Finite Element Analysis

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Lecture No. # 01

Welcome to the series of lectures, on finite element analysis. Before I start, I will show you list of references, which you may refer during this course.

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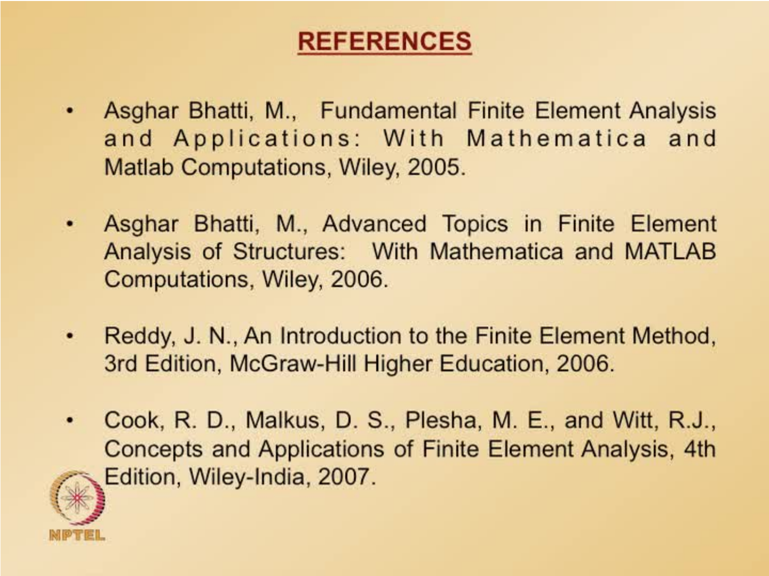

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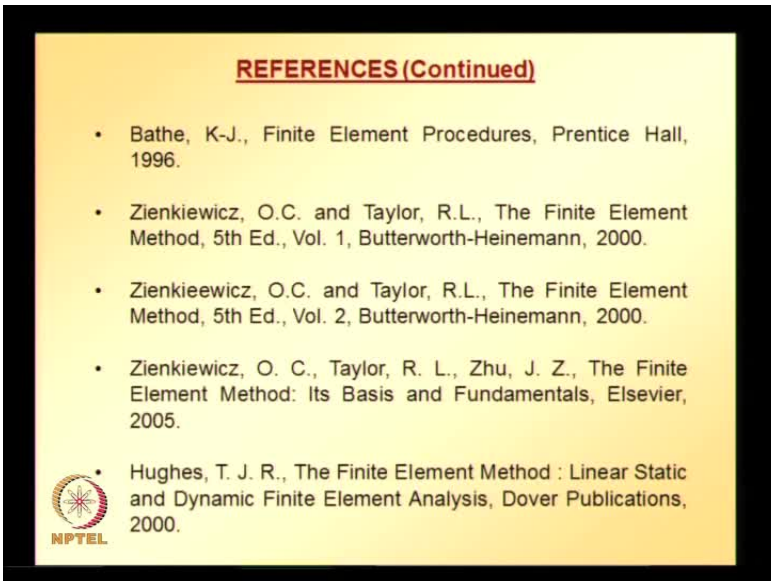

Here are the four references and rests of the references are with this.

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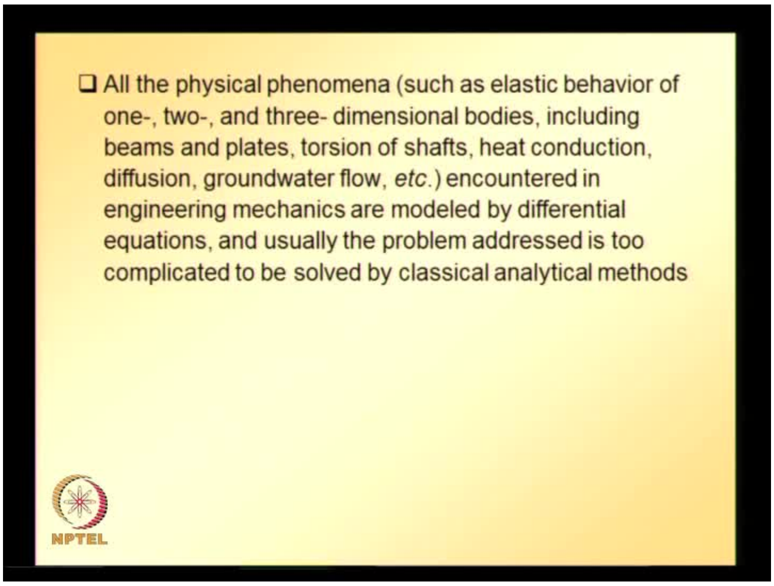

Let us, start with the lecture, to start with all the physical phenomena, such as elastic behavior of one, two, three dimensional bodies, including beams, plates, torsion of shafts, heat conduction, diffusion, groundwater flow, etcetera. That, we encounter in real engineering mechanics are modeled by differential equations. And solving these differential equations over the domain, subjected to some boundary conditions is usually too complicated to solve by classical analytical methods.

So, this transformation of a physical phenomenon into differential equation involves differential calculus and mathematical concepts. So, once we have this physical phenomena expressed in terms of a differential equations and set of boundary conditions. Then they can adopt the finite element method to solve the problem for the solution at any point over the domain that we are interested.

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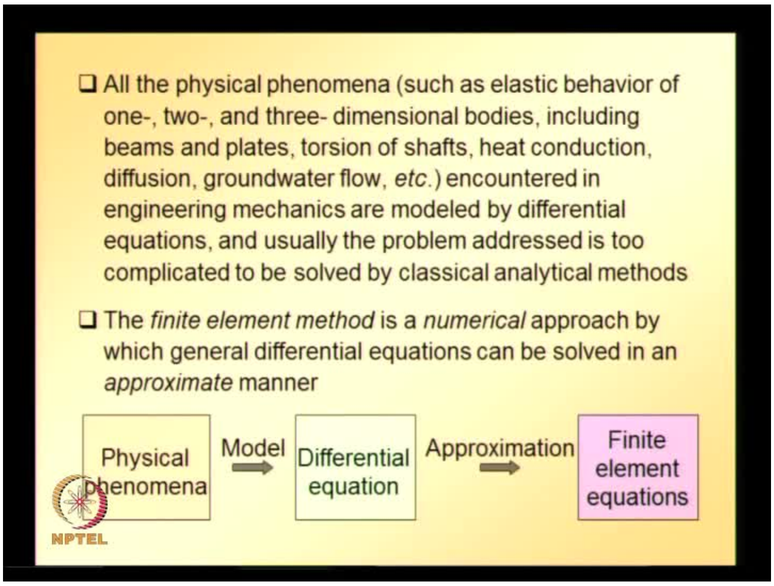

So, finite element method is a numerical approach by which a general differential equation can be solved in an approximate manner. Please keep in mind, the solution that we are going to get from finite element method is an approximate and in some cases, this approximate solution may match with exact solution. So, basically, the entire process involves these steps, first: the physical phenomena needs to be addressed or needs to be converted into a mathematical model in the form of a differential equation, that needs to be solved over your domain subjected to some boundary conditions, once we have the physical phenomena expressed in this form and this fields. This physical phenomena can be any one of those listed above, that is, it can be any problem in structural mechanics, beams, plates, torsion of shafts and heat conduction problems, diffusion and groundwater flow problems. All these problems or any of these problems needs to be converted into a mathematical form. In the form of differential equation that needs to be solved over a domain subjected to some boundary conditions.

Once we have the physical phenomena expressed in that form, we can use some of the classical solution techniques in conjunction with finite element method. To get the finite element equations and which we can solve for the unknowns or the solution at any point on the domain. So, basically finite element method involves discretization of the entire solution domain into some number of elements and key points, in these elements are called nodes. So, depending on the approximation that we are the accuracy in the approximation that we are looking for we can use elements having different number of nodes. So, the entire solution domain is divided into finite elements and solution is approximated at key points called nodes. And once we obtain the solution at the nodes, we can interpolate at any point in the solution domain using, what are called interpolation functions or finite element shape functions.

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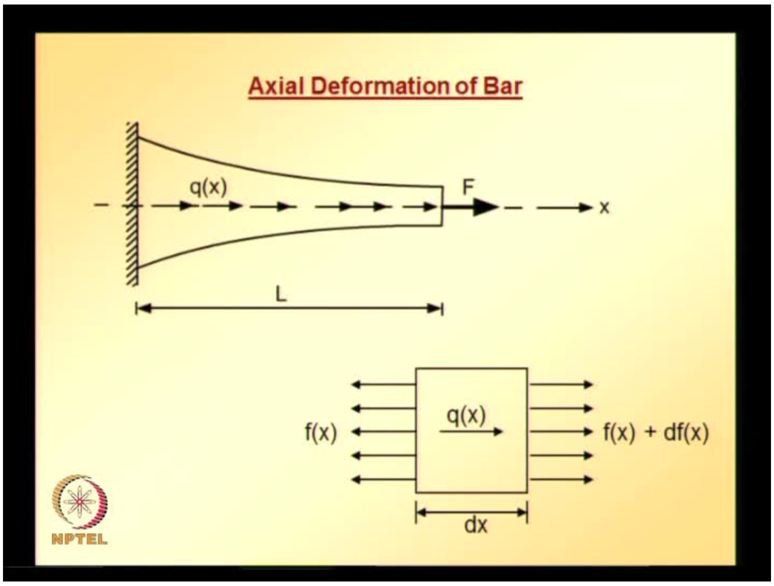

So, all those details will be looking into this course. So, to understand these steps, let us take a simple example, axial deformation of a bar. So, the ideas presented, so far can be illustrated through this example, here a bar is shown, whose cross sectional areas vary over the length l and it is subjected to a body force or distributed load q over it is length entire length and a tip load at the extreme point, where x is equal to l. So, if you see this bar at x is equal to zero, the bar is fixed and x is equal to l, a tip load is applied. So, the governing differential equation explaining the physical phenomena here, that is, axial deformation of bar can be derived or the differential equation, which represents that phenomena can be derived by considering equilibrium of forces acting on a infinitesimal bar segment as shown in the second figure. And please note that the sign convention that we are following is tension is positive. So, if we look at the equilibrium of the infinitesimal element that is shown. We can sum up all the forces acting in the x direction body or distributed force or body force q x is acting over length d x f x is acting on the right hand side and f x plus f d y is acting on the left hand side.

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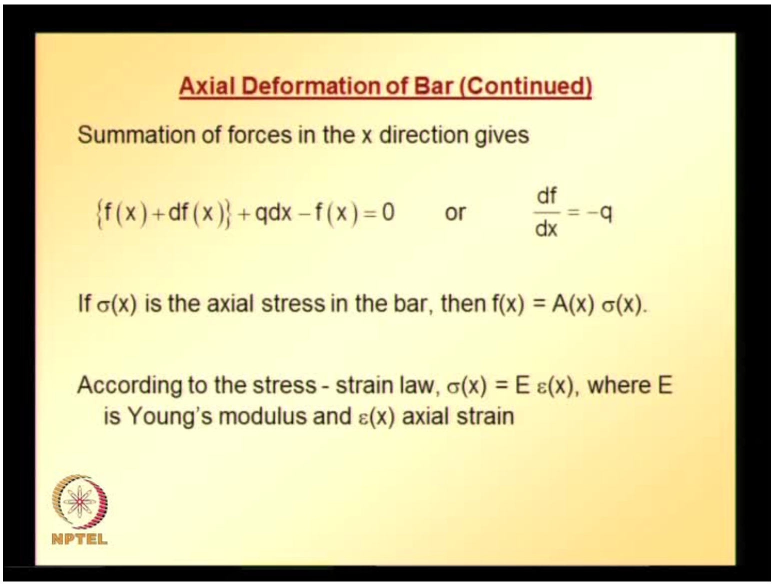

So, summing up all the forces, along x direction; summation of forces in the x direction gives us this equation- which can be simplified to this form derivative of f with respect to x is equal to minus q and we know that if sigma x is the axial stress in the bar. Force is force effects is given byarea of cross section time sigma from structural mechanics and also according to stress strain law, sigma is equal to young's modulus time strain; strain here, is denoted with epsilon stress with sigma young's modulus with e and also.

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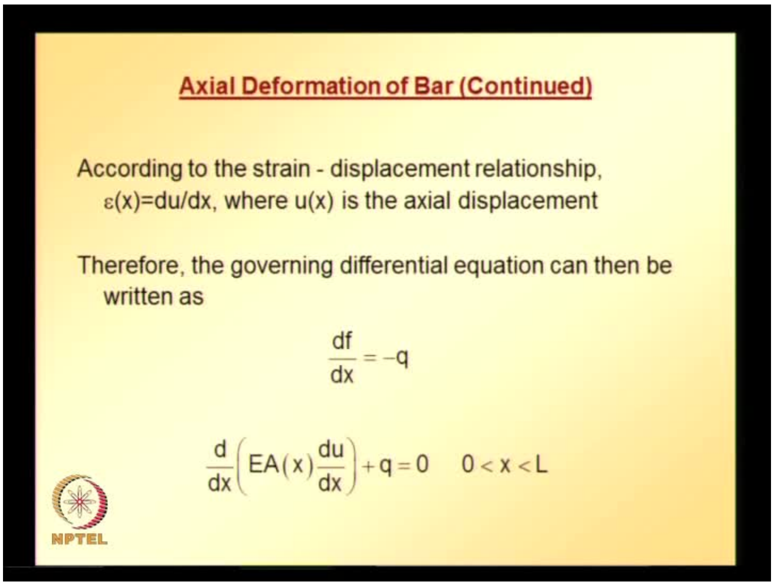

We know that according to strain-displacement relationship, strain epsilon is derivative of u or derivative of displacement with respect to x, where u is axial displacement and so substituting all this that is f is equal to stress times area of cross section. Stress is given by young's modulus time strain.

Strain is given by first derivative of axial displacement substituting all this information into the equation that we have derivative of f with respect x is equal to minus q. We get this differential equation, which is a second order differential equation and this differential equation or at any or every point along the length of the bar needs to satisfy this differential equation for it to be in equilibrium. So, we need to solve this differential equation and subjected to boundary conditions. Boundary conditions, we can easily figure out for this particular problem, if you recall a bar is fixed at x is equal to zero and a tip load is applied at x is equal to l that means, axial displacement evaluated at x is equal to zero is zero and point load that is young's modulus times area of cross section evaluated at x is equal to l, times derivative of axial displacement with respect to x evaluated at x is equal to l should be equal to the tip load. So, these are the two boundary conditions.

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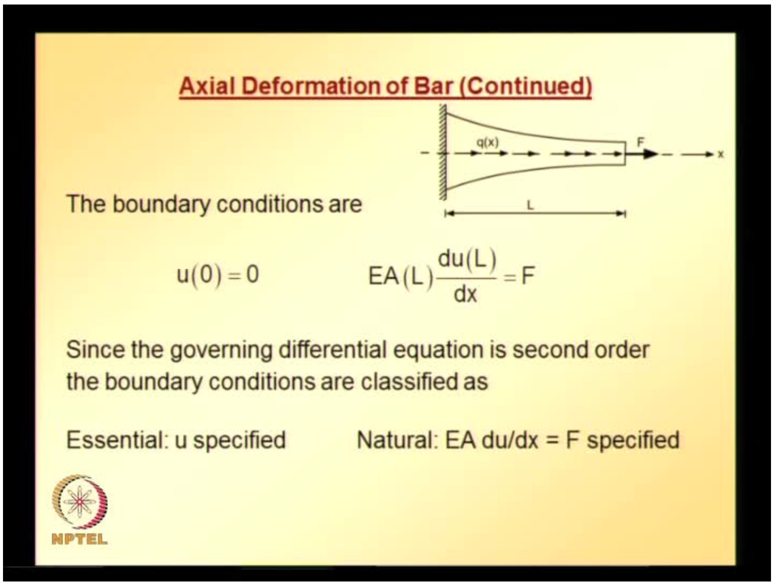

So, boundary conditions are these so, we need to solve the differential equation representing the physical phenomena of axial deformation of a bar, over the domain x going from zero to l subjected to these two boundary conditions. And we noted already that the differential equation is second order differential equation. So, to solve the differential equation, we require two boundary conditions. So, just now we obtain the two boundary conditions and these boundary conditions can be classified into two types u evaluated at x is equal to zero, which involves or which is a zeroth order equation is can be classified as essential boundary condition. And the second boundary condition that is e times a evaluated at x is equal to l derivative of u with respect x, evaluated at x is equal to l is equal to f that involves first order, that is, first order equation or that involves first order derivative and that can be classified as natural boundary condition. So, the two boundary conditions can be classified in the manner, that is, mentioned there that is, essential u specified natural e times a times derivative of u with respect x is equal to f specified. So, for a general boundary value problem of order two p, how to classify these boundary conditions?

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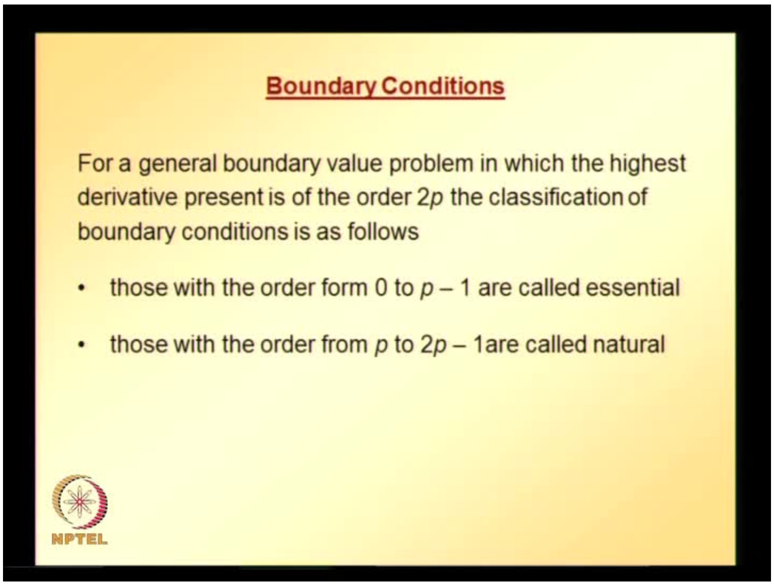

For a general boundary value problem in which highest derivative present is of order two pthe classification of boundary conditions is as follows: those boundary conditions of order zero two p minus one are called essential; those boundary conditions of order p to two p minus one are natural boundary conditions. So, this is how for a general boundary value problem of order two p boundary conditions can be classified into essential and natural. In case, the information about the physical phenomena which the differential equation is representing is not available.

So, with this before we proceed to the derivation of finite element equations, let us look at some of the approximate methods for solving boundary value problems, there are several classical methods for obtaining approximate solutions of boundary value problems and which forms base or which forms the foundation for finite element methods or any other method like boundary element method.

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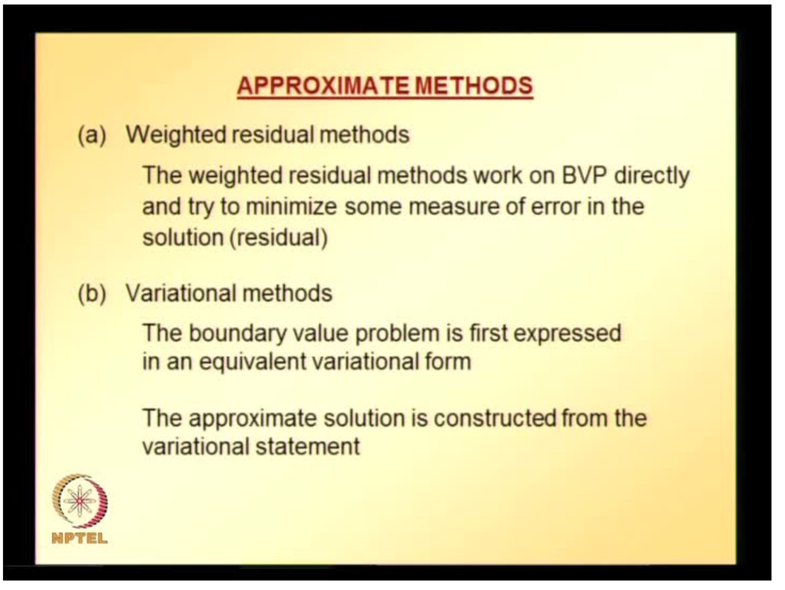

So, now, let us look at some of the approximate methods and these are very important because as i just mention they form foundation for understanding finite element method, because as you can see in the later part of the lectures these some of these methods that we are using or that we are going to see now we are going to substitute finite element approximation into these into the equations that we obtain using these methods.

So, basically, all the classical methods can be categorized into two groups: one is called weighted residual methods - in weighted residual methods or in the weighted residual methods work on the boundary value problem directly and try to minimize some measure of error in the solution which is called residual. And the other category of methods or variational methods in this methods, the boundary, the boundary value problem is first expressed in an equivalent variational form or weak form, and the approximate solution is constructed from this variational statement or weak form. And in both this class of method both classes of methods require a general form of solution to be assumed first. And this general form of solution can be polynomial based trail solution or it can be or it can have some other quantities like trigonometric terms. And this trail solution must include some unknown parameters, which can be determined based on the techniques or based on the technique that we are going to adopt for each of these methods.

So, let us look at these two categories of methods, one by one and solve a problem using these two category of methods and compare how the solution matches with exacts solution. To do that let us take a boundary value problem just for comparison purpose of different methods.

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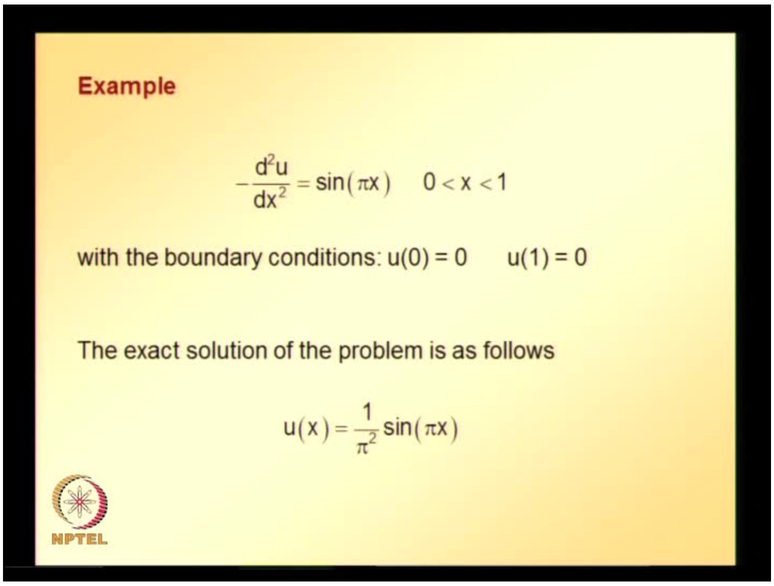

So, let us take this boundary value problem, this is a second order differential equation, which needs to be solved over the domain x going from zero to one subjected to the boundary conditions. Since, this is a second order differential equation, as you can guess we require two boundary conditions to solve this problem and the two boundary conditions that are given here turns out to be essential boundary conditions, if you recall the thumb rule that we just looked at, if a boundary value problem is of order two p those boundary conditions of order zero to p minus one are essential, and those boundary conditions of order p to two p minus one are natural boundary condition. If you apply that thumb rule here taking p is equal to one we can easily verify that these two boundary conditions are essential boundary conditions anyway we can solve this problem exactly by integrating it twice and getting the integration constants by substituting the boundary conditions.

So, when we do that the exact solution of the problem turns out to be this. And this exact solution is going to be used for comparison purposes with the other approximate or with this. This is solution of other approximate techniques that we are going to look at.

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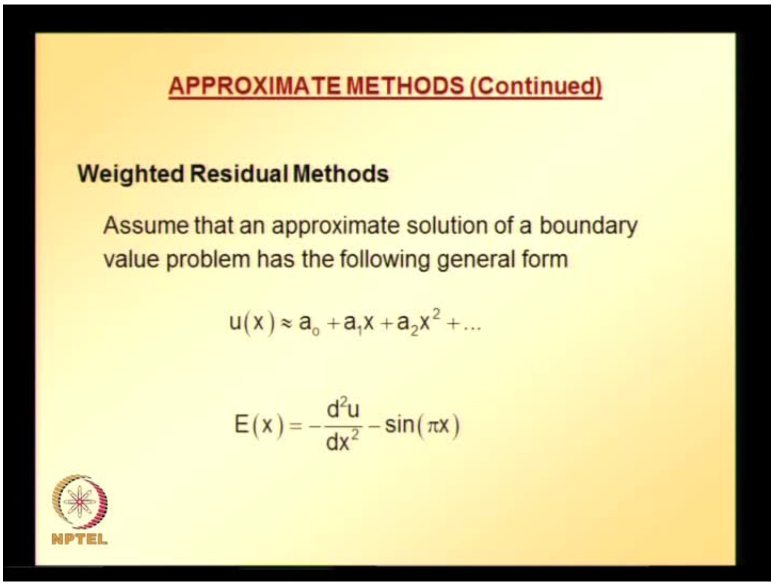

So, let us start with weighted residual method of weighted residuals or weighted residual method assume an approximate solution of boundary value problem. And let us assume a polynomial based trial solution and where a naught a 1, a 2, and so on are some unknown parameters which needs to be determined as a part of solution. And when the solution is substituted in the given boundary value problem or into the given differential equation right hand side is not going to match with the left hand side and the difference, that is, difference between right hand side and left hand side is this, residual denoted with e and it is going to be a function of x and once we have this residual at a point, we can sum up the residual at all points and we can get the total residual. But the problem which doing that is residual at one point may be positive and residual at another point may be negative over the domain x going from zero to one.

So, residual positive residual gets cancel with negative residual and there is a a danger of error cancellation and to avoid such situation, we are going to use or we are going to multiply this residual with a weight function and integrate over the problem domain that is zero to one.

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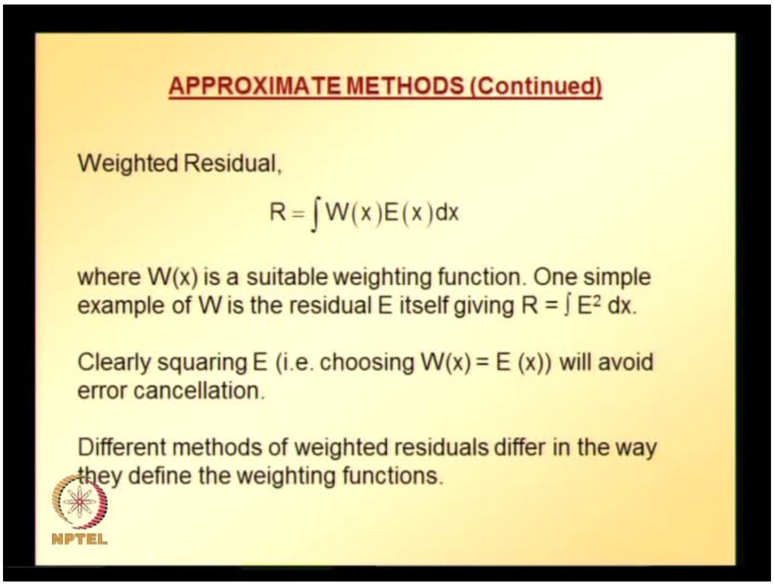

So, weighted residual is denoted with r integral weight function times residual integrated over the problem domain. And this weight function is a suitable weight function the simplest weight function is residual itself, because the main purpose of this weight function is to avoid error cancellation or the main purpose is to make error positive. So, we can if we take a quantity whether it is positive or negative, when we square it then you are going to get a positive quantity. So, that is why the simplest weight function that we can have is residual itself. So, as it is mentioned clearing squaring e that is choosing weight function is equal to residual itself will avoid error cancellation and depending on the weight function that we choose different methods of weighted residuals are there. So only difference between different methods of weighted residuals is the way in which the weight function is defined.

So, once the weight function or once the weight function is decided and we can define weighted residual and then the goal is to pick the unknown parameters, because when we substitute the trial solution or approximate solution into the residual weighted residual is going to be function of unknown parameters. So, our goal is to minimize the weighted residual or to find the unknown parameters a naught a 1 a 2 such a way that weighted residual is minimum. Sometimes we need to include even boundary residual, which will be seeing later when we solve the problems.

Since, the approximate solution is not going to satisfy boundary conditions also accurately or exactly a residual is going to come in to picture as a part of boundary. So, that also needs to be included in the calculation of weighted residual for minimization of weighted residual.

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So, some of the commonly employed weighted residual methods are: least square method, collocation method, Galerkin method. Now, let us solve the differential equation problem or boundary value problem that we looked at whose exact solution we already know we can solve that problem using each of these methods least square method collocation method and method.

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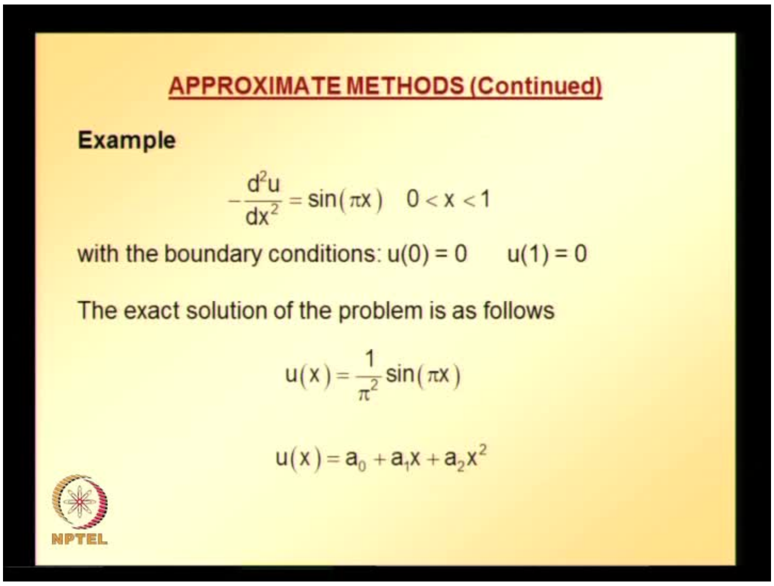

So, this is the example that we are looking at and the exact solution is this one. So, let us start with a polynomial based trial solution and let us take a quadratic trial solution. So, it is up to us to what order the we can choose so here we will solve this problem using quadratic trial solution, that is, let u be a naught plus a 1 x plus a 2 x square. And this trail solution before we substitute in to the residual this sub trial solution needs to be or the trial solution that we use needs to be admissible trial solution and admissible trial solution is a trial solution, which satisfies essential boundary conditions. So, once we assume a trial solution like this, we need to make sure that it satisfies essential boundary conditions. So, before we proceed with this trial solution, we need to make sure that this satisfies the condition that u evaluated at x is equal is zero, u evaluated at x is equal to one is zero. So, substituting or forcing those conditions on this trial solution, we can determine or we can eliminate some how the unknown parameters or unknown coefficients out of a naught a 1 a 2.

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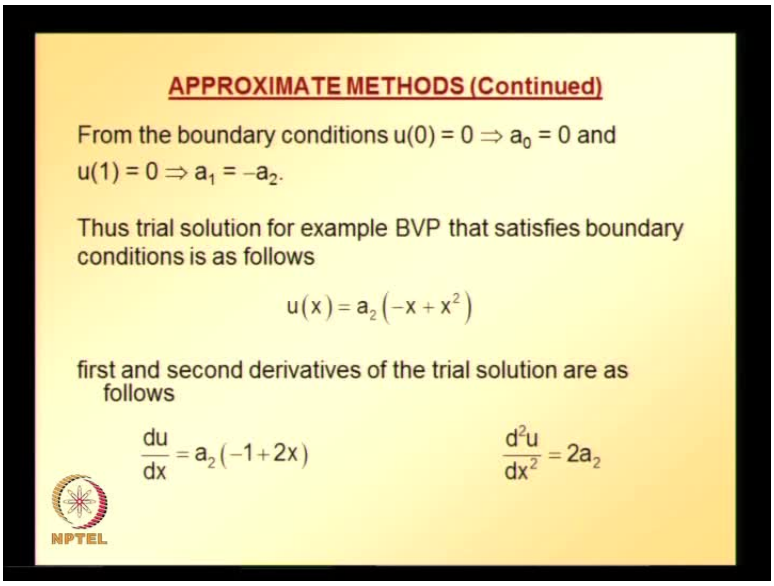

So, doing that we get this a naught is equal to zero one is equal to minus a 2. So, the trial solution by after substituting this information into the trial solution that we started out with quadratic trail solution a naught plus a 1 x plus a 2 x square; a naught is zero a 1 is equal to minus a 2 substituting that we get this. So, this is an admissible trial solution. So, we can substitute this into the residual before we substitute if you recall residual involve second derivative of u with respect x. So, we need to find first, second derivative of trial solution with respect to x or first and second derivatives of trial solution are shown here.

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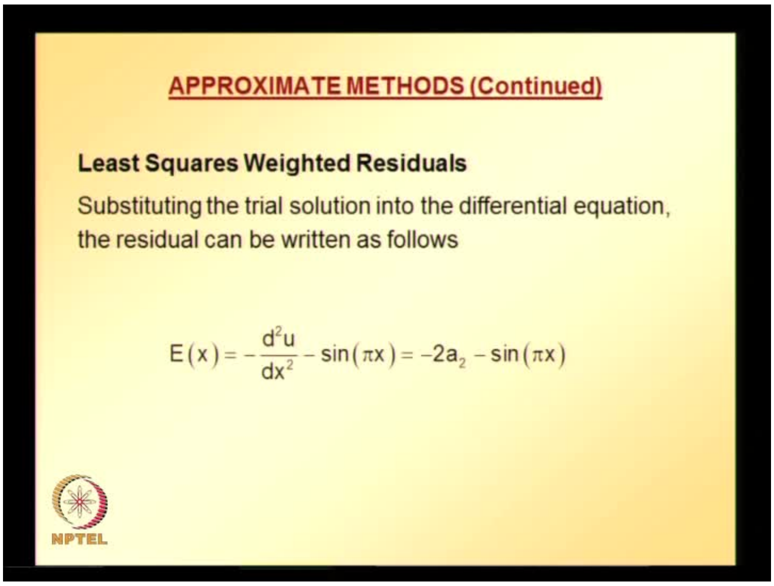

So, once we have this, we can actually substitute into the residual in least square weighted residual method substituting trial solution into the differential equation residual can be written as follows. And as you can see, here residual is having only one unknown coefficient need that needs to be determine a 2 and in least square weighted residual method weight function is residual itself.

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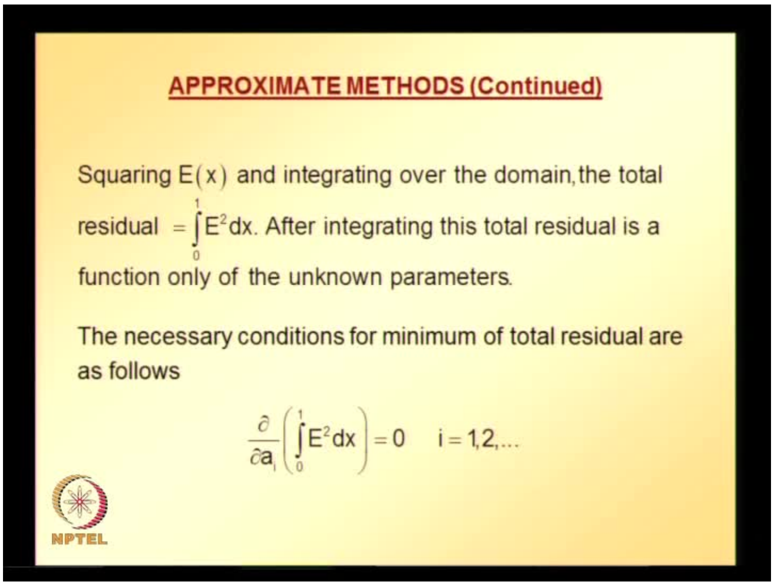

So, squaring residual and integrating over the domain, total residual is going to be integral zero to one residual square d x. After integrating this total residual it is going to be function of only a 2 because residual is function of a 2 it is going to the function of only one unknown parametera two and the necessary condition for residual or total residual to be minimum is... At the stage, recall that if a function f which is function of x is given to you and if you are asked to find what value of x makes this function minimum the first thing that you are going to do is take the first derivative of f with respect x equated to zero solve for x and that value of x gives you minimum or maximum. And at least that first derivative of f with respect to x is equal to zero that equation is the necessary condition for finding minimum of function minimum of x value, which makes function or x value for the which makes the function minimum.

So that is the necessary condition that you can recall same logic, we are going to use here so i want to find a two such a way that this total residual is minimum and total residual is function of a two. So, the necessary condition is going to be the partial derivative or derivative of this total residual with respect to this unknown parameter is going is equal to zero. So, the necessary condition for minimum of total residual is as follows partial derivative of total residual with respect to unknown parameters. Here in this particular example that we are looking at residual as only one unknown parameter a 2. But in some other example, we have more number of unknowns in that case, to determine all those unknown parameters or unknown coefficients, when it requireas many number of equations as the number of unknowns that needs to be determined.

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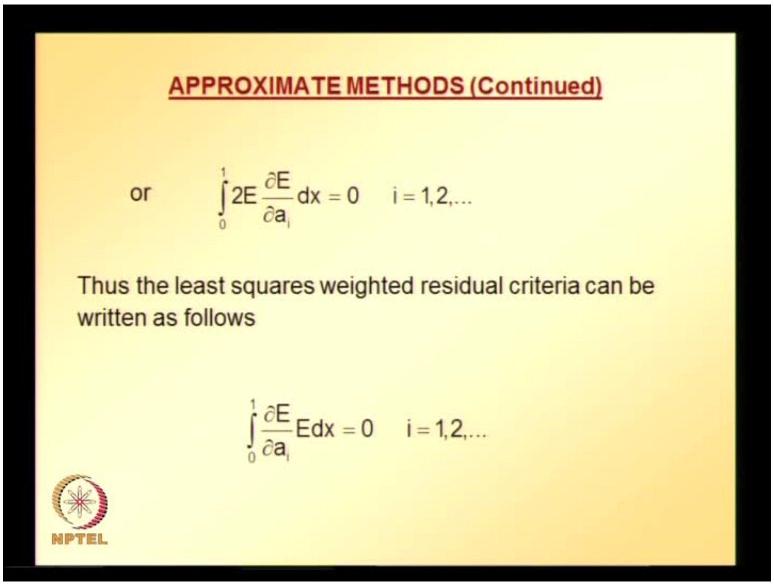

So, this is the necessary condition. And this necessary condition can further be simplified by noting that integration and differentiation are interchangeable and taking differentiation inside we get this equation and here u is sorry two is can be cancelled out on both sides and we get this equation. So, weighted least square weighted residual criteria can be written like integral zero to one partial derivative of a with respect to a times e d x is equal to zero.

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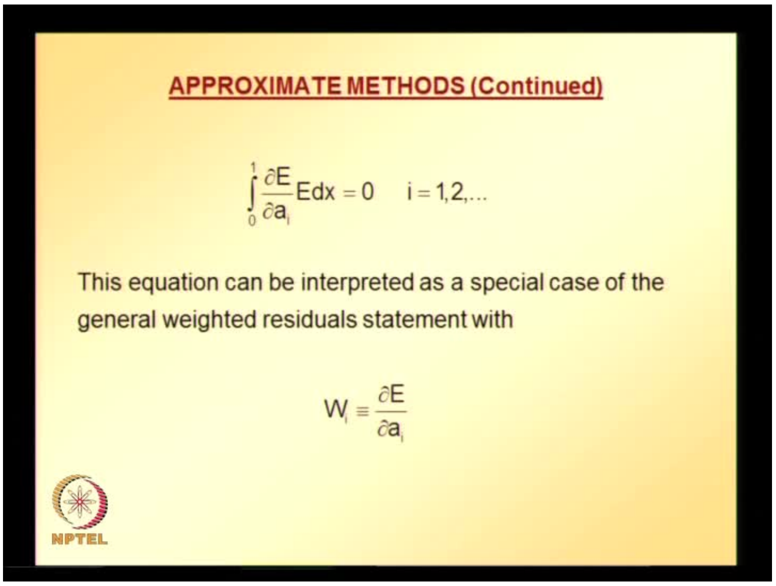

So, this is a least square weighted residual criteria and whatever statement i made earlier that can be revised based on this equation. This equation can be interpreted as a special case of general weighted residual statement, where weight function is w i is partial derivative of residual with respect to the unknown coefficient a i . So, this is about least square weighted residual method. And now, let us go back and solve the problem that we are looking at the boundary value problem that we are looking at. Now, this is how weight function can be determined in least square weighted residual method.

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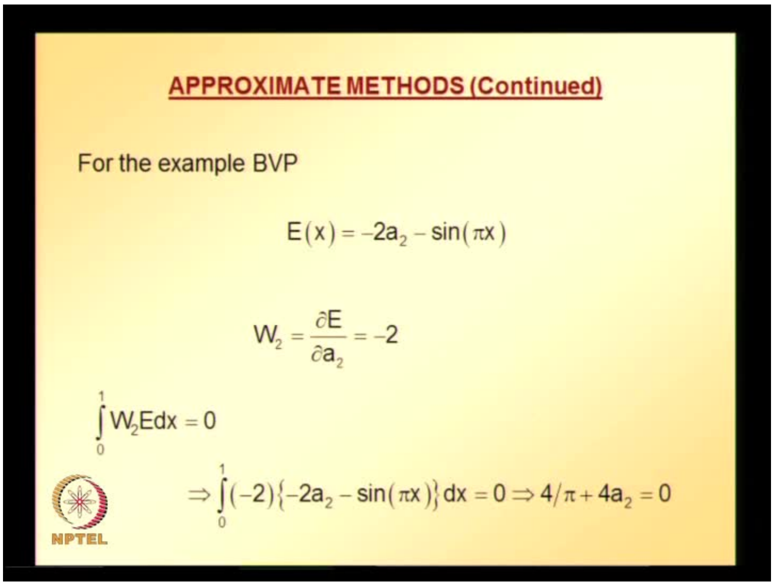

So, applying this to the problem that we are looking at for the boundary value problem that we had looking at residual is given by this. So, weight function is partial derivative of this residual with respect to the unknown coefficient a 2 it gives us minus 2. So, once we have this weight function or weight w 2, we can take this w 2 and multiply with the residual and integrate over the problem domain between zero and one equated to zero here residual is a function of a 2.

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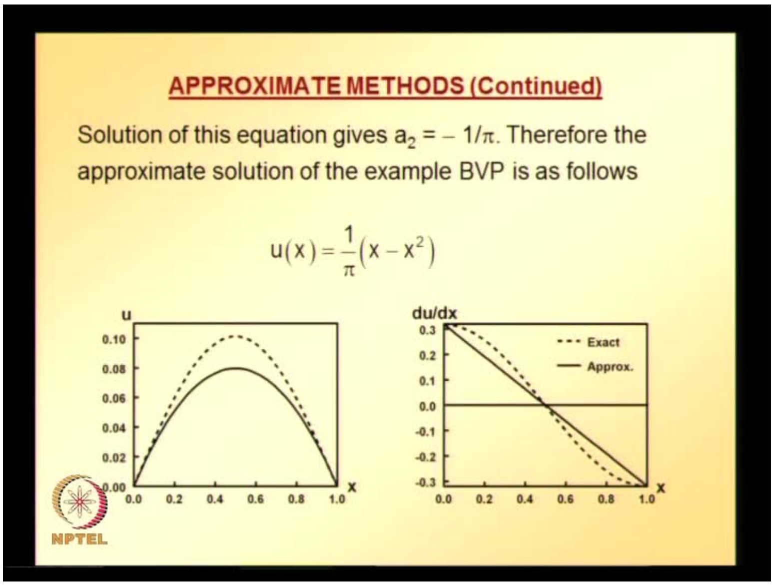

So, we are going to get an equation only one equation in terms of a 2 and we can solve this equation and find a 2. And this a 2, again we can substitute back in to the admissible trail solution to get the approximate solution of the boundary value problem that we looked at. And this is an approximate solution and how this is going to match with exact solution that we can get an idea, if we plot the approximate and exact solution for the problem, for this particular problem. Please note that this approximate solution that we obtained is based on least square weighted residual method starting with quadratic trial solution that you need to keep in mind. And if you want more accurate solution, then you may need to start with a polynomial having more number of coefficients like cubic or quartic or quintic.

If it is cubic, then you will be having four unknown coefficients out of which two can be determined using the essential boundary conditions that are given and rest two you need to get two equations based on, minimization of total residual and two equation, two unknowns can be solved to get the approximate solution. Similarly, if it is quartic or fourth order polynomial with the starting polynomial is fourth order polynomial, then two coefficients can be determined using essential boundary conditions rest of the three coefficients can be determined using, minimization of total residual with respect to the unknown coefficients for a fourth order polynomial will be having five number of coefficients.

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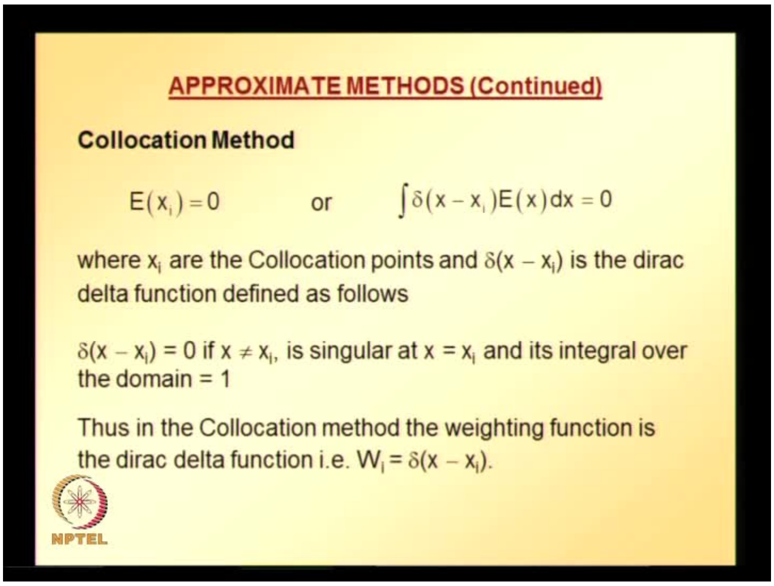

So, now, let us look at the other weighted residual method that is collocation method. Basically in collocation method we will force residual to be zero at a particular at some chosen point and that point is totally the selection of that point istotally arbitrary. So, residual at x i is equal to zero, this is that is we are actually forcing residual at point x i equal to zero and please note that this equation can be written in this manner integral dirac delta function times residual dirac delta function, here is delta x minus x iintegral of this or this integral is same as e evaluated or residual evaluated at x is equal to x i, Where x i is the collocation point and delta x minus x i is the dirac delta function and i hope you know what is dirac delta function, if not here is the definition: delta x minus x i is equal to zero if x is not equal to x i that is singular at x is equal to x i and it is integral over the domain the entire domain is equal to one. So, if we compare this with the weighted residual method the concept of weighted residual, the definition that we looked at we can easily notice that the weighting function here for collocation method is dirac delta function.

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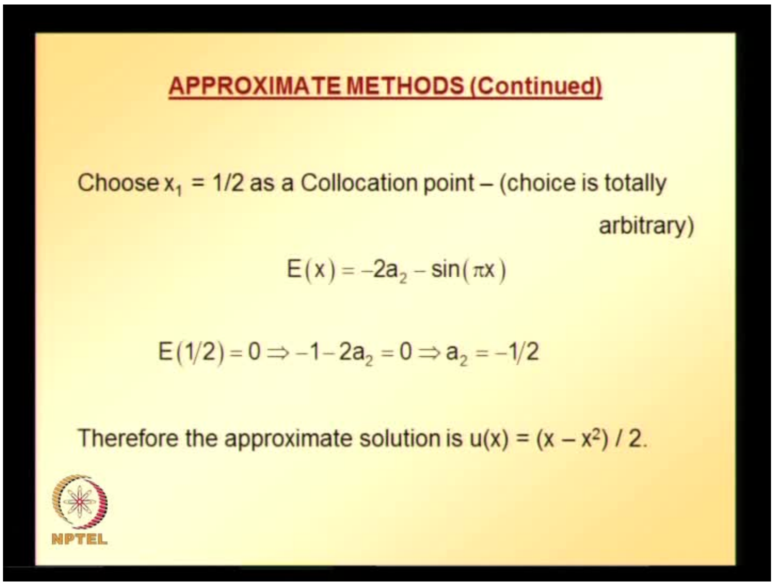

So, now, let us go back to the problem that boundary value problem that we are looking at. Let us choose one collocation point, because in the admissible trial solution, we require to determine one unknown parameter or coefficient a 2. So, it is enough, if we get one equation. So, that we can solve for that unknown coefficient so, one equation we obtain can we can get one equation by choosing, one collocation point if we decide to adopt collocation method.

Suppose, if more number of coefficients needs to be determined we need to select as many number of coefficients needs to be determined that many number of collocation points that you keep in mind. So, here collocation point be selected as point five or half and please note that this selection is totally arbitrary, if some of you want to select some other collocation point, you are welcome to that and your solution may be different from what is show, what is going to be shown here. So, once we select collocation point, five residual is given by this, we already noted this. Now, substitute in this x is equal to point five and force residual to zero, we get one equation, one unknown, we can solve for a 2 and then once we get a 2, we can substitute this into the admissible trial solution to get the approximate solution. So, approximate solution using collocation method turns out to be x minus x square divided by two, to see how this solution matches with the exact solution, we can again overlay the plots of exact solution and also approximate solution.

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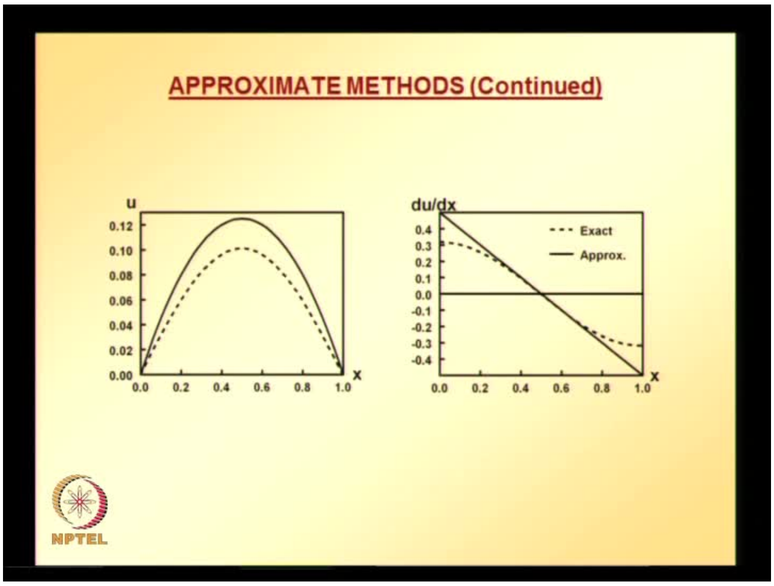

I forgot to mention even in the previous plot that we that we have seen for least square weighted residual method, when we are solving this problem even the derivative of the trials or derivative with approximate solution is also shown. So, solution of u and derivative of u the comparison with exact is shown here. And please note that this is the solution that we obtain using collocation method by, starting with a quadratic trial solution and choosing collocation point as point five is somebody is choosing a different collocation point. You are going to get a different solution and accuracy may also be different so that is one of the limitation. So, depending on there is no unique solution or you may not be getting unique solution, if you choose different collocation points. Whereas, in the least square weighted residual method, if somebody is starting with quadratic trial solution and do it properly they should get the same solution as what i showed here. But collocation method it depends on the collocation point that is being chosen and again to improve the accuracy somebody can start with instead of quadratic trial solution they can take cubic quartic or quintic. As a starting point and they can proceed and only thing is they need they may need to get as many number of equations as unknown coefficient that needs to be determined, that is, as many number of collocation points as the number of unknowns that needs to be determine.

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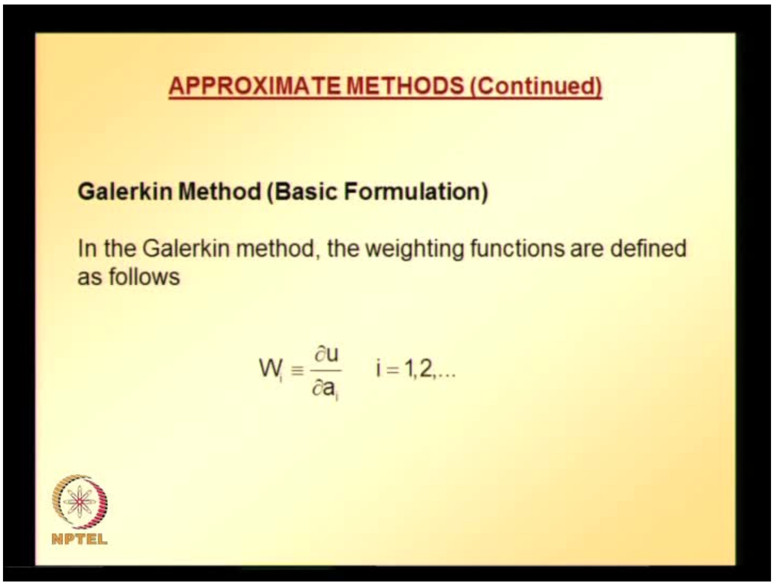

So, now, let us look at the other weighted residual method, that is, Galerkin method and this is what we are going to use in conjunction with finite element method. So, in Galerkin method weight function is defined as follows: w i is equal to or w i is defined as: partial derivative of u with respect to a i. Please recall, that in least square weighted residual method, weight function is partial derivative of residual with respect to the unknown coefficients or unknown parameters, where as in collocation method weight function is dirac delta function. And now for Galerkin method weight function is partial derivative of trial solution with respect to the unknown parameters or unknown coefficients. So, with this substituting this into the weighted residual definition, we can easily evaluate total weighted residual and minimize the total residual with respect to the unknown parametersto determine the unknown parameters.

So, unknown parameters in the trial solution are computed using the following criteria, where w i is defined as partial derivative of u trial solution with respect to the unknown coefficients.

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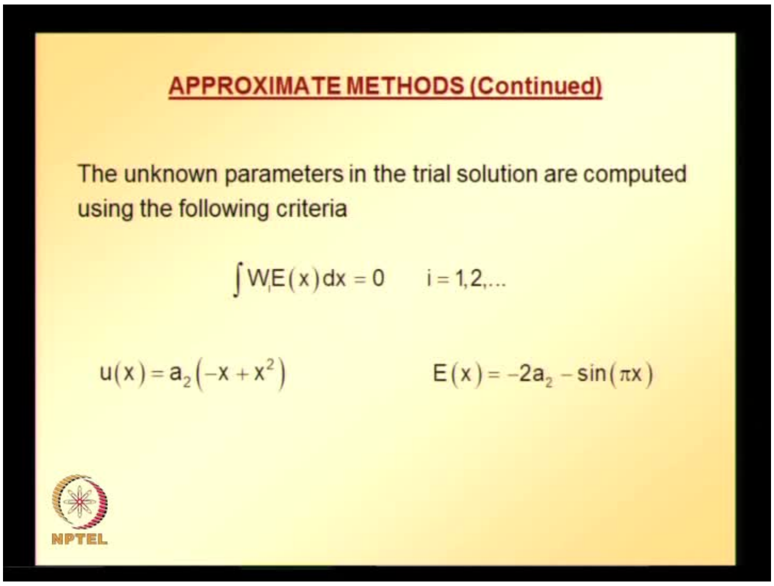

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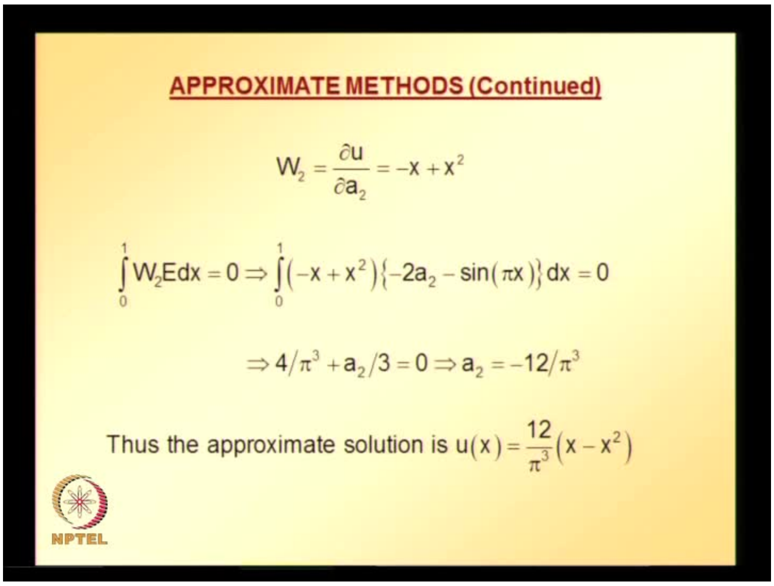

So, this is admissible trial solution u is equal to a two times minus x plus x square and residual is given by this and once we have the admissible trial solution, they can find what is weight function for Galerkin method weight function for this particular problem for Galerkin method is partial derivative of u with respect to a two gives minus x plus x square. So, once we have this weight function, we can evaluate this integral that is integral zero to one w two times residual is equal to zero, we can evaluate this we get one equation and one unknown in terms of a 2 and we can solve this one equation one unknown and get find what is a 2? So once we have this a 2 we can substitute back this a 2 into the admissible trial solution and we can get the approximate solution. So, once again this is the solution this is the approximate solution that we obtained starting with quadratic trial solution using Galerkin method. Now, let us see, how this solution compares very exact by comparing both solution and also it is derivative.

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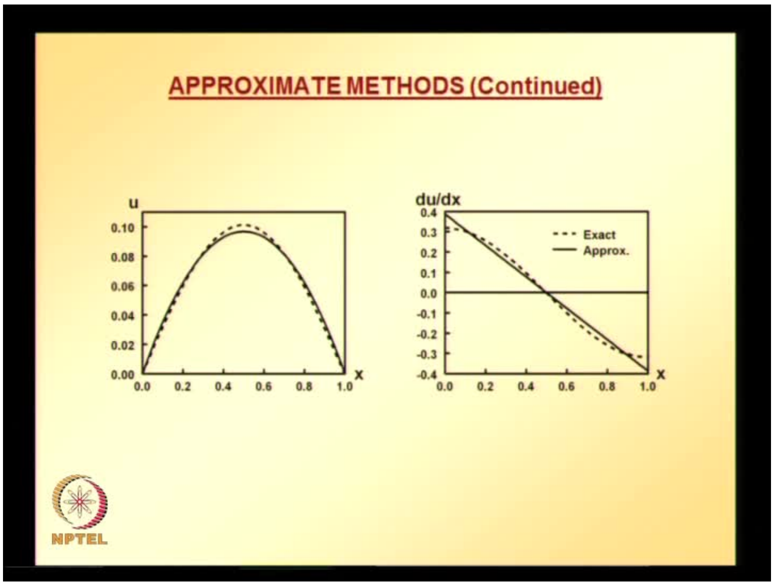

So, this is how it compares, if you recall or if you just go back and see how the solution that we obtain using least square weighted residual method and collocation method how that matched the solution obtain using those methods, how it match with exact solution and how this is matching, we can see here the solution is fairly good and the because it is matching very well with the exact solution. So, that is why Galerkin method is superior to least square weighted residual method. And collocation method, please note that we put the same amount of effort we started with quadratic trial solution and still we got a better solution using Galerkin method and if one wants to improve this solution that we just obtained using Galerkin method with quadratic trial solution, they can do it by starting or by selecting cubic quartic or quintic polynomial as a trial solution only thing is as the number of coefficients in the trial solution increases effort increases.

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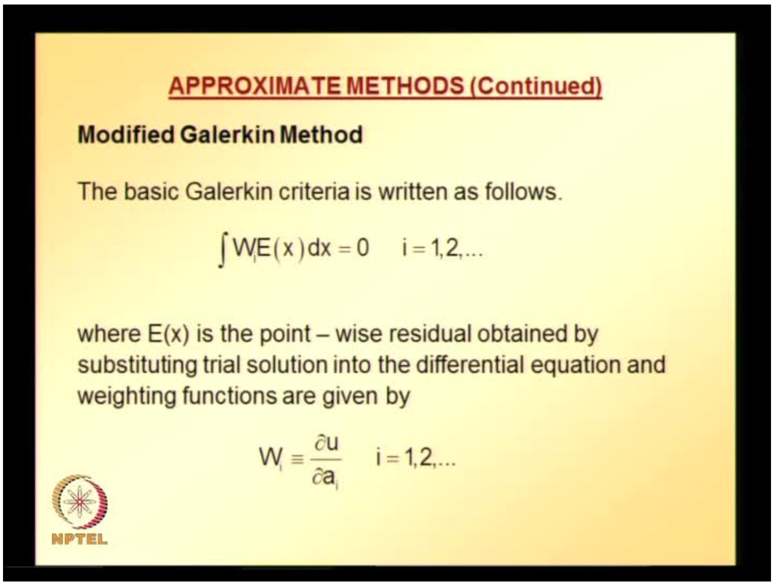

So, now, let us look a small modification to this Galerkin method, which is called modified Galerkin method the basic Galerkin. Galerkin criteria is this one just now, we looked at and where e is nothing but, point wise residual obtained by substituting trial solution with the differential equation and weighting functions are given by this one and please if you recall the residual that we have seen in the boundary value problem involve second derivative of trial solution. So, computation of residual requires differentiating trial solution twice or it can be for different for some other case, it can be as number of times as the order of the highest derivative in the boundary value problem. So, therefore, to get the second derivative we need to start at least with a trial solution, which is quadratic because we are looking for second derivative; when we are calculating the residual in the previous problem in some other problem it may be it depends on the order of the difference highest derivative that is appearing. So, we are unnecessarily increasing the demand on the starting trial solution. So, the computation of residual requires differentiating the solution as many times as the order of highest derivative in the boundary value problem.

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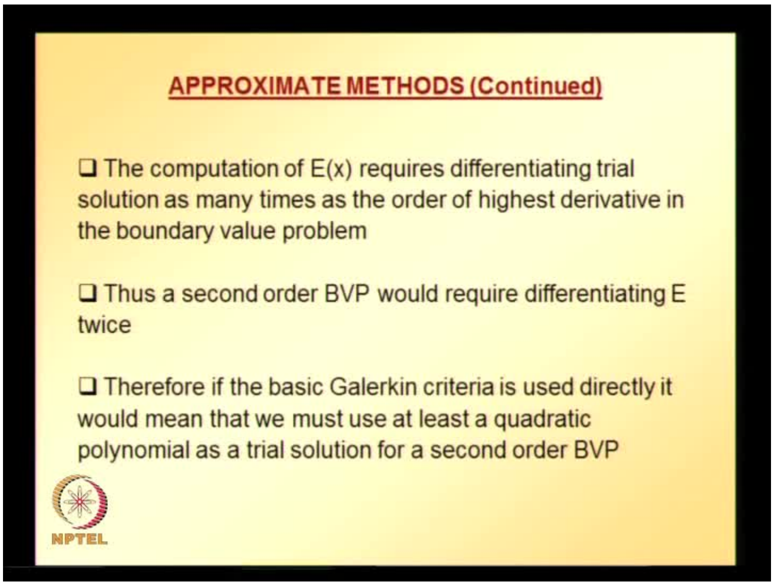

Thus, a second order, boundary value problem require differentiating residual twice. Therefore, if basic Galerkin criteria is used directly it would mean thatyou must use at least quadratic polynomial as a trial solution for second order boundary value problem.

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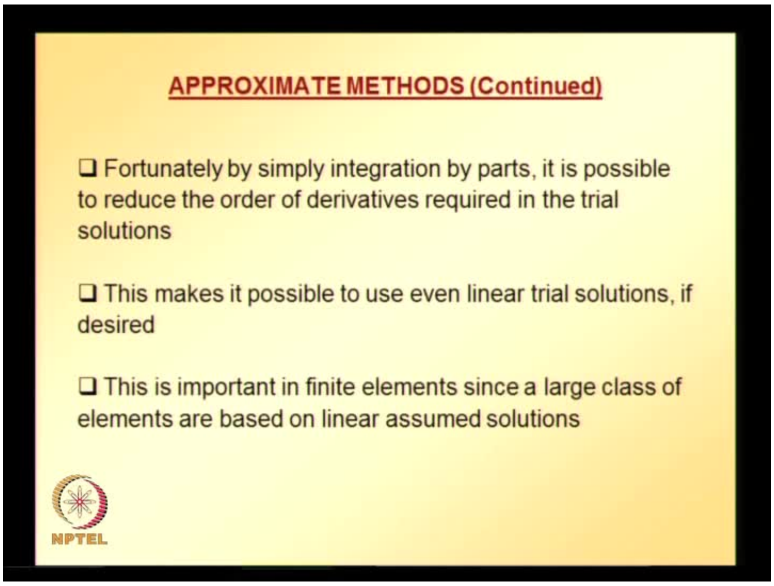

So, is there a way that we can reduce this demand on the starting trial solution, that is, the question and there is an answer to that also fortunately by simple integration by parts it is possible to reduce the order of derivatives required in the trial solution. And this makes it possible to even use linear trial solution if desired. And this is very important in finite element finite elements since a large class of elements are based on linear assumed solutions. This point it will be clearer for you at later stages of this course.

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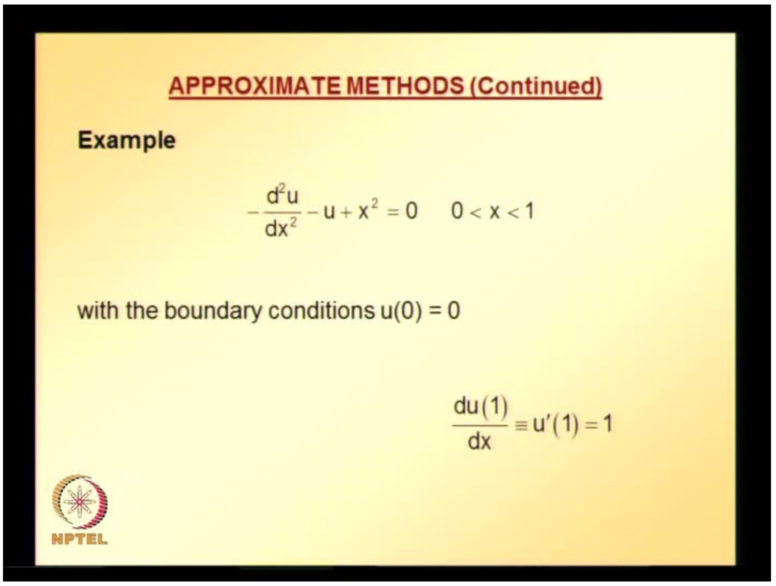

So, now, let us look at how to proceedso let us take this example, for a while forget about the previous example that we are looking at the boundary value problem, which involves sine function. And, now, let us look at this boundary value problem which is a second order boundary value problem and we need to find approximate solution of this over the domain x going from zero to one. And it needs to satisfy two boundary conditions u evaluated at x is equal to zero is zero and first derivative of u evaluated at x is equal to one is one and if you see compare with the thumb rule that we already noted it turns out that one of the boundary condition that is u evaluated at x is equal to zero is zero is essential and the other boundary condition, which involves first derivative of u is natural boundary condition that is just a side node.

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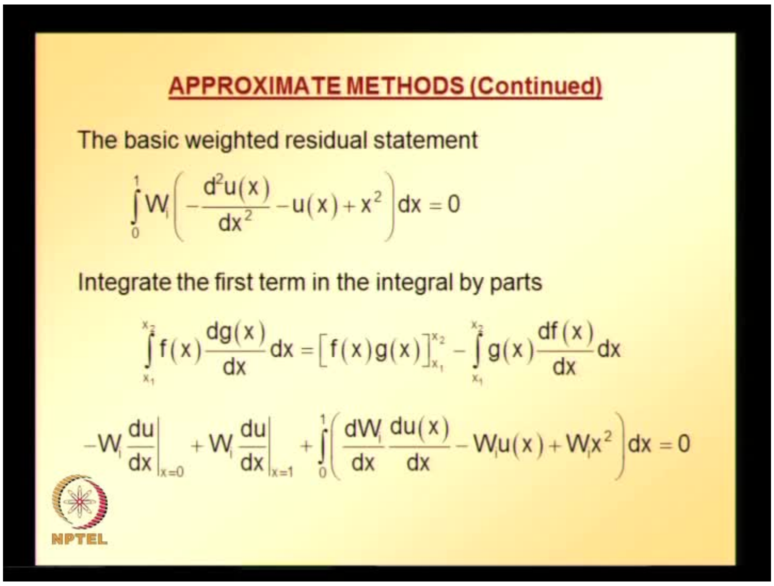

So, now applying the Galerkin criteria, basic Galerkin criteria over this weight function times the difference between the left hand side and right hand side or the residual integrated over the problem domain equated to zero. Since, this is the basic Galerkin criteria and here the demand on the starting trial solution we want to reduce it.

So, what we can do is we can integrate the term which is having highest derivative and in this case it is the first term integrate the first term by parts and integration by parts formula is reproduced here and if we have f times derivative of g with respect x integrated between x 1 to x 2 it can be written as f times g evaluated between the limits x one x two and integral x 1 to x 2 g are minus integral g times first derivative of f with respect x. So, this is integration by parts. So, using this formula over the first term we can reduced or we can rewrite the first term in this first term without having the second derivative of u. So, the derivative of u and the first term can be reduced using integration by parts and after doing that and substituting back into the first equation we get the equation which is shown at the end. So, if you see this last equation you can see that the demand on the starting trial solution is reduced. So, we can work with the last equation and if we work with the last equationwe can even solve this problem using linear trial solution. So that is the advantage of modified Galerkin method, so this is what modified Galerkin method is and later stages whenever we refer Galerkin method that refers to the modified Galerkin method.