Lecture 1: Finite Elements for Elastic Stability

7.1.1 Introduction

There are two types of structural failure associated: namely (i) material failure and (ii) geometry or configuration failure. In case of material failure, the stresses exceed the permissible values which may lead to formation of cracks In configuration failure, the structure is unable to maintain its designed configuration under the external disturbance even though the stresses are in permissible range. The stability loss due to tensile forces falls in broad category of material instability. The loss of stability under compressive load is termed as structural or geometric instability which is commonly known as buckling. Thus, buckling is considered by a sudden failure of a structural element subjected to large compressive stress, where the actual compressive stress at the point of failure is less than the ultimate allowable compressive stresses. Buckling is a wide term that describes a range of mechanical behaviours. Generally, it refers to an incident where a structural element in compression deviates from a behaviour of elastic shortening within the original geometry and undergoes large deformations involving a change in member shape for a small increase in force. Various types of buckling may occur such as flexural buckling, torsional buckling, torsional flexural buckling etc. Here, the use of finite element technique for elastic stability analysis of various structural members will be briefly discussed.

7.1.2 Buckling of Truss Members

For a truss member, the axial strain can be expressed in terms of its displacements

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \tag{7.1.1}$$

Here, u and v are the displacements in local X and Y directions respectively. Here, the strain-displacement relation is nonlinear. The total potential energy in the member with uniform cross sectional area and subjected to external forces can be written as

$$\Pi = \frac{EA}{2}\int_0^L \varepsilon_x^{\,2}\,dx - \left(P_1 v_1 + P_2 v_2\right) \tag{7.1.2}$$

Where, P_1 and P_2 are the external forces in nodes 1 and 2, v_1 and v_2 are the vertical displacements in nodes 1 and 2, E and A are the modulous of elasticity and cross sectional area respectively (Fig. 7.1.1). The length of the member is considered to be L.

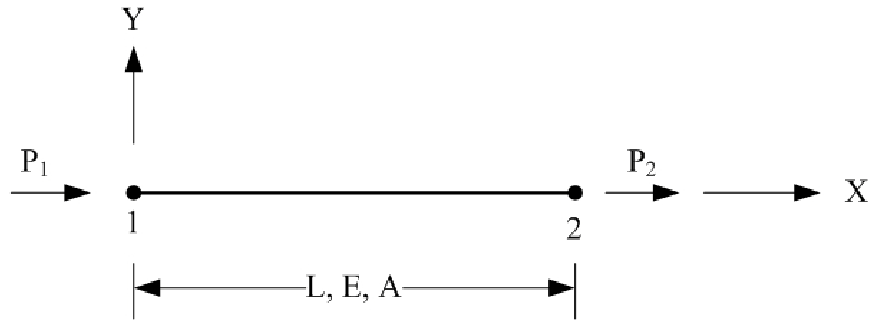

Fig.7.1.1 Truss element in local coordinate system

The above expression can be rewritten in terms of displacement variables as

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\right)^2 dx - \left(P_1 v_1 + P_2 v_2\right)$$

$$= \frac{EA}{2}\int_0^L \left\{\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^3\right) + \frac{1}{4}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]^2\right\} dx - \left(P_1 v_1 + P_2 v_2\right) \tag{7.1.3}$$

Neglecting higher order terms and considering $\mathrm{EA}\frac{\partial u}{\partial x} = P$ the above expression can simplify to the following form.

$$\Pi = \frac{EA}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{P}{2}\int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx - \left(P_1 v_1 + P_2 v_2\right) \tag{7.1.4}$$

Considering nodal displacements at nodes 1 and 2 as u_1, v_1 and u_2, v_2, the displacement at any point inside the element can be represented in terms of its interpolation functions and nodal displacements. The shape function for a two node truss element is

$$[N] = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right] \tag{7.1.5}$$

Equation expressed in eq. (7.1.5) is expressed in terms of the nodal displacement vectors $\{u_i\}$ and $\{v_i\}$ with the help of interpolation function.

$$\Pi = \frac{1}{2}\int_0^L \{u_i\}^{\mathrm{T}} [N']^{\mathrm{T}} EA [N'] \{u_i\} dx + \frac{P}{2}\int_0^L \{v_i\}^{\mathrm{T}} [N']^{\mathrm{T}} [N'] \{v_i\} dx - \left(P_1 v_1 + P_2 v_2\right) \tag{7.1.6}$$

Where,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left([N]\{u_i\}\right) = [N']\{u_i\} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left([N]\{v_i\}\right) = [N']\{v_i\} \tag{7.1.7}$$

Making potential energy (eq. 7.1.6) stationary one can find the equilibrium equation as

$$\{F\} = \int_0^L [N']^{\mathrm{T}} EA[N']dx\{u_i\} + P\int_0^L [N']^{\mathrm{T}} [N']dx\{v_i\} \quad (7.1.8)$$

Or,

$$\{F\} = [k_A]\{u_i\} + \mathrm{P}[k_G]\{v_i\} \quad (7.1.9)$$

Where, $[k_A]$ is the axial stiffness of the member and $[k_G]$ is the geometric stiffness of the member in its local coordinate system and can be derived as follows.

$$[k_A] = \int_0^L [N']^{\mathrm{T}} EA[N']dx = AE\int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (7.1.10)$$

$$[k_G] = \int_0^L [N']^{\mathrm{T}} [N']dx = \int_0^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (7.1.11)$$

However, in a generalised form to accommodate both the direction, these stiffness matrices can be written as

$$[k_A] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } [k_G] = \frac{1}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (7.1.12)$$

The generalised stiffness matrix of a plane truss member in global coordinate system can be derived using the transformation matrix as derived module 4, lecture 1. The transformation matrix can be recalled and written here as follows.

$$[T] = \begin{bmatrix} \cos\theta & sin\theta & 0 & 0 \\ -sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & sin\theta \\ 0 & 0 & -sin\theta & \cos\theta \end{bmatrix}$$

(7.1.13)

Thus, the stiffness matrices with respect to global coordinate system will become

$$[K_A] = [T]^T [k_A][T] \text{ and } [K_G] = [T]^T [k_G][T] \quad (7.1.14)$$

Here, $[K_A]$ and $[K_G]$ are the axial stiffness and geometric stiffness of the member in global coordinate system. The force-displacement relationship in global coordinate system can be written from eq. (7.1.9) as

$$\{F\} = [K_A]\{d\} + \mathrm{P}[K_G]\{d\} = [[K_A] + \mathrm{P}[K_G]]\{d\} \tag{7.1.15}$$

Where, $\{d\}$ is the displacement vector in global coordinates. If the external force is absent in eq. (7.1.15), the value of P will be considered to be undetermined as.

$$[K_A]\{d\} + P[K_G]\{d\} = 0 \tag{7.1.16}$$

The above equation can be solved as an eigenvalue problem to calculate buckling load *P*.

7.1.3 Buckling of Beam-Column Members

Let consider a pin ended column under the action of compressive force *P*. The elastic and geometric stiffness matrices can be developed from 1st principles for a beam-column element which can be used in the linear elastic stability analysis of frameworks. Considering small deflection approximation to the curvature, the total potential energy is computed from the following.

$$\Pi = \frac{EI}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \qquad = \int_0^L \left[\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 + \frac{P}{2}\left(\frac{dw}{dx}\right)^2\right] dx \tag{7.1.17}$$

Here, *w* is the transverse displacement and *I* is the moment of inertia of the member.

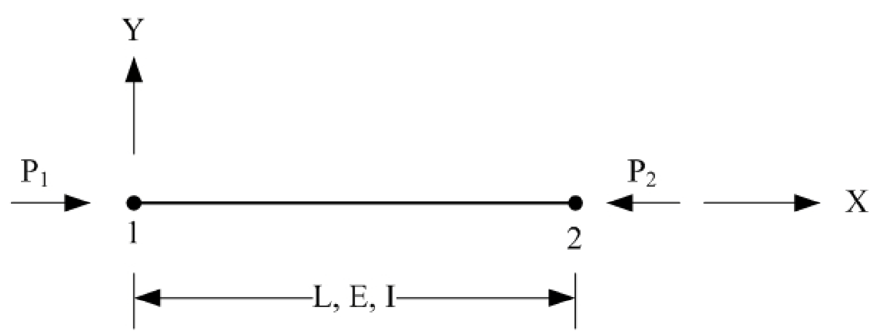

Fig. 7.1.2 Beam-column member

Considering nodal displacements at nodes 1 and 2 as w_1, θ_1 and w_2, θ_2, the displacement at any point inside the element can be represented in terms of its interpolation functions and nodal displacements.

$$w = [N_1 \quad N_2 \quad N_3 \quad N_4]\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = [N]\{d\} \tag{7.1.18}$$

Thus, the above energy equation can be rewritten as

$$\Pi = \frac{1}{2}\int_0^L \left[\{d\}^T [N'']^T EI[N'']\{d\} + \{d\}P[N']^T[N']\{d\}\right] dx \tag{7.1.19}$$

Where,

$$\frac{d^2w}{dx^2} = \frac{d^2}{dx^2}\left([N]\{d\}\right) = [N'']\{d\} \quad \text{and} \quad \frac{d}{dx} = \frac{wd}{dx}\left([N]\{d\}\right) = [N']\{d\} \qquad (7.1.20)$$

Applying the variational principle one can express

$$\{F\} = \frac{\partial \Pi}{\partial \{d\}} = \int_0^l \left(\left[N''\right]^T EI\left[N''\right] + P\left[N'\right]^T\left[N'\right]\right)dx\{d\} \qquad (7.1.21)$$

Thus, the stiffness matrix will be obtained as follows which have two terms.

$$[k] = \int_0^l \left(\left[N''\right]^T EI\left[N''\right]\right)dx + P\int_0^l \left(\left[N'\right]^T\left[N'\right]\right)dx = [k_F] + P[k_G] \qquad (7.1.22)$$

The first term resembles ordinary stiffness matrix for the bending of a beam. So this matrix is called flexural stiffness matrix. The second matrix is known as geometric stiffness matrix as it only depends on the geometrical parameters. Thus, the flexural stiffness matrix [k_F] and geometric stiffness matrix [k_G] can be derived from the following expressions.

$$[k_F] = \int_0^l \left(\left[N''\right]^T EI\left[N''\right]\right)dx \quad \text{and} \quad [k_G] = \int_0^l \left(\left[N'\right]^T\left[N'\right]\right)dx \qquad (7.1.23)$$

The above matrices can be derived from the assumed interpolation function. For example, the interpolation functions for a two node beam element are expressed by the following equation.

$$[N] = \left[\left(1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3}\right), \left(x - 2\frac{x^2}{L} + \frac{x^3}{L^2}\right), \left(3\frac{x^2}{L^2} - 2\frac{x^3}{L^3}\right), \left(-\frac{x^2}{L} + \frac{x^3}{L^2}\right)\right] \qquad (7.1.24)$$

Now, the first and second order derivative of the above function will become

$$[N'] = \left[\left(-6\frac{x}{L^2} + 6\frac{x^2}{L^3}\right), \left(1 - 4\frac{x}{L} + 3\frac{x^2}{L^2}\right), \left(6\frac{x}{L^2} - 6\frac{x^2}{L^3}\right), \left(-2\frac{x}{L} + 3\frac{x^2}{L^2}\right)\right] \qquad (7.1.25)$$

$$[N''] = \left[\left(-\frac{6}{L^2} + 12\frac{x}{L^3}\right), \left(-\frac{4}{L} + 6\frac{x}{L^2}\right), \left(\frac{6}{L^2} - 12\frac{x}{L^3}\right), \left(-\frac{2}{L} + 6\frac{x}{L^2}\right)\right] \qquad (7.1.26)$$

Using the above expressions in eq. (7.1.23), flexural and geometric stiffness matrices can be derived and obtained as follows.

$$[K_F] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad (7.1.27)$$

$$[K_G] = \frac{1}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ 36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \qquad (7.1.28)$$

Due to external forces {F}, one can find the displacement vectors from the following equation.

$$\{F\} = [k]\{d\} = [k_F]\{d\} + P[k_G]\{d\} \qquad (7.1.29)$$

The above is the beam-column equation in finite element form. In the absence of transverse load, the member will become column and P will be considered to be undetermined.

$$[k_F]\{d\} + P[k_G]\{d\} = 0 \qquad (7.1.30)$$

Denoting P as -λ we can reach to the familiar eigenvalue problem as given below.

$$[k_F]\{d\} = \lambda[k_G]\{d\} \tag{7.1.31}$$

Solving above equation, values of λ and associated nodal displacement vectors can be obtained. Mathematically this can be expressed as

$$([k_F] - \lambda[k_G])\{d\} = 0 \tag{7.1.32}$$

Thus,

$$\left|[k_F] - \lambda[k_G]\right| = 0 \tag{7.1.33}$$

For the matrix of size *n*, one can find *n*+1 degree polynomial in λ. The smallest root of the above equation will become the first approximate buckling load. From this value λ, one can find a set of ratios for the nodal displacement components. From this, the first buckling mode shape can be calculated. Higher mode approximations can also be found in a similar process. The procedure to determine the critical load by the above method is illustrated in the following example.

Example 7.1.1

Consider a column with one end clamped and other end free as shown in Fig. 7.1.2.

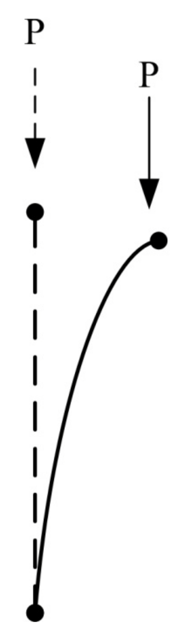

Fig. 7.1.3 Column with one end fixed and other end free

Now, the finite element equation of the column considering single element will be

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} - \lambda\frac{1}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ 36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} = 0$$

The boundary conditions for this member are given by

$$\text{At } x = 0,\ d_1 = 0 \text{ and } \theta_1 = \frac{dw}{dx} = 0$$

Thus, according to the above boundary conditions, the first and second rows as well as columns of the equation are deleted and rewritten in the following form.

$$\left|\frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix} - \frac{\lambda}{30L}\begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix}\right| = 0$$

Or

$$\left(\frac{4EI}{L} - \frac{2\lambda L}{15}\right)\left(\frac{12EI}{L^3} - \frac{6\lambda}{5L}\right) - \left(-\frac{6EI}{L^2} + \frac{\lambda}{10}\right)^2 = 0$$

Solving the above expression, the critical value of λ_{cr} and thus the critical value of force P_{cr} will become as

$$\lambda_{cr} = P_{cr} = 2.486\frac{EI}{L^2}$$

It is important to note that the exact value for such clamped-free column is

$$P_{cr} = \frac{\pi^2 EI}{L_e^{\,2}} = \frac{\pi^2 EI}{(2L)^2} = 2.467\frac{EI}{L^2}$$

The finite element result has slight deviation from the exact result. This difference can be minimized by the increase of number of elements in the column as we know, more we subdivide the continuum, better we can obtain the result close to the exact one.

7.1.4 Buckling of Plate Bending Elements

The elastic stability analysis of rectangular plates is discussed in this section. The total potential energy for plate are expressed as

$$\pi = \frac{D}{2}\int_{-a}^{a}\int_{-b}^{b}\left\{(\nabla^2 \mathrm{w})^2 + 2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\right]\right\}dxdy$$
$$-\frac{1}{2}\int_{-a}^{a}\int_{-b}^{b}\left\{F_x\left(\frac{\partial w}{\partial x}\right)^2 + 2F_{xy}\frac{\partial w}{\partial y}\frac{\partial w}{\partial x} + F_y\left(\frac{\partial w}{\partial y}\right)^2\right\}dxdy \qquad (7.1.34)$$

Here, F_x, F_y, and F_{xy}, are the in-plane edge load and compressive load is considered as positive. For, finite element formulation the deflection in above expression needs to convert in terms of nodal displacements in the element. Following the derivations in beam-column member (eqs. 7.1.18 to 7.1.22), the flexural and geometric stiffness for the plate element can be derived. Thus, the above expression can be derived to the following form using interpolation functions.

$$\pi = \frac{1}{2}\{d\}^T[k_F]\{d\} - \frac{F_x}{2}\{d\}^T[k_{Gx}]\{d\} - \frac{F_y}{2}\{d\}^T\left[kK_{Gy}\right]\{d\} - \frac{F_{xy}}{2}\{d\}^T\left[k_{Gxy}\right]\{d\} \qquad (7.1.35)$$

Here $[k_F]$ is the flexural stiffness matrix. The other stiffness matrices are analogous to the geometric stiffness matrices of the plate and can be expressed as

$$k_{Gx} = \iint \lfloor N'_x \rfloor \{N'_x\} dx\, dy$$

$$k_{Gy} = \iint \lfloor N'_y \rfloor \{N'_y\} dx\, dy \tag{7.1.36}$$

$$k_{Gxy} = \iint \lfloor N'_x \rfloor \{N'_y\} dx\, dy$$

Where $[N'_x]$ and $[N'_y]$ indicate partial derivative of $[N]$ with respect to x and y respectively. Thus, the equation of buckling becomes

$$[k_F]\{d\} - F_x[k_{Gx}]\{d\} - F_y[k_{Gy}]\{d\} - F_{xy}[k_{Gxy}]\{d\} = 0 \tag{7.1.37}$$

If the in-plane loads have a constant ratio to each other at all time during their buildup, the above equation can be expressed as follows

$$[k_F]\{d\} = P^*(\alpha[k_{Gx}] + \beta[k_{Gy}] + \gamma[k_{Gxy}])\{d\} \tag{7.1.38}$$

The term P^* is called the load factor, and $\alpha, \beta,$ and γ are constants relating the in-plane loads in the plate member. Solving the above expression, the buckling mode shapes are possible to determine.

Lecture 2: Finite Elements in Fluid Mechanics

7.2.1 Governing Fluid Equations

The fluid mechanics topic covers a wide range of problems of interest in engineering applications. Basically fluid is a material that conforms to the shape of its container. Thus, both the liquids and gases are considered as fluid. However, the physical behaviour of liquids and gases is very different. The differences in behaviour lead to a variety of subfields in fluid mechanics. In case of constant density of liquid, the flow is generally referred to as incompressible flow. The density of gases not constant and therefore, their flow is compressible flow. The Navier–Stokes equations are the fundamental basis of almost all fluid dynamics related problems. Any single-phase fluid flow can be defined by this expression. The general form of motion of a two dimensional viscous Newtonian fluid may be expressed as

$$\frac{1}{\rho}p_{,i} + \dot{v}_i + v_j v_{i,j} - \nu v_{i,ji} = f_i \qquad (7.2.1)$$

Here,

ν = kinematic viscosity

ρ = mass density of fluid

v_i = Velocity components

f_i = Body forces

p = fluid pressure

The suffix ,j and ,ji are the derivatives along j and j & i direction respectively. The dot represents the derivative with respect to time. Neglecting non linear convective terms, viscosity and body forces, eq. (7.2.1) can be simplified as:

$$p_{,i} + \rho\dot{v}_i = 0 \qquad (7.2.2)$$

Now, the continuity equation of the fluid is expressed by

$$\dot{p} + \rho c^2 v_{k,k} = 0 \qquad (7.2.3)$$

Here, *c* is the acoustic wave speed in fluid. In the above expression, two sets of variables, the velocity and the pressure are used to describe the behaviour of fluid. Now it is possible to combine equation (7.2.2) and (7.2.3) to obtain a single variable formulation. For the small amplitude of fluid motion, one can assume

$$v_i = \dot{u}_i \qquad (7.2.4)$$

Where u_i is the displacement component of fluid. To obtain single variable formulation, eq. (7.2.4) may be substituted into eq. (7.2.3) and one can get

$$\dot{p} + \rho c^2 \dot{u}_{k,k} = 0 \qquad (7.2.5)$$

Integrating eq. (5) w. r. t. time we have

$$p = -\rho c^2 u_{k,k} \tag{7.2.6}$$

Now differentiating the above expression w. r. t. x_i following expression will be arrived:

$$p_{,i} = -\rho c^2 u_{k,ki} \tag{7.2.7}$$

Substituting the above in to eq. (7.2.2) one can have

$$\rho \dot{v}_i - \rho c^2 u_{k,ki} = 0 \tag{7.2.8}$$

Thus, eq. (7.2.8) is expressed in terms of displacement variables only and known as displacement based equation.

Similarly, it is possible to obtain the fluid equation in terms of pressure variable only. Differentiating eq. (7.2.3) w. r. t. Time, the following expression can be obtained.

$$\ddot{p} + \rho c^2 \dot{v}_{k,k} = 0 \tag{7.2.9}$$

Again, differentiating eq. (7.2.2) w. r. t. x_i, we have

$$p_{,ii} + \rho \dot{v}_{i,i} = 0 \tag{7.2.10}$$

From eqs. (7.2.9) and (7.2.10), the pressure based single variable expression can be arrived as given below.

$$\ddot{p} - c^2 p_{,ii} = 0 \tag{7.2.11}$$

The above expression is basically the Helmholtz wave equation for a compressible fluid having acoustic speed *c*.

$$\nabla^2 p - \frac{1}{c^2}\ddot{p} = 0 \tag{7.2.12}$$

Thus, the general form of fluid equations of 2D linear steady state problems can be expressed by the Helmholtz equation. For incompressible fluid *c* becomes infinitely large. Hence for incompressible fluid, eq. (7.2.12) can be written as

$$\nabla^2 p = 0 \tag{7.2.13}$$

For the ideal, irrotational fluid flow problems, the field variables are the streamline, ψ and potential ϕ functions which are governed by Laplace's equations

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{7.2.14}$$

Derivation of the above expression and many other related equations can be found in details in fluid mechanics related text books.

7.2.2 Finite Element Formulation

The equation of motion of fluid can be expressed in various ways and some of those are shown in previous section. Finite element form of those expressions can be derived using various methods considering pressure, displacement, velocity, velocity potential, stream functions and their combinations. Here, displacement and pressure based formulations will be derived using finite element method.

7.2.2.1 Displacement based finite element formulation

Consider the equation (7.2.8) which can be expressed only in terms of displacement variables.

$$\rho \ddot{u}_i - \rho c^2 u_{k,ki} = 0 \qquad (7.2.15)$$

Here, u is the displacement vector. Now, the weak form of the above equation will become

$$\int_\Omega w_i \left(\rho \ddot{u}_i - \rho c^2 u_{k,ki} \right) d\Omega = 0 \qquad (7.2.16)$$

Performing integration by parts on the second terms, one can arrive at the following expression:

$$\int_\Omega w_i \;\rho \ddot{u}_i \; d\Omega - \int w_i \;\rho c^2 u_{k,k} d\Gamma + \int w_{i,i} \rho c^2 u_{k,k} d\Omega = 0$$

$$\text{or} \int_\Omega w_i \;\rho \ddot{u}_i \; d\Omega + \int_\Omega w_{i,i} \;\rho c^2 u_{k,ki} d\Omega = \int_\Gamma w_i \rho c^2 u_{k,k} d\Gamma \qquad (7.2.17)$$

Now from earlier relation (eq. 7.2.6) we have, $p = -\rho c^2 u_{k,k}$. Thus, the above equation may be written as:

$$\int_\Omega w_i \;\rho \ddot{u}_i \; d\Omega + \int_\Omega w_{i,i} \;\rho c^2 u_{k,ki} d\Omega = -\int_\Gamma w_i p d\Gamma \qquad (7.2.18)$$

In case of fluid filled rigid tank, the weighting function w_i must satisfy the condition $w_i n_i = 0$ on its rigid boundary. Therefore, the above eq. will become

$$\int_\Omega \left(w_i \rho \ddot{u}_i + w_{i,i} \rho c^2 u_{k,k} \right) d\Omega = -\int_{\Gamma_p} w_i \; n_i \; p\Gamma \qquad (7.2.19)$$

For finite element implementation of the above expression, let consider the interpolation function as *N* and $\bar{u}$ as the nodal displacement vector. Thus,

$$u = N\bar{u} \text{ and } w = N\bar{w} \quad (7.2.20)$$

Now the divergence of the displacement vector can be expressed as:

$$u_{i,i} = Lu = LN\bar{u} = B\bar{u} \quad (7.2.21)$$

Where $L = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$ is the differential operator. Thus eq. (7.2.19) may be written as

$$w^T \int_{\Omega} \left[N^T \rho N \ddot{\bar{u}} + B^T \rho c^2 B \bar{u} \right] d\Omega = -w^T \int_{\Gamma_p} N^T n \bar{p} \; d\Gamma \quad (7.2.22)$$

Or,

$$[M]\{\ddot{\bar{u}}\} + [K]\{\bar{u}\} = \{F\} \quad (7.2.23)$$

Where,

$$[M] = \int_{\Omega} \rho [N]^T [N] d\Omega$$

$$[K] = \int_{\Omega} \rho c^2 [B]^T [B] d\Omega \quad (7.2.24)$$

$$\{F\} = -\int_{\Gamma_p} [N]^T \, n \; \{\bar{p}\} \; d\Gamma$$

Using eq. (7.2.23), the displacements in fluid domain can be determined under external forces applying proper boundary conditions.

7.2.2.2 Pressure based finite element formulation

The Helmholtz equation (7.2.12) for a compressible fluid in two dimension can be used to determine the pressure distribution in the fluid domain using finite element technique.

$$p,ii - \frac{1}{c^2}\ddot{p} = 0 \quad (7.2.25)$$

The weak form of the above expression can be written as

$$\int_{\Omega} w_i \left(p,ii - \frac{1}{c^2}\ddot{p} \right) d\Omega = 0 \quad (7.2.26)$$

Now, performing integration by parts on the first term, the following expression can be obtained.

$$\int_{\Gamma} w_i \;\; p,i \; d\Gamma - \int_{\Omega} w_{i,i} \;\; p,i \; d\Omega - \int_{\Omega} \frac{1}{c^2} w_i \;\; \ddot{p} d\Omega = 0 \quad (7.2.27)$$

Thus,

$$\frac{1}{c^2} \int w_i \ddot{p} \; d\Omega + \int_{\Omega} w_{i,i} p_{,i} d\Omega = \int_{\Gamma} w_i \;\; p_{,i} d \; \Gamma \quad (7.2.28)$$

Assuming interpolation function as *N* and $\overline{p}$ as the nodal pressure vector, the pressure (*p*) at any point can be written as: $p = N\overline{p}$ and similarly, $w = N\overline{w}$. The divergence of the pressure can be expressed as: $p,i = Lp = LN\overline{p} = B\overline{p}$, where, $L = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$. Again,

$$\begin{aligned} w_{i,i} &= Lw = LN\overline{w} = B\overline{w} \\ w_i\ddot{p} &= \left[N\overline{w}\right]^T\left[N\ddot{\overline{p}}\right] = \overline{w}^T N^T N\ddot{\overline{p}} \end{aligned} \tag{7.2.29}$$

Thus, eq. (7.2.28) will become:

$$\frac{1}{c^2}\int_\Omega \overline{w}^T N^T N\ddot{\overline{p}}d\Omega + \int_\Omega \overline{w}^T B^T B\overline{p}d\Omega = \int_\Gamma w^T N^T \frac{\partial p}{\partial n}d\Gamma \tag{7.2.30}$$

Or,

$$[E]\{\ddot{\overline{p}}\} + [G]\{\overline{p}\} = \{B\} \tag{7.2.31}$$

Where,

$$\begin{aligned} [E] &= \frac{1}{c^2}\int_\Omega [N]^T[N]d\Omega \\ [G] &= \int_\Omega [B]^T[B]d\Omega \\ \{B\} &= \int_\Gamma [N]^T\frac{\partial p}{\partial n}d\Gamma \end{aligned} \tag{7.2.32}$$

Applying boundary conditions, eq. (7.2.31) can be solved to calculate the dynamic pressure developed in the fluid under applied accelerations on the domain.

7.2.3 Finite Element Formulation of Infinite Reservoir

Let consider an infinite reservoir adjacent to a dam like structure. In such case, if the dam is vibrated, the hydrodynamic pressure will be developed in the reservoir which can be calculated using above method. For finite element analysis, it is necessary to truncate such infinite domain at a certain distance away from structure to have a manageable computational domain. The reservoir has four sides (Fig. 7.2.1) and as a result four types of boundary conditions need to be specified.

$$\{B\} = \{B\}_1 + \{B\}_2 + \{B\}_3 + \{B\}_4 \tag{7.2.33}$$

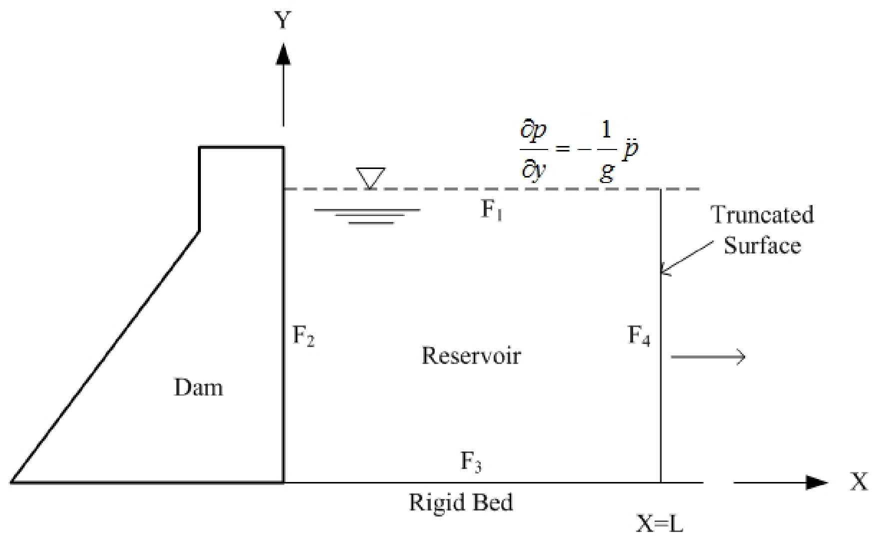

Fig. 7.2.1 Reservoir and its boundary conditions

(i) At the free surface (*Γ*$_1$)

Neglecting the effects of surface waves of the water, the boundary condition of the free surface may be expressed as

$$p(x, H, t) = 0 \qquad (7.2.34)$$

Here, *H* is the depth of the reservoir. However, sometimes, the effect of surface waves of the water needs to be considered at the free surface. This can be approximated by assuming the actual surface to be at an elevation relative to the mean surface and the following linearised free surface condition may be adopted.

$$\frac{1}{g}\ddot{p} + \frac{\partial p}{\partial y} = 0 \qquad (7.2.35)$$

Thus, the above expression may be written in finite element form as

$$\{B_1\} = \frac{\partial p}{\partial n} = -\frac{1}{g}[R_1]\{\ddot{p}\} \qquad (7.2.36)$$

In which,

$$[R_1] = \int_{\Gamma_1} [N]^T [N] d\Gamma \qquad (7.2.37)$$

(ii) At the dam-reservoir interface (*Γ*$_2$)

At the dam-reservoir interface, the pressure should satisfy

$$\frac{\partial p}{\partial n}(0, y, t) = \rho a e^{i\omega t} \qquad (7.2.38)$$

where $ae^{i\omega t}$ is the horizontal component of the ground acceleration in which, ω is the circular frequency of vibration and $i=\sqrt{-1}$, n is the outwardly directed normal to the elemental surface along the interface. In case of vertical dam-reservoir interface $\partial p/\partial n$ may be written as $\partial p/\partial x$ as both will represent normal to the element surface. For an inclined dam-reservoir interface $\partial p/\partial x$ cannot represent the normal to the element surface. Therefore, to generalize the expressions $\partial p/\partial n$ is used in eq. (7.2.38). If $\{a\}$ is the vector of nodal accelerations of generalized coordinates, $\{B_2\}$ may be expressed as

$$\{\mathrm{B}_2\}=-\rho[\mathrm{R}_2]\{a\} \tag{7.2.39}$$

where,

$$[\mathrm{R}_2]=\sum\int_{\Gamma_2}[\mathrm{N}_\mathrm{r}]^\mathrm{T}[\mathrm{T}][\mathrm{N}_\mathrm{d}]d\Gamma \tag{7.2.40}$$

Here, $[T]$ is the transformation matrix for generalized accelerations of a point on the dam reservoir interface and $[N_d]$ is the matrix of shape functions of the dam used to interpolate the generalized acceleration at any point on their interface in terms of generalized nodal accelerations of an element.

(iii) At the reservoir bed interface (Γ_3)

At the interface between the reservoir and the elastic foundation below the reservoir, the accelerations should not be specified as rigid foundation because they depend on the interaction between the reservoir and the foundation. However, for the sake of simplicity, the reservoir bed can be assumed as rigid and following boundary condition may be adopted.

$$\{\mathrm{B}_3\}=\frac{\partial \mathrm{p}}{\partial \eta}(\mathrm{x},0,\mathrm{t})=0 \tag{7.2.41}$$

(iv) At the truncation boundary (Γ_4)

The specification of the far boundary condition is one of the most important features in the finite element analysis of a semi-infinite or infinite reservoir. This is due to the fact that the developed hydrodynamic pressure, which affects the response of the structure, is dependent on the truncation boundary condition. The infinite fluid domain may truncated at a finite distance away from the structure for finite element analysis satisfying Sommerfeld radiation boundary condition. Application of Sommerfeld radiation condition at the truncation boundary leads to

$$\{\mathrm{B}_4\}=\frac{\partial \mathrm{p}}{\partial \mathrm{x}}(\mathrm{L},\mathrm{y},\mathrm{t})=0 \tag{7.2.42}$$

Here, L represents the distance between the structure and the truncation boundary. Thus, the hydrodynamic pressure developed on the dam-reservoir interface can be calculated under external excitation by the use of finite element technique.

Lecture 3: Dynamic Analysis

The finite element method is a powerful device for analyzing the dynamic response of structures as in case of static analyses. The responses such as displacements, velocities, strains, stresses become time dependent in dynamic analysis. The dynamic equation of motion of a structure can be derived from Hamilton's principle using Lagrange equation.

7.3.1 Hamilton's Principle

Hamilton's principle is a simple but powerful tool to derive discritized dynamic system equations. It can be stated as "Of all the possible time histories of displacement the most accurate solution makes the Lagrangian functional a minimum provided the following conditions are satisfied."

- the essential or the kinematic boundary conditions,
- the compatibility equations, and
- the conditions at initial (t_1) and final time (t_2).

First condition of the above ensures that the displacement constraints are fulfilled. Second condition makes sure that the displacements are continuous in the problem domain and the third condition necessitates the displacement history to satisfy the constraints at the initial and final times. Hamilton's principle allows one to assume any set of displacements, as long as it satisfies the above three conditions. It is basically a variational formulation method. Just like in the other variational formulations, a functional is used here. Mathematically, Hamilton's principle states:

$$\delta\int_{t_1}^{t_2} L\mathrm{dt} = 0 \tag{7.3.1}$$

The Langrangian functional, *L*, is defined as follows using a set of permissible time histories of displacement.

$$\mathrm{L} = \mathrm{T} - \Pi \tag{7.3.2}$$

Where T is the kinetic energy and Π is the potential energy which is basically elastic strain energy. The kinetic energy, T is defined in the integral form as

$$\mathrm{T} = \frac{1}{2}\int_{\Omega} \dot{\mathbf{x}}^{\mathrm{T}}\rho\dot{\mathbf{x}}\,\mathrm{d}\Omega \tag{7.3.3}$$

Here Ω represents the whole volume of the domain, ρ is the mass density of structure and $\dot{\mathbf{x}}$ is the set of time histories of velocities. The strain energy in the entire domain of elastic structure can be expressed as

$$\Pi = \frac{1}{2}\int_{\Omega} \varepsilon^{\mathrm{T}}\boldsymbol{\sigma}\,\mathrm{d}\Omega = \frac{1}{2}\int_{\Omega} \varepsilon^{\mathrm{T}}\mathbf{D}\varepsilon\,\mathrm{d}\Omega \tag{7.3.4}$$

Here $\boldsymbol{\varepsilon}$ are the strains obtained from the set of time histories of displacements. Here, L can be expressed in terms of the generalized variables $\mathrm{x}_1, \mathrm{x}_2 \cdots\cdots \mathrm{x}_n$ and $\dot{\mathrm{x}}_1, \dot{\mathrm{x}}_2 \cdots\cdots \dot{\mathrm{x}}_n$. Here, x_i is the displacement and the velocity is expressed by

$$\dot{x}_i = \frac{d}{dt}(x_i) \tag{7.3.5}$$

Considering $\mathbf{x_i}$ as generalised displacement, the Lagrange's equation of motion are expressed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad \text{where } i = 1 \text{ to } n \tag{7.3.6}$$

7.3.2 Finite Element Form in MDOF System

Consider a two-degree of freedom system as shown in the figure below.

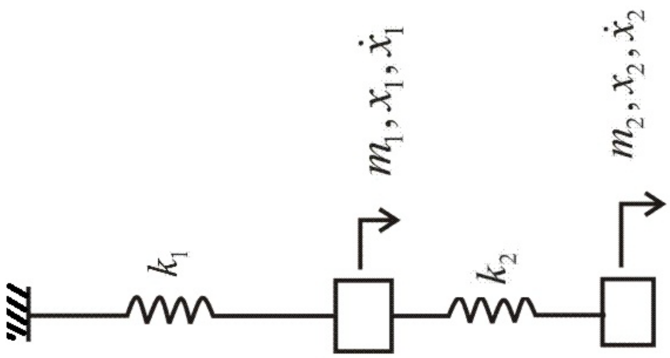

Fig. 7.3.1 Two Degree Freedom System

The kinetic and potential energy can be expressed by following form:

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \\ \Pi &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(x_2 - x_1\right)^2 \end{aligned} \tag{7.3.7}$$

Now, substituting the values in Lagrangean L, we can obtain

$$L = T - \Pi = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 + k_2x_1x_2 - \frac{1}{2}k_2x_1^2 \tag{7.3.8}$$

Thus,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} &= m_1\ddot{x}_1 + k_1x_1 - k_2x_2 + k_2x_1 \\ &= m_1\ddot{x}_1 + k_1x_1 - k_2\left(x_2 - x_1\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} &= m_2\ddot{x}_2 + k_2x_2 - k_2x_1 \\ &= m_2\ddot{x}_2 + k_2\left(x_2 - x_1\right) = 0 \end{aligned} \tag{7.3.9}$$

Above equations can be written in the matrix form as shown below:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{7.3.10}$$

Thus, for a multi degree freedom system, the above expression can be written as

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\} \tag{7.3.11}$$

For a steady state condition, one can consider $x = a \sin \omega t$. Thus, $\ddot{x} = -\omega^2 a \sin \omega t = -\omega^2 x$. As a result, the above equation can be re-written as

$$[K]\{x\} - \omega^2 [M]\{x\} = \{0\}$$

$$\left[[K] - \omega^2 [M]\right]\{x\} = \{0\}$$

$$\left[\omega^2\right] = [M]^{-1}[K] \tag{7.3.12}$$

Thus, for a single degree of freedom system, the fundamental frequency can be found as

$$\omega = \sqrt{\frac{K}{M}} \tag{7.3.13}$$

7.3.3 Solid Body with Distributed Mass

The velocity vector $\dot{\mathbf{x}}$ can be expressed in terms of the elemental nodal velocities with the help of interpolation functions as $\dot{\mathbf{x}} = N\dot{x}$. Here, $\dot{x}$ is the nodal velocity, and N is the interpolation function. Thus the expression of kinetic energy can be re-written as

$$T = \frac{1}{2}\int_{\Omega} \dot{\mathbf{x}}^T \rho \dot{\mathbf{x}} \, d\Omega = \frac{1}{2}\dot{x}^T \int_{\Omega} N^T \rho N \, d\Omega \, \dot{x} \tag{7.3.14}$$

In the above equations, the mass matrix can be obtained as

$$M = \frac{1}{2}\int_{\Omega} N^T \rho N \, d\Omega \tag{7.3.15}$$

Volume integral has to be taken over the volume of the element to find the mass matrix of that element.

7.3.3.1 Mass matrix for truss element

In case of truss member, the degrees of freedom at each node is one because of its only axial deformation. Thus for a two node truss element, the interpolation function will become

$$[N] = [1-\xi \quad \xi]$$

Here, $\xi = x/L$ and thus, $dx = Ld\xi$. Hence, the mass matrix can be expressed as

$$[M]=\rho\int_{\Omega} N^T N d\Omega=\rho A\int_0^{\ell} N^T N dx=\rho AL\int_0^{l} N^T N d\xi$$

$$=\rho AL\int_0^{l}\begin{bmatrix}(1-\xi)\\ \xi\end{bmatrix}\begin{bmatrix}(1-\xi) & \xi\end{bmatrix}d\xi=\rho AL\int_0^{l}\begin{bmatrix}(1-\xi)^2 & \xi(1-\xi)\\ \xi(1-\xi) & \xi^2\end{bmatrix}d\xi \tag{7.3.16}$$

$$=\rho AL\begin{bmatrix}\frac{-(1-\xi)^3}{3} & \frac{\xi^2}{2}-\frac{\xi^3}{3}\\ \frac{\xi^2}{2}-\frac{\xi^3}{3} & \frac{\xi^3}{3}\end{bmatrix}_0^1=\rho AL\begin{bmatrix}\frac{1}{3} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{3}\end{bmatrix}=\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Thus, the consistent mass matrix will be

$$[M]=\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} \tag{7.3.17}$$

Whereas the lumped mass matrix can be expressed as follows.

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{7.3.18}$$

Now, to obtain a generalised mass matrix of a two dimensional truss member, consider Fig. 7.3.2.

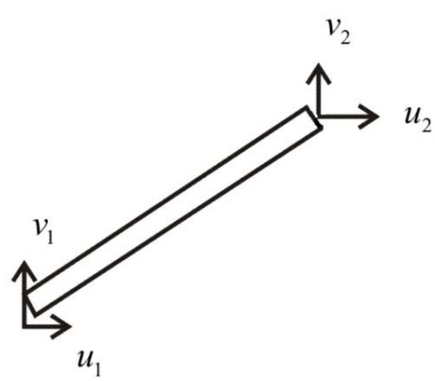

Fig. 7.3.2 Two Degree Freedom System

The displacement vector and the shape function for such case are expressed as

$$\{d\}=\begin{Bmatrix}u_1\\ v_1\\ u_2\\ v_2\end{Bmatrix} \quad \text{and} \quad [N]=\begin{bmatrix}N_1 & 0 & N_2 & 0\\ 0 & N_1 & 0 & N_2\end{bmatrix} \tag{7.3.19}$$

Thus, the mass matrix can be derived as given below.

$$[\mathrm{M}] = \rho\int_{\Omega} \mathrm{N}^{\mathrm{T}}\mathrm{N}d\Omega = \rho\mathrm{AL}\int_0^l \begin{bmatrix} 1-\xi & 0 \\ 0 & 1-\xi \\ \xi & 0 \\ 0 & \xi \end{bmatrix} \begin{bmatrix} 1-\xi & 0 & \xi & 0 \\ 0 & 1-\xi & 0 & \xi \end{bmatrix} d\xi \qquad (7.3.20)$$

After performing integration, the mass matrix will be obtained as

$$[\mathrm{M}] = \frac{\rho\mathrm{AL}}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad (7.3.21)$$

The lumped mass matrix of two dimensional truss member will become

$$[\mathrm{M}] = \frac{\rho\mathrm{AL}}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7.3.22)$$

7.3.3.2 Mass matrix for beam element

The beam element has two degrees of freedom at each node, one transverse displacement (v) and the other one is an angle of rotation (θ). So, a beam with two nodes has a total of four degrees of freedom. From earlier notes in module 4, lecture 3, the interpolation function of a beam is obtained as

$$\begin{aligned} &\mathrm{N}_1 = 1 - \frac{3}{\mathrm{L}^2}\mathrm{x}^2 + \frac{2}{\mathrm{L}^3}\mathrm{x}^3, \qquad \mathrm{N}_2 = \mathrm{x} - \frac{2}{\mathrm{L}}\mathrm{x}^2 + \frac{\mathrm{x}^3}{\mathrm{L}^2}, \\ &\mathrm{N}_3 = \frac{3\mathrm{x}^2}{\mathrm{L}^2} - \frac{2\mathrm{x}^3}{\mathrm{L}^3} \text{ and } \qquad \mathrm{N}_4 = -\frac{\mathrm{x}^2}{\mathrm{L}} + \frac{\mathrm{x}^3}{\mathrm{L}^2} \end{aligned} \qquad (7.3.23)$$

The consistent mass matrix can be following earlier process.

$$[\mathrm{M}] = \rho\int_{\Omega} \mathrm{N}^{\mathrm{T}}\mathrm{N}d\Omega = \rho\mathrm{A}\int_0^l [\mathrm{N}]^{\mathrm{T}}[\mathrm{N}]dx$$

Substitution for the interpolation functions and performing the required integrations, one can arrive the following mass matrix for the beam element.

$$[\mathrm{M}] = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \qquad (7.3.24)$$

Similarly, the lumped mass matrix of the beam element is

$$[M] = \frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} \qquad (7.3.25)$$

7.3.3.3 Mass matrix for plane frame element

The plane frame element has three degrees of freedom at each node: (i) axial displacement (u), (ii) transverse displacement (v) and (iii) angle of rotation (θ). So it has a total of six degrees of freedom for a two node plane frame element and its shape functions are given by

$$N_1 = 1 - \frac{x}{L}, \quad N_2 = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3, \quad N_3 = x - \frac{2}{L}x^2 + \frac{x^3}{L^2},$$
$$N_4 = \frac{x}{L}, \quad N_5 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \text{ and } \quad N_6 = -\frac{x^2}{L} + \frac{x^3}{L^2} \qquad (7.3.26)$$

Similar to earlier case, one can derive the consistent mass matrix of a two node plane frame member which will become as given below.

$$[M] = \rho AL\begin{bmatrix}1/3 & 0 & 0 & 1/6 & 0 & 0\\ & 13/35 & 11L/210 & 0 & 9/70 & -13L/420\\ & & L^2/105 & 0 & 13L/420 & -L^2/140\\ & & & 1/3 & 0 & 0\\ Sym. & & & & 13/35 & -11L/210\\ & & & & & L^2/105\end{bmatrix} \qquad (7.3.27)$$

The lumped mass matrix of such two dimensional frame element will be

$$[M] = \frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \qquad (7.3.28)$$

7.3.3.4 Mass matrix for space frame element

In case of space frame, each node has six degrees of freedom: (i) three displacements and (ii) three rotations. Therefore, for a two node space frame element there will be twelve degrees of freedom and the consistent mass matrix of the element in the local *xyz* system can be derived as

$$[M]=\rho AL\begin{bmatrix}\frac{1}{3} & & & & & & & & & & & \\ 0 & \frac{13}{35} & & & & & & & & & & \\ 0 & 0 & \frac{13}{35} & & & & & & & & & \\ 0 & 0 & 0 & \frac{J}{3A} & & & & & & & & \\ 0 & 0 & -\frac{11}{210}L & 0 & \frac{L^2}{105} & & & & & & & \\ 0 & \frac{11}{210}L & 0 & 0 & 0 & \frac{L^2}{105} & & & & & & \\ \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & & & & & \\ 0 & \frac{9}{70} & 0 & 0 & 0 & \frac{13}{420}L & 0 & \frac{13}{35} & & & & \\ 0 & 0 & \frac{9}{70} & 0 & -\frac{13}{420}L & 0 & 0 & 0 & \frac{13}{35} & & & \\ 0 & 0 & 0 & \frac{J}{6A} & 0 & 0 & 0 & 0 & 0 & \frac{J}{3A} & & \\ 0 & 0 & \frac{13}{420}L & 0 & -\frac{L^2}{140} & 0 & 0 & 0 & \frac{11}{210}L & 0 & \frac{L^2}{105} & \\ 0 & -\frac{13}{420}L & 0 & 0 & 0 & -\frac{L^2}{140} & 0 & -\frac{11}{210}L & 0 & 0 & 0 & \frac{L^2}{105}\end{bmatrix}$$

(7.3.29)

In a similar process, one can derive the mass matrix of other types of structures such as two dimensional plane stress/strain element, three dimensional solid element, plate bending element, shell element etc.

7.3.4 Time History Analysis

The dynamic equation of motion of a multi-degree freedom system can be written as

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\} \tag{7.3.30}$$

Where, [C] is the damping matrix and {F(t)} is the time dependent force vector. The element matrices of the above equation can be expressed as follows.

$$[\mathrm{M}]=\frac{1}{2}\int_{\Omega}[\mathrm{N}]^{\mathrm{T}}\rho[\mathrm{N}]\,d\Omega$$
$$[\mathrm{C}]=\frac{1}{2}\int_{\Omega}[\mathrm{N}]^{\mathrm{T}}\mu[\mathrm{N}]\,d\Omega \qquad (7.3.31)$$
$$[\mathrm{K}]=\frac{1}{2}\int_{\Omega}[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}]\mathrm{B}\,d\Omega$$

In a linear dynamic system, these values remain constant throughout the time history analysis. Damping is a mechanism which dissipates energy, causing the amplitude of free vibration to decay with time. The various types of damping which influences structural dynamical behaviour are viscous damping, hysteresis damping, radiation damping etc. Viscous damping exerts force proportional to velocity, as exhibited by the term. A formulation for this kind of problem was developed by Rayleigh. In this case, the energy dissipated is proportional to frequency and to the square of amplitude. Viscous damping is provided by surrounding gas or liquid or by the viscous damper attached to the structure. Radiation damping refers to energy dissipation to a practically unbounded medium, such as soil that supports structure. Among the various types of physical damping, viscous damping is easy represent computationally in dynamic equations. Fortunately, damping in structural problems is usually small enough regardless of its actual source. Its effect on structural response is modelled well by regarding it as viscous. The Rayleigh damping defines the global damping matrix $[C]$ as a linear combination of the global mass and stiffness matrices.

$$[C]=\alpha'[M]+\beta'[K] \qquad (7.3.32)$$

Here, α' and β' are the stiffness and mass proportional damping constants respectively. The relationship between the fraction of critical damping ratio, ξ, α and β at frequency ω is given by

$$\xi=\frac{1}{2}\left(\alpha\omega+\frac{\beta}{\omega}\right) \qquad (7.3.33)$$

Damping constants α' and β' are determined by choosing the fractions of critical damping (ξ_1' and ξ_2') at two different frequencies (ω_1 and ω_2) and solving simultaneously as follows

$$\alpha'=\frac{2\omega_1\omega_2\left(\xi_2'\omega_2-\xi_1'\omega_1\right)}{\left(\omega_2^2-\omega_1^2\right)}$$
$$\beta'=\frac{2\left(\xi_1'\omega_2-\xi_2'\omega_1\right)}{\left(\omega_2^2-\omega_1^2\right)} \qquad (7.3.34)$$

There are several numerical integration schemes are available to obtain the time history response of the structural system. Amongst, the implicit type of methods such as the Newmark's method is one of the most popular one.