation of compressed storage of the stiffness matrix is desirable both from the standpoint of fitting into memory (immediate access portion of the computer) and for computational efficiency. We will discuss the banded-symmetric format, which is not necessarily the most efficient format but is relatively simple to implement on the computer.

Another method, based on the concept of the skyline of the stiffness matrix, is often used to improve the efficiency in solving the equations. *The* **skyline** *is an envelope that begins with the first nonzero coefficient in each column of the stiffness matrix* (Figure B–5). In skylining, only the coefficients between the main diagonal and the skyline are stored (normally by successive columns) in a one-dimensional array. In general, this procedure takes even less storage space in the computer and is more efficient in terms of equation solving than the conventional banded format. (For more information on skylining, consult References [10–12].)

A matrix is **banded** if the nonzero terms of the matrix are gathered about the main diagonal. To illustrate this concept, consider the plane truss of Figure B–4.

From Figure B–4, we see that element 2 connects nodes 1 and 4. Therefore, the 2×2 submatrices at positions 1–1, 1–4, 4–1, and 4–4 of Figure B–5 have nonzero coefficients. Figure B–5 represents the total stiffness matrix of the plane truss. The X's denote nonzero coefficients. From Figure B–5, we observe that the nonzero terms are within the band shown. When we use a banded storage format, only the main diagonal and the nonzero upper codiagonals need be stored as shown in Figure B–6. Note that any codiagonal with a nonzero term requires storage of the whole

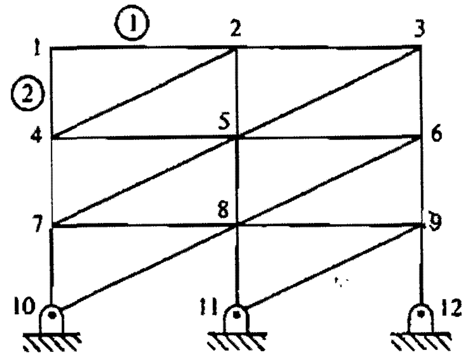

Figure B–4 Plane truss for bandwidth illustration

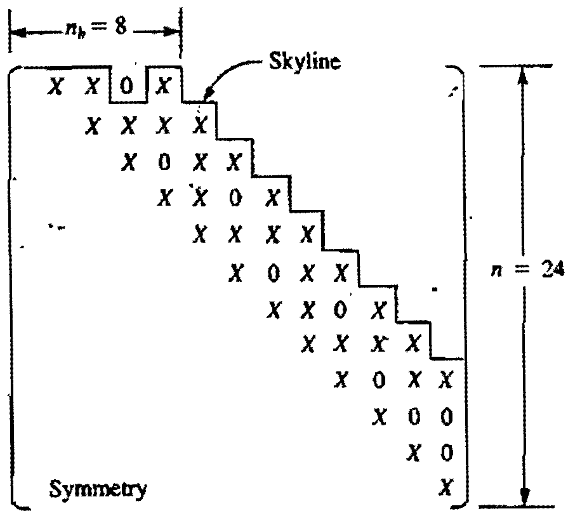

Figure B–5 Stiffness matrix for the plane truss of Figure B–4, where X denotes, in general, blocks of 2 × 2 submatrices with nonzero coefficients

codiagonal *and* any codiagonals between it and the main diagonal. The use of banded storage is efficient for computational purposes. The Scientific Subroutine Package gives a more detailed explanation of banded compressed storage [4].

We now define the semibandwidth n_b as $n_b = n_d(m + 1)$, where n_d is the number of degrees of freedom per node and m is the maximum difference in node numbers determined by calculating the difference in node numbers for each element of a finite element model. In the example for the plane truss of Figure B–4, $m = 4 - 1 = 3$ and $n_d = 2$, so $n_b = 2(3 + 1) = 8$.

Execution time (primarily equation-solving time) is a function of the number of equations to be solved. It has been shown [5] that when banded storage of global stiffness matrix $\underline{K}$ is not used, execution time is proportional to $(1/3)n^3$, where n is the number of equations to be solved, or, equivalently, the size of $\underline{K}$. When banded storage of $\underline{K}$ is used, the execution time is proportional to $(n)n_b^2$. The ratio of time of execution without banded storage to that with banded storage is then

$\longleftarrow n_b = 8 \longrightarrow$

$$\begin{bmatrix} X & X & 0 & X \\ X & X & X & X \\ X & 0 & X & X \\ X & X & 0 & X \\ X & X & X & X \\ X & 0 & X & X \\ X & X & 0 & X \\ X & X & X & X \\ X & 0 & X & X \\ X & 0 & 0 & 0 \\ X & 0 & 0 & 0 \\ X & 0 & 0 & 0 \end{bmatrix}$$

Figure B–6 Banded storage format of the stiffness matrix of Figure B–5

$(1/3)(n/n_b)^2$. For the plane truss example, this ratio is $(1/3)(24/8)^2 = 3$. Therefore, it takes about three times as long to execute the solution of the example truss if banded storage is not used.

Hence, to reduce bandwidth we should number systematically and try to have a minimum difference between adjacent nodes. A small bandwidth is usually achieved by consecutive node numbering across the shorter dimension, as shown in Figure B–4. Some computer programs use the banded-symmetric format for storing the global stiffness matrix, $\underline{K}$.

Several automatic node-renumbering schemes have been computerized [6]. This option is available in most general-purpose computer programs. Alternatively, the wavefront or frontal method is becoming popular for optimizing equation solution time. In the **wavefront method**, elements, instead of nodes, are automatically renumbered.

In the wavefront method, the assembly of the equations alternates with their solution by Gauss elimination. The sequence in which the equations are processed is determined by element numbering rather than by node numbering. The first equations eliminated are those associated with element 1 only. Next, the contributions of stiffness coefficients of the adjacent element, element 2, are added to the system of equations. If any additional degrees of freedom are contributed by elements 1 and 2 only—that is, if no other elements contribute stiffness coefficients to specific degrees of freedom—these equations are eliminated (condensed) from the system of equations. As one or more additional elements make their contributions to the system of equations and additional degrees of freedom are contributed only by these elements, those degrees of freedom are eliminated from the solution. This repetitive alternation between assembly and solution was initially seen as a wavefront that sweeps over the structure in a pattern determined by the element numbering. For greater efficiency of this method, consecutive element numbering should be done across the structure in a direction that spans the smallest number of nodes.

The wavefront method, though somewhat more difficult to understand and to program than the banded-symmetric method, is computationally more efficient. A banded solver stores and processes any blocks of zeros created in assembling the stiffness matrix. In the wavefront method, these blocks of zero coefficients are not stored or processed. Many large-scale computer programs are now using the wavefront method to solve the system of equations. (For additional details of this method, see References [7–9].) Example B.4 illustrates the wavefront method for solution of a truss problem.

Example B.4

For the plane truss shown in Figure B–7, illustrate the wavefront solution procedure.

We will solve this problem in symbolic form. Merging k's for elements 1, 2, and 3 and enforcing boundary conditions at node 1, we have

$$
\begin{array}{cc|cccc}
d_{2x} & d_{2y} & d_{3x} & d_{3y} & d_{4x} & d_{4y}
\end{array}
$$

$$
\left[\begin{array}{cc|cccc}
k_{33}^{(1)}+k_{11}^{(2)}+k_{11}^{(3)} & k_{34}^{(1)}+k_{12}^{(2)}+k_{12}^{(3)} & k_{13}^{(3)} & k_{14}^{(3)} & k_{13}^{(2)} & k_{14}^{(2)} \\
k_{43}^{(1)}+k_{21}^{(2)}+k_{21}^{(3)} & k_{44}^{(1)}+k_{22}^{(2)}+k_{22}^{(3)} & k_{23}^{(3)} & k_{24}^{(3)} & k_{23}^{(2)} & k_{24}^{(2)} \\
\hline
k_{31}^{(2)} & k_{32}^{(3)} & k_{33}^{(3)} & k_{34}^{(3)} & k_{33}^{(2)} & k_{34}^{(2)} \\
k_{41}^{(3)} & k_{42}^{(3)} & k_{43}^{(3)} & k_{44}^{(3)} & k_{43}^{(2)} & k_{44}^{(2)} \\
k_{31}^{(2)} & k_{32}^{(2)} & 0 & 0 & 0 & 0 \\
k_{41}^{(2)} & k_{42}^{(2)} & 0 & 0 & 0 & 0
\end{array}\right]
\begin{Bmatrix} d_{2x} \\ d_{2y} \\ \hline d'_{3x} \\ d'_{3y} \\ d'_{4x} \\ d'_{4y} \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ \hline 0 \\ -P \\ 0 \\ 0 \end{Bmatrix}
\tag{B.4.1}
$$

Eliminating d_{2x} and d_{2y} (all stiffness contributions from node 2 degrees of freedom have been included from these elements; these contributions are from elements 1–3) by static condensation or Gauss elimination yields

$$
[k'_c]\begin{Bmatrix} d'_{3x} \\ d'_{3y} \\ d'_{4x} \\ d'_{4y} \end{Bmatrix} = \{F'_c\}
\tag{B.4.2}
$$

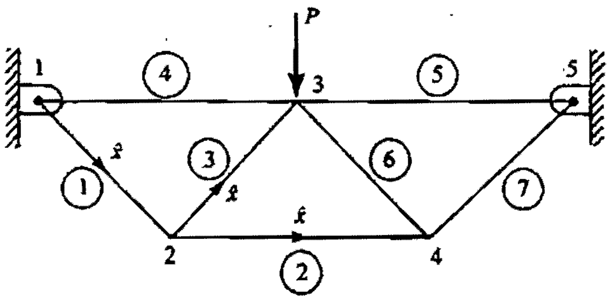

Figure B–7 Truss for wavefront solution

where the condensed stiffness and force matrices are (also see Section 7.5)

$$[k'_c] = [K'_{22}] - [K'_{21}][K'_{11}]^{-1}[K'_{12}] \tag{B.4.3}$$

$$\{F'_c\} = \{F'_2\} - [K'_{21}][K'_{11}]^{-1}\{F'_1\} \tag{B.4.4}$$

where primes on the degrees of freedom, such as d'_{3x} in Eq. (B.4.1), indicate that all stiffness coefficients associated with that degree of freedom have not yet been included. Now include elements 4–6 for degrees of freedom at node 3. The resulting equations are

$$\begin{array}{cccc} d_{3x} & d_{3y} & d_{4x} & d_{4y} \end{array}$$

$$\left[\begin{array}{cc|cc} k'_{c11} + k^{(4)}_{33} + k^{(5)}_{11} + k^{(6)}_{11} & k^{(4)}_{34} + k^{(5)}_{12} + k^{(6)}_{12} + k'_{c12} & k^{(6)}_{13} + k'_{c13} & k^{(6)}_{14} + k'_{c14} \\ k'_{c21} + k^{(4)}_{34} + k^{(5)}_{21} + k^{(6)}_{21} & k^{(4)}_{44} + k^{(5)}_{22} + k^{(6)}_{22} + k'_{c22} & k^{(6)}_{23} + k'_{c23} & k^{(6)}_{24} + k'_{c24} \\ \hline k'_{c31} + k^{(6)}_{31} & k'_{c32} + k^{(6)}_{32} & k'_{c33} + k^{(6)}_{33} & k'_{c34} + k^{(6)}_{34} \\ k'_{c41} + k^{(6)}_{41} & k'_{c42} + k^{(6)}_{42} & k'_{c43} + k^{(6)}_{43} & k'_{c44} + k^{(6)}_{44} \end{array}\right]$$

$$\times \begin{Bmatrix} d_{3x} \\ d_{3y} \\ \hline d'_{4x} \\ d'_{4y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ \hline 0 \\ 0 \end{Bmatrix} \tag{B.4.5}$$

Using static condensation, we eliminate d_{3x} and d_{3y} (all contributions from node 3 degrees of freedom have been included from each element) to obtain

$$[k''_c]\begin{Bmatrix} d'_{4x} \\ d'_{4y} \end{Bmatrix} = \{F''_c\} \tag{B.4.6}$$

where

$$[k''_c] = [K''_{22}] - [K''_{21}][K''_{11}]^{-1}[K''_{12}] \tag{B.4.7}$$

$$\{F''_c\} = \{F''_2\} - [K''_{21}][K''_{11}]^{-1}\{F''_1\} \tag{B.4.8}$$

Next we include element 7 contributions to the stiffness matrix. The condensed set of equations yield

$$[k'''_c]\begin{Bmatrix} d_{4x} \\ d_{4y} \end{Bmatrix} = \{F'''_c\} \tag{B.4.9}$$

$$[k'''_c] = [K'''_{22}] - [K'''_{21}][K'''_{11}]^{-1}[K'''_{12}] \tag{B.4.10}$$

where

$$\{F'''_c\} = \{F'''_2\} - [K'''_{21}][K'''_{11}]^{-1}\{F'''_1\} \tag{B.4.11}$$

The elimination procedure is now complete, and we solve Eq. (B.4.9) for d_{4x} and d_{4y}. Then we back-substitute d_{4x} and d_{4y} into Eq. (B.4.5) to obtain d_{3x} and d_{3y}. Finally, we back-substitute d_{3x} through d_{4y} into Eq. (B.4.1) to obtain d_{2x} and d_{2y}. Static condensation and Gauss elimination with back-substitution have been used to solve the set of equations for all the degrees of freedom. The solution procedure has then proceeded as though it were a wave sweeping over the structure, starting at node 2, engulfing node 2 and elements with degrees of freedom at node 2, and then sweeping through node 3 and finally node 4. ■

We now describe a practical computer scheme often used in computer programs for the solution of the resulting system of algebraic equations. The significance of this scheme is that it takes advantage of the fact that the stiffness method produces a banded $\underline{K}$ matrix in which the nonzero elements occur about the main diagonal in $\underline{K}$. While the equations are solved, this banded format is maintained.

Example B.5

We will now use a simple example to illustrate this computer scheme. Consider the three-spring assemblage shown in Figure B–8. The assemblage is subjected to forces at node 2 of 100 lb in the x direction and 200 lb in the y direction. Node 1 is completely constrained from displacement in both the x and y directions, whereas node 3 is completely constrained in the y direction but is displaced a known amount δ in the x direction.

Our purpose here is not to obtain the actual $\underline{K}$ for the assemblage but rather to illustrate the scheme used for solution. The general solution can be shown to be given by

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & k_{33} & k_{34} & k_{35} & k_{36} \\ & & & k_{44} & k_{45} & k_{46} \\ & & & & k_{55} & k_{56} \\ \text{Symmetry} & & & & & k_{66} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} = 100 \\ F_{2y} = 200 \\ F_{3x} \\ F_{3y} \end{Bmatrix} \tag{B.4.12}$$

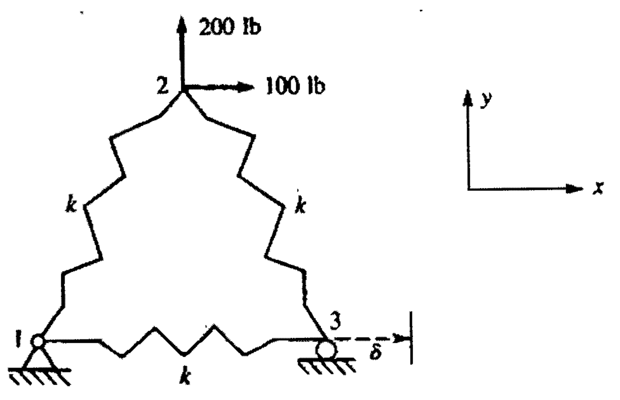

Figure B–8 Three-spring assemblage

where $\underline{K}$ has been left in general form. Upon our imposing the boundary conditions, the computer program transforms Eq. (B.4.12) to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{33} & k_{34} & 0 & 0 \\ 0 & 0 & k_{43} & k_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 100 - k_{35}\delta \\ 200 - k_{45}\delta \\ \delta \\ 0 \end{Bmatrix} \qquad \text{(B.4.13)}$$

From Eq. (B.4.13), we can see that $d_{1x} = 0$, $d_{1y} = 0$, $d_{3y} = 0$, and $d_{3x} = \delta$. These displacements are consistent with the imposed boundary conditions. The unknown displacements, d_{2x} and d_{2y}, can be determined routinely by solving Eq. (B.4.13).

We will now explain the computer scheme that is generally applicable to transform Eq. (B.4.12) to Eq. (B.4.13). First, the terms associated with the known displacement boundary condition(s) within each equation were transformed to the right side of those equations. In the third and fourth equations of Eq. (B.4.12), $k_{35}\delta$ and $k_{45}\delta$ were transformed to the right side, as shown in Eq. (B.4.13). Then the right-side force term corresponding to the known displacement row was equated to the known displacement. In the fifth equation of Eq. (B.4.12), where $d_{3x} = \delta$, the right-side, fifth-row force term F_{3x} was equated to the known displacement δ, as shown in Eq. (B.4.13). For the homogeneous boundary conditions, the affected rows of $\underline{F}$, corresponding to the zero-displacement rows, were replaced with zeros. Again, this is done in the computer scheme only to obtain the nodal displacements and does not imply that these nodal forces are zero. We obtain the unknown nodal forces by determining the nodal displacements and back-substituting these results into the original Eq. (B.4.12). Because $d_{1x} = 0$, $d_{1y} = 0$, and $d_{3y} = 0$ in Eq. (B.4.12), the first, second, and sixth rows of the force matrix of Eq. (B.4.13) were set to zero. Finally, for both nonhomogeneous and homogeneous boundary conditions, the rows and columns of $\underline{K}$ corresponding to these prescribed boundary conditions were set to zero except the main diagonal, which was made unity. That is, the first, second, fifth, and sixth rows and columns of $\underline{K}$ in Eq. (B.4.12) were set to zero, except for the main diagonal terms, which were made unity. Although doing so is not necessary, setting the main diagonal terms equal to 1 facilitates the simultaneous solution of the six equations in Eq. (B.4.13) by an elimination method used in the computer program. This modification is shown in the $\underline{K}$ matrix of Eq. (B.4.13). ■

▲ References

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▲ Problems

B.1 Determine the solution of the following simultaneous equations by Cramer's rule.

$$1x_1 + 3x_2 = 5$$
$$4x_1 - 1x_2 = 12$$

B.2 Determine the solution of the following simultaneous equations by the invers method.

$$1x_1 + 3x_2 = 5$$
$$4x_1 - 1x_2 = 12$$

B.3 Solve the following system of simultaneous equations by Gaussian elimination.

$$\begin{aligned} x_1 - 4x_2 - 5x_3 &= 4 \\ 3x_2 + 4x_3 &= -1 \\ -2x_1 - 1x_2 + 2x_3 &= -3 \end{aligned}$$

B.4 Solve the following system of simultaneous equations by Gaussian elimination.

$$\begin{aligned} 2x_1 + 1x_2 - 3x_3 &= 11 \\ 4x_1 - 2x_2 + 3x_3 &= 8 \\ -2x_1 + 2x_2 - 1x_3 &= -6 \end{aligned}$$

B.5 Given that

$$x_1 = 2y_1 - y_2 \qquad z_1 = -x_1 - x_2$$

$$x_2 = y_1 - y_2 \qquad z_2 = 2x_1 + x_2.$$

a. Write these relationships in matrix form.
b. Express $\underline{z}$ in terms of $\underline{y}$.
c. Express $\underline{y}$ in terms of $\underline{z}$.

B.6 Starting with the initial guess $\underline{X}^T = [1 \quad 1 \quad 1 \quad 1 \quad 1]$, perform five iterations of the Gauss–Seidel method on the following system of equations. On the basis of the results of these five iterations, what is the exact solution?

$$\begin{aligned} 2x_1 - 1x_2 \qquad\qquad\qquad &= -1 \\ -1x_1 + 6x_2 - 1x_3 \qquad\qquad &= 4 \\ -2x_2 + 4x_3 - 1x_4 \qquad &= 4 \\ -1x_3 + 4x_4 - 1x_5 &= 6 \\ -1x_4 + 2x_5 &= -2 \end{aligned}$$

B.7 Solve Problem B.1 by Gauss–Seidel iteration.

B.8 Classify the solutions to the following systems of equations according to Section B.2 as unique, nonunique, or nonexistent.

a. $2x_1 - 4x_2 = 2$
$-9x_1 + 12x_2 = -6$

b. $10x_1 + 1x_2 = 0$
$5x_1 + \frac{1}{2}x_2 = 3$

c. $2x_1 + 1x_2 + 1x_3 = 6$
$3x_1 + 1x_2 - 1x_3 = 4$
$5x_1 + 2x_2 + 2x_3 = 8$

d. $1x_1 + 1x_2 + 1x_3 = 1$
$2x_1 + 2x_2 + 2x_3 = 2$
$3x_1 + 3x_2 + 3x_3 = 3$

B.9 Determine the bandwidths of the plane trusses shown in Figure PB–9. What conclusions can you draw regarding labeling of nodes?

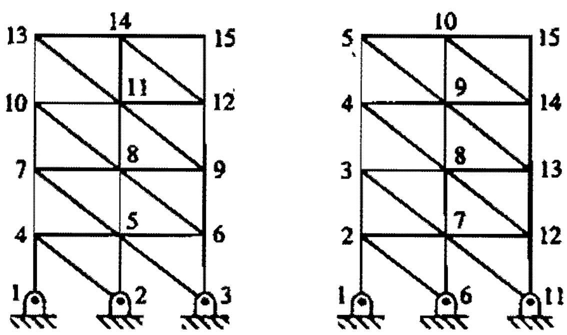

Figure PB–9