Lecture 1: Introduction to Plate Bending Problems

6.1.1 Introduction

A plate is a planer structure with a very small thickness in comparison to the planer dimensions. The forces applied on a plate are perpendicular to the plane of the plate. Therefore, plate resists the applied load by means of bending in two directions and twisting moment. A plate theory takes advantage of this disparity in length scale to reduce the full three-dimensional solid mechanics problem to a two-dimensional problem. The aim of plate theory is to calculate the deformation and stresses in a plate subjected to loads. A flat plate, like a straight beam carries lateral load by bending. The analyses of plates are categorized into two types based on thickness to breadth ratio: thick plate and thin plate analysis. If the thickness to width ratio of the plate is less than 0.1 and the maximum deflection is less than one tenth of thickness, then the plate is classified as thin plate. The well known as Kirchhoff plate theory is used for the analysis of such thin plates. On the other hand, Mindlin plate theory is used for thick plate where the effect of shear deformation is included.

6.1.2 Notations and Sign Conventions

Let consider plates to be placed in XY plane. Representation of plate surface slopes $W_{,x}$, $W_{,y}$ by right hand rule produces arrows that point in negative Y and positive X directions respectively. Both surface slopes and rotations are required for plate elements. Signs and subscripts of rotations and slopes are reconciled by replacing $\boldsymbol{\theta_x}$ by $\boldsymbol{\Psi_y}$ and $\boldsymbol{\theta_y}$ by $\boldsymbol{-\Psi_x}$

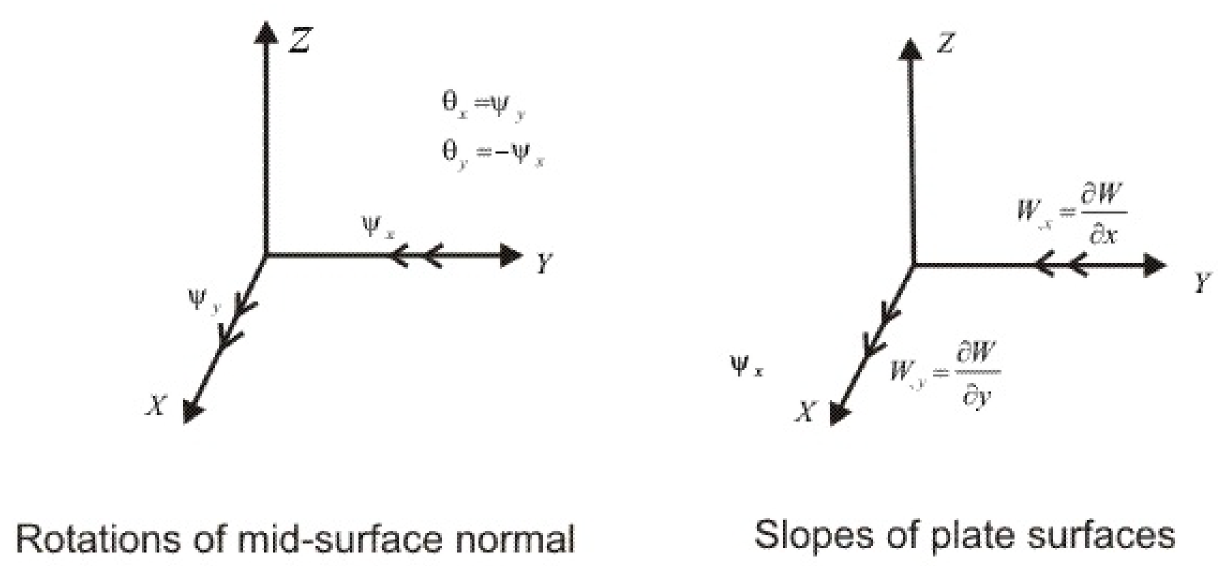


Fig. 6.1.1 Notations and sign conventions

6.1.3 Thin Plate Theory

Classical thin plate theory is based upon assumptions initiated for beams by Bernoulli but first applied to plates and shells by Love and Kirchhoff. This theory is known as Kirchhoff's plate theory. Basically, three assumptions are used to reduce the equations of three dimensional theory of elasticity to two dimensions.

1. The line normal to the neutral axis before bending remains straight after bending.
2. The normal stress in thickness direction is neglected. i.e., $\sigma_z = 0$. This assumption converts the 3D problem into a 2D problem.
3. The transverse shearing strains are assumed to be zero. i.e., shear strains γ_{xz} and γ_{yz} will be zero. Thus, thickness of the plate does not change during bending.

The above assumptions are graphically shown in Fig. 6.1.2.

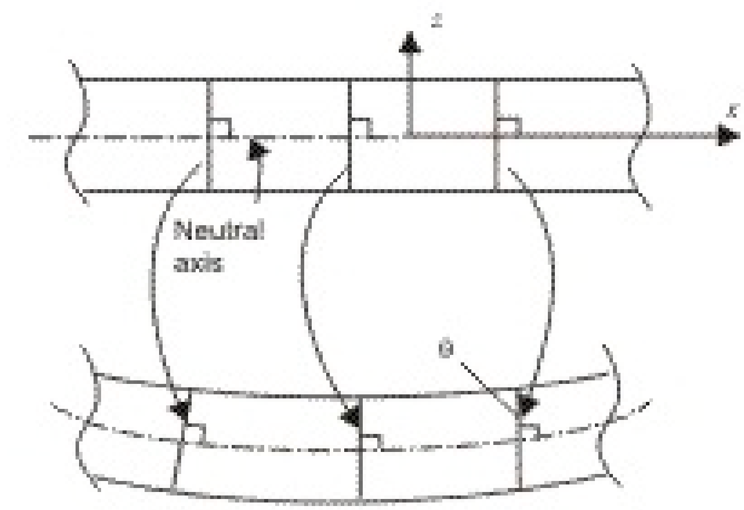


Fig. 6.1.2 Kirchhoff plate after bending

6.1.3.1 Basic relationships

Let, a plate of thickness t has mid-surface at a distance $\frac{t}{2}$ from each lateral surface. For the analysis purpose, X-Y plane is located in the plate mid-surface, therefore z=0 identifies the mid-surface. Let **u, v, w** be the displacements at any point (x, y, z).

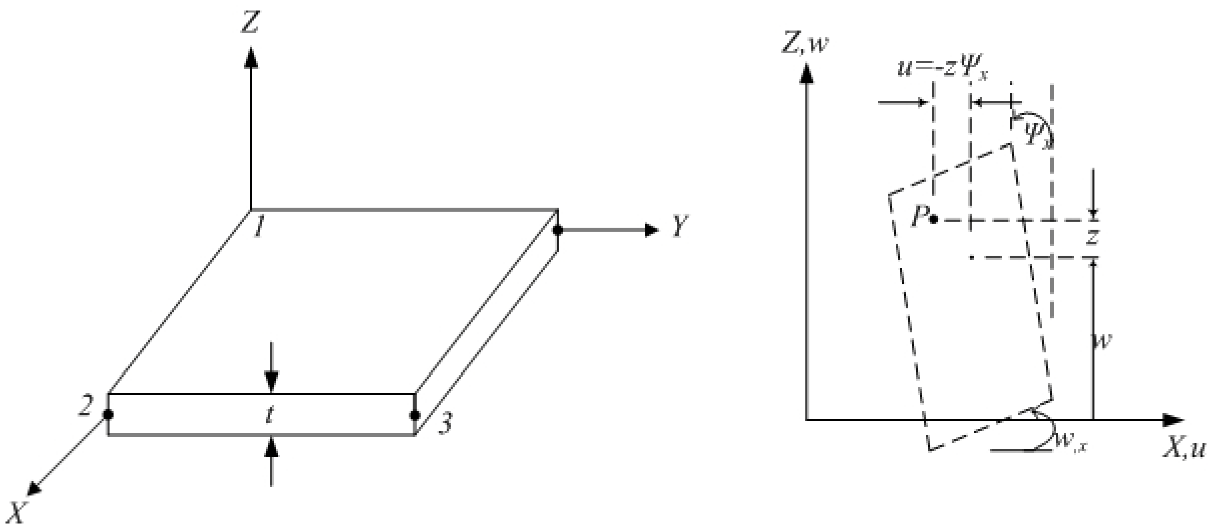


Fig. 6.1.3 Thin plate element

Then the variation of *u* and *v* across the thickness can be expressed in terms of displacement *w* as

$$u = -z\frac{\partial w}{\partial x} \qquad v = -z\frac{\partial w}{\partial y} \tag{6.1.1}$$

Where, *w* is the deflection of the middle plane of the plate in the *z* direction. Further the relationship between, the strain and deflection is given by,

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} = z\,\chi_x \\ \varepsilon_y &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} = z\,\chi_y \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y} = z\,\chi_{xy} \end{aligned} \tag{6.1.2}$$

where,

- **ε** corresponds to direct strain
- **γ** corresponds to shear strain
- **χ** corresponds to curvature along respective directions.

Or in matrix form, the above expression can written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = -z\begin{bmatrix} \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial y^2} \\ \frac{\partial^2}{\partial x \partial y} \end{bmatrix} w \tag{6.1.3}$$

Or,

$$\varepsilon = -z\Delta w \tag{6.1.4}$$

Where, ε is the vector of in-plane strains, and Δ is the differential operator matrix.

6.1.3.2 Constitutive equations

From Hooke's law,

$$\sigma = [D]\varepsilon \tag{6.1.5}$$

Where,

$$[D] = \frac{E}{(1-\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \tag{6.1.6}$$

Here, [D] is equal to the value defined for 2D solids in plane stress condition (i.e., $\sigma_z = 0$).

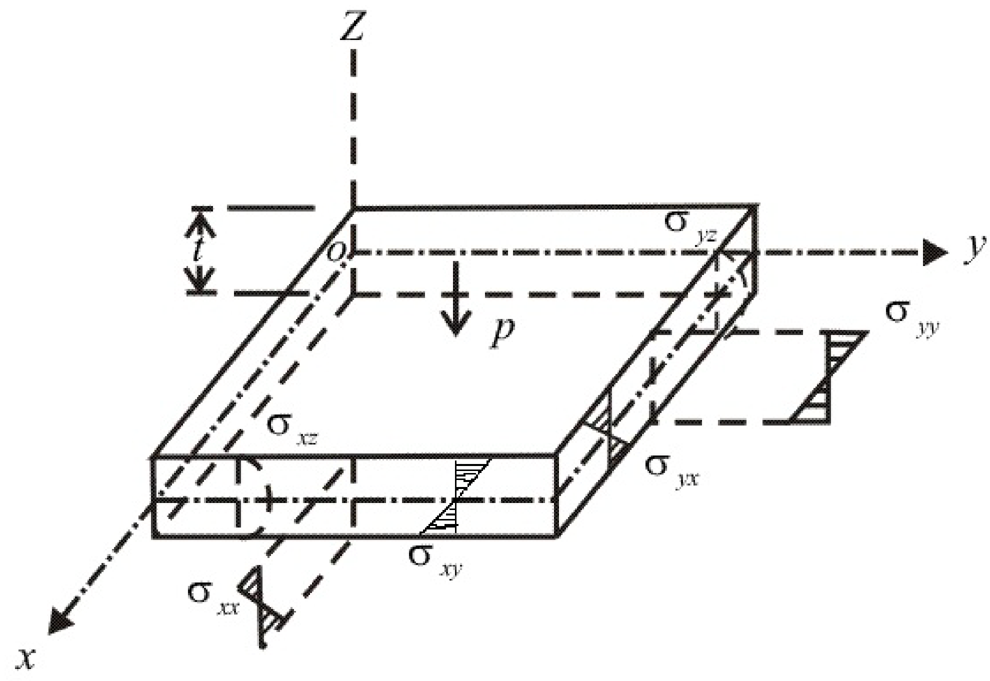
6.1.3.3 Calculation of moments and shear forces

Let consider a plate element of $dx \times dy$ and with thickness *t*. The plate is subjected to external uniformly distributed load *p*. For a thin plate, body force of the plate can be converted to an equivalent load and therefore, consideration of separate body force is not necessary. By putting eq. (6.1.4) in eq. (6.1.5),

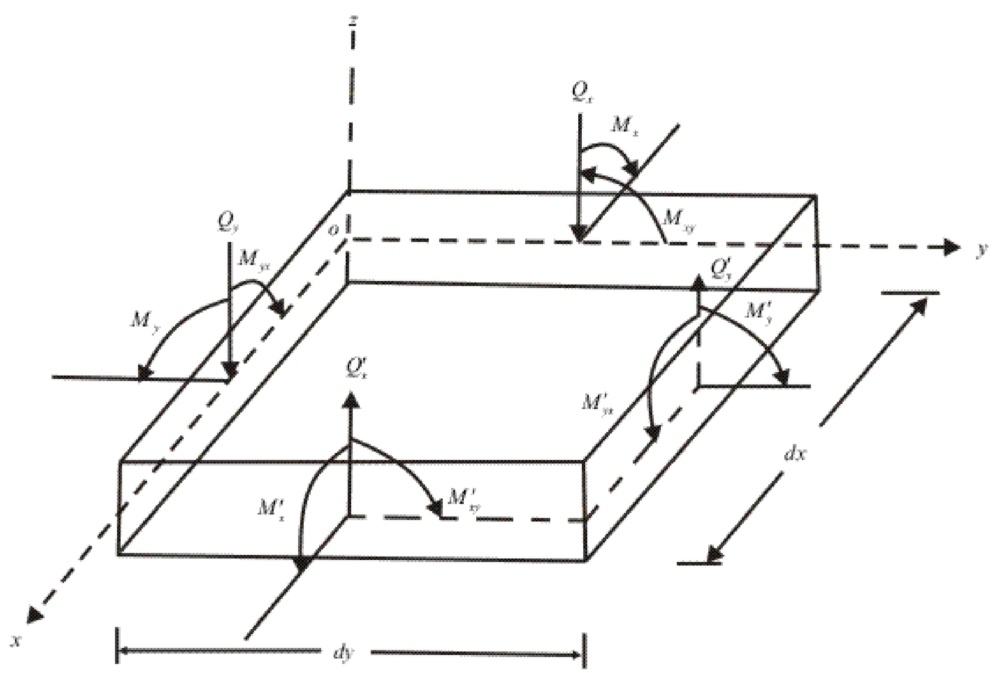
$$\sigma = -z[D]\Delta w \tag{6.1.7}$$

It is observed from the above relation that the normal stresses are varying linearly along thickness of the plate (Fig. 6.1.4(a)). Hence the moments (Fig. 6.1.4(b)) on the cross section can be calculated by integration.

$$M = \begin{Bmatrix} M_X \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \sigma z dt = -\left(\int_{-t/2}^{t/2} z^2 dt \right) [D]\Delta w = -\frac{t^3}{12}[D]\Delta w \tag{6.1.8}$$



(a) Stresses in plate



(b) Forces and moments in plate

Fig. 6.1.4 Forces on thin plate

On expansion of eq. (6.1.8) one can find the following expressions.

$$M_X = -\frac{Et^3}{12\left(1-\upsilon^2\right)}\left(\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}\right) = D_P\left(\chi_x+\upsilon\chi_y\right)$$

$$M_y = -\frac{Et^3}{12\left(1-\upsilon^2\right)}\left(\frac{\partial^2 w}{\partial y^2}+\upsilon\frac{\partial^2 w}{\partial x^2}\right) = D_P\left(\chi_y+\upsilon\chi_x\right) \tag{6.1.9}$$

$$M_{xy} = M_{yx} = \frac{Et^3}{12\left(1+\upsilon\right)}\frac{\partial^2 w}{\partial x\partial y} = -\frac{D_P\left(1-\upsilon\right)}{2}\chi_{xy}$$

Where, D_P is known as flexural rigidity of the plate and is given by,

$$D_P = \frac{Et^3}{12\left(1-\upsilon^2\right)} \tag{6.1.10}$$

Let consider the bending moments vary along the length and breadth of the plate as a function of *x* and *y*. Thus, if M_x acts on one side of the element, $M_x^{'} = M_x + \frac{\partial M_x}{\partial x}dx$ acts on the opposite side. Considering equilibrium of the plate element, the equations for forces can be obtained as

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p=0 \tag{6.1.11}$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x \tag{6.1.12}$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y \tag{6.1.13}$$

Using eq.6.1.9 in eqs.6.1.12 & 6.1.13, the following relations will be obtained.

$$Q_x = -D_P\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{6.1.14}$$

$$Q_y = -D_P\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{6.1.15}$$

Using eqs. (6.1.14) and (6.1.15) in eq. (6.1.11) following relations will be obtained.

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=-\frac{p}{D_P} \tag{6.1.16}$$

6.1.4 Thick Plate Theory

Although Kirchhoff hypothesis provides comparatively simple analytical solutions for most of the cases, it also suffers from some limitations. For example, Kirchhoff plate element cannot rotate independently of the position of the mid-surface. As a result, problems occur at boundaries, where the undefined transverse shear stresses are necessary especially for thick plates. Also, the Kirchhoff theory is only applicable for analysis of plates with smaller deformations, as higher order terms of strain-displacement relationship cannot be neglected for large deformations. Moreover, as plate deflects its transverse stiffness changes. Hence only for small deformations the transverse stiffness can be assumed to be constant.

Contrary, Reissner–Mindlin plate theory (Fig. 6.1.5) is applied for analysis of thick plates, where the shear deformations are considered, rotation and lateral deflections are decoupled. It does not require the cross-sections to be perpendicular to the axial forces after deformation. It basically depends on following assumptions,

1. The deflections of the plate are small.
2. Normal to the plate mid-surface before deformation remains straight but is not necessarily normal to it after deformation.
3. Stresses normal to the mid-surface are negligible.

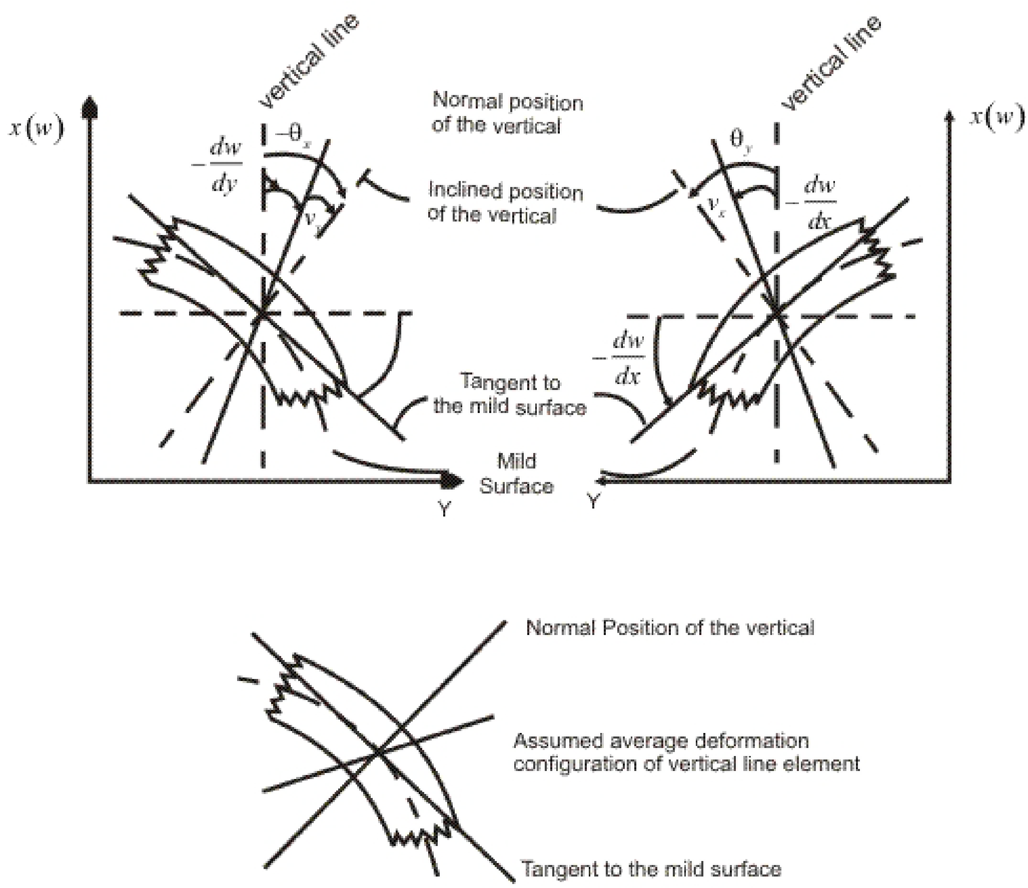


Fig. 6.1.5 Bending of thick plate

Thus, according to Mindlin plate theory, the deformation parallel to the undeformed mid surface, *u* and *v*, at a distance *z* from the centroidal axis are expressed by,

$$u = z\theta_y \tag{6.1.17}$$

$$\mathrm{v} = -\mathrm{z}\theta_x \tag{6.1.18}$$

Where θ_x and θ_y are the rotations of the line normal to the neutral axis of the plate with respect to the *x* and *y* axes respectively before deformation. The curvatures are expressed by

$$\chi_x = \frac{\partial \theta_y}{\partial x} \tag{6.1.19}$$

$$\chi_y = -\frac{\partial \theta_x}{\partial y} \tag{6.1.20}$$

Similarly the twist for the plate is given by,

$$\chi_{xy} = \left(\frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x} \right) \tag{6.1.21}$$

Using eqs.6.1.9-6.1.10, the bending stresses for the plate is given by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{Et^3}{12\left(1-\upsilon^2\right)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \tag{6.1.22}$$

Or

$$\{M\} = [D]\{\chi\} \tag{6.1.23}$$

Further, the transverse shear strains are determined as

$$\gamma_{xz} = \theta_y + \frac{\partial w}{\partial x} \tag{6.1.24}$$

$$\gamma_{yz} = -\theta_x + \frac{\partial w}{\partial y} \tag{6.1.25}$$

The shear strain energy can be expressed as

$$\mathrm{U_s} = \frac{1}{2}\alpha GA \iint_A \left[(\gamma_x)^2 + \left(\gamma_y\right)^2\right] dx\, dy$$

$$= \frac{1}{2}\alpha GA \iint_A \left[\left(\theta_y + \frac{\partial w}{\partial x}\right)^2 + \left(-\theta_x + \frac{\partial w}{\partial y}\right)^2\right] dx\, dy \tag{6.1.26}$$

Where, $\mathrm{G} = \frac{E}{2(1+\mu)}$. The shear stresses are

$$\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{2(1+\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} \tag{6.1.27}$$

Hence the resultant shear stress is given by,

$$\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \frac{Et\alpha}{2(1+\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \gamma_x \\ \gamma_y \end{Bmatrix} \tag{6.1.28}$$

Or,

$$\{Q\} = [D_s]\{\gamma\} \tag{6.1.29}$$

Here "α" is the numerical correction factor used to characterize the restraint of cross section against warping. If there is no warping i.e., the section is having complete restraint against warping then α = 1 and if it is having no restraint against warping then α = 2/3. The value of α is usually taken to be $\pi^2/12$ or 5/6. Now, the stress resultant can be combined as follows.

$$\begin{Bmatrix} \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [D] & 0 \\ 0 & [D_s] \end{bmatrix} \begin{Bmatrix} \{\chi\} \\ \{\gamma\} \end{Bmatrix} \tag{6.1.30}$$

Or,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\mu)} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_x \\ \gamma_y \end{Bmatrix} \tag{6.1.31}$$

Or,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\mu)} \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \frac{\partial \theta_y}{\partial x} \\ -\frac{\partial \theta_x}{\partial y} \\ \frac{\partial \theta_y}{\partial y} - \frac{\partial \theta_x}{\partial x} \\ \theta_y + \frac{\partial w}{\partial x} \\ -\theta_x + \frac{\partial w}{\partial y} \end{Bmatrix} \tag{6.1.32}$$

The above relation may be compared with usual stress-strain relation. Thus, the stress resultants and their corresponding curvature and shear deformations may be considered analogous to stresses and strains.

6.1.5 Boundary Conditions

For different boundaries of the plate (Fig. 6.1.6), suitable conditions are to be incorporated in plate equation for solving the governing differential equations. For example, following conditions need to satisfy along *y* direction of the plate for various boundaries.

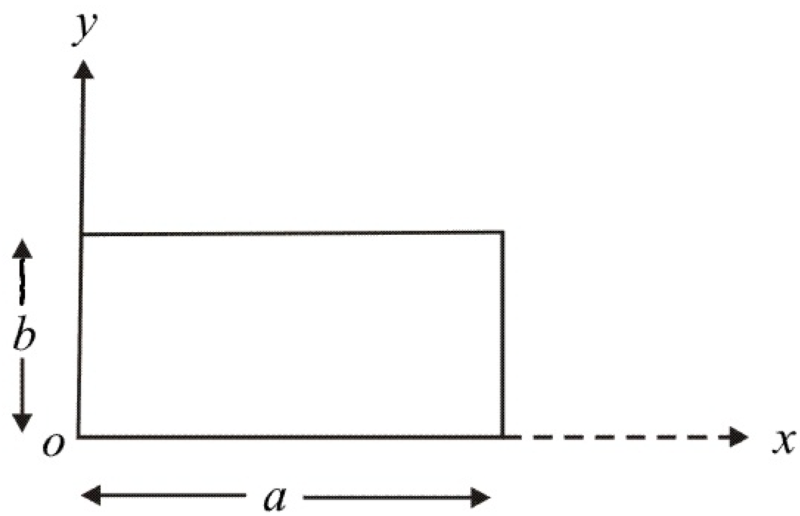


Fig. 6.1.6 Plate with four boundaries

1. Simply support edge (Along *y* direction)

   $$w(x,y)=0,\ M_x=0 \quad [x=const \ \ \&\ 0\le y\le b]$$

2. Clamped Edge (Along *y* direction)

   $$w(x,y)=0,\ \frac{\partial w}{\partial x}(x,y)=0 \qquad [x=const \ \ \&\ 0\le y\le b]$$

3. Free Edge (Along *y* direction)

   $$M_x=0,\ Q_x+\frac{\partial M_{xy}}{\partial x}=0, \qquad [x=const \ \ \&\ 0\le y\le b]$$

Similar to the above, the boundary conditions along *x* direction can also be obtained. Once the displacements *w(x,y)* of the plate at various positions are found, the strains, stresses and moments developed in the plate can be determined by using corresponding equations.

Lecture 2: Finite Element Analysis of Thin Plate

6.2.1 Triangular Plate Bending Element

A simplest possible triangular bending element has three corner nodes and three degrees of freedom per nodes (w, θ_x, θ_y) as shown in Fig. 6.2.1.

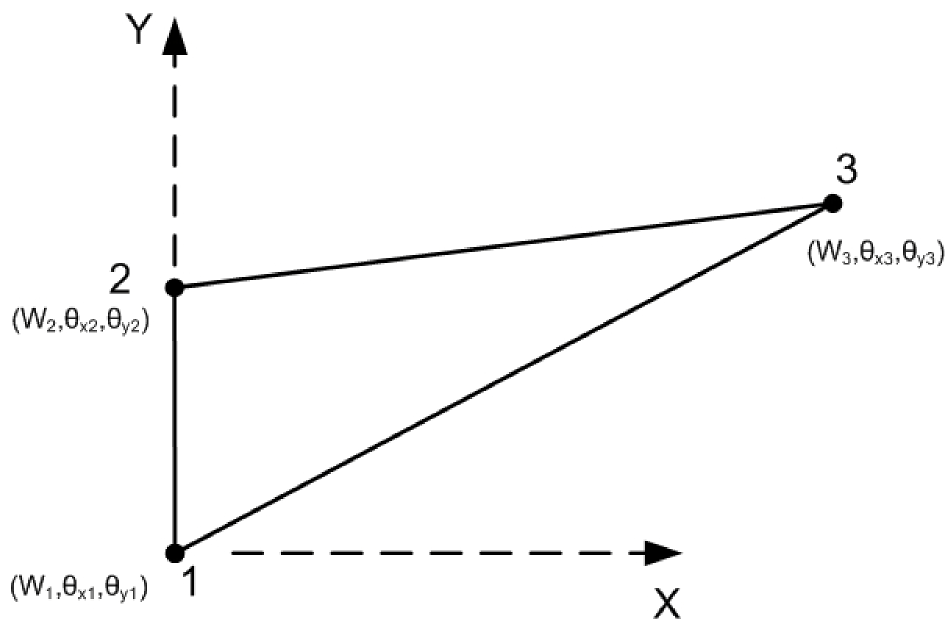


Fig. 6.2.1 Triangular plate bending element

As nine displacement degrees of freedom present in the element, we need a polynomial with nine independent terms for defining, w(x,y) . The displacement function is obtained from Pascal's triangle by choosing terms from lower order polynomials and gradually moving towards next higher order and so on.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & x & & y & & & \\ & & x^2 & & xy & & y^2 & & \\ & x^3 & & x^2y & & xy^2 & & y^3 & \\ x^4 & & x^3y & & x^2y^2 & & xy^3 & & y^4 \end{array}$$

Thus, considering Pascal triangle, and in order to maintain geometric isotropy, we may consider the displacement model in terms of the complete cubic polynomial as,

$$w(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + \alpha_6 x^3 + \alpha_7 \left(x^2 y + xy^2\right) + \alpha_8 y^3 \qquad (6.2.1)$$

Corresponding values for $(\theta_x, \theta_y)$are,

$$\theta_x = \frac{\partial w}{\partial y} = \alpha_2 + \alpha_4 x + 2\alpha_5 y + \alpha_7 \left(x^2 + 2xy\right) + 3\alpha_8 y^2 \tag{6.2.2}$$

$$\theta_y = -\frac{\partial w}{\partial x} = -\alpha_1 - 2\alpha_3 x - \alpha_4 y - 3\alpha_6 x^2 - \alpha_7 \left(2xy + y^2\right) \tag{6.2.3}$$

In the matrix form,

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & \left(x^2 y + xy^2\right) & y^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & \left(x^2 + 2xy\right) & 3y^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -\left(2xy + y^2\right) & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \tag{6.2.4}$$

Putting the nodal displacements and rotations for the triangular plate element as shown in Fig. 6.2.1 in the above equation, one can express following relations.

$$\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & y_2 & 0 & 0 & y_2^2 & 0 & 0 & y_2^3 \\ 0 & 0 & 1 & 0 & 0 & 2y_2 & 0 & 0 & 3y_2^2 \\ 0 & -1 & 0 & 0 & -y_2 & 0 & 0 & -y_2^2 & 0 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 & x_3^3 & \left(x_3^2 y_3 + x_3 y_3^2\right) & y_3^2 \\ 0 & 0 & 1 & 0 & x_3 & -2y_3 & 0 & \left(2x_3 y_3 + x_3^2\right) & 3y_3^2 \\ 0 & -1 & 0 & -2x_3 & -y_3 & 0 & 3x_3^2 & \left(-y_3^2 + 2x_3 y_3\right) & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \tag{6.2.5}$$

Or,

$$\{\alpha\} = [\Phi]^{-1} \{d_i\} \tag{6.2.6}$$

Further, the curvature of the plate element can be written as

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{2\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{6.2.7}$$

Again, from eq. (6.2.1) the following equations can be obtained.

$$\frac{\partial^2 w}{\partial x^2} = 2\alpha_3 + 6\alpha_6 x + 2\alpha_7 y$$
$$\frac{\partial^2 w}{\partial y^2} = 2\alpha_5 + 2\alpha_7 x + 6\alpha_8 y \qquad (6.2.8)$$
$$\frac{\partial^2 w}{\partial x \partial y} = \alpha_4 + 2\alpha_7 (x + y)$$

The above equation is expressed in matrix form as

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -2x & -6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} \qquad (6.2.9)$$

Or,

$$\{\chi\} = [B]\{\alpha\}$$

Thus,

$$\{\chi\} = [B][\Phi]^{-1}\{d\} \qquad (6.2.10)$$

Further for isotropic material,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{Et^3}{12(1-\upsilon^2)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \qquad (6.2.11)$$

Or

$$\{M\} = [D]\{\chi\} = [D][B][\Phi]^{-1}\{d\} \qquad (6.2.12)$$

Now the strain energy stored due to bending is

$$U = \frac{1}{2}\int_0^a \int_0^b [\chi]^T \{M\} dxdy = \frac{1}{2}\int_0^a \int_0^b \{d\}^T [\phi^{-1}]^T [B]^T [D][B][\phi^{-1}]\{d\} dxdy \qquad (6.2.13)$$

Hence the force vector is written as

$$\{F\} = \frac{\partial U}{\partial \{d\}} = [\phi^{-1}]^T \int_0^a \int_0^b [B]^T [D][B] dxdy \, [\phi^{-1}]\{d\} = [k]\{d\} \qquad (6.2.14)$$

Thus, $[k]$ is the stiffness matrix of the plate element and is given by

$$[k] = [\phi^{-1}]^T \int_0^a \int_0^b [B]^T[D][B]dxdy\, [\phi^{-1}] \qquad (6.2.15)$$

For a triangular plate element with orientation as shown in Fig. 6.2.2, the stiffness matrix defined in local coordinate system $[k]$ can be transformed into global coordinate system.

$$[K] = [T]^T[k]\,[T] \qquad (6.2.16)$$

Where, $[K]$ is the elemental stiffness matrix in global coordinate system and $[T]$ is the transformation matrix given by

$$[T]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_x & m_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_y & m_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_x & m_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_y & m_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_x & m_x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & l_y & m_y \end{bmatrix} \qquad (6.2.17)$$

Here, (l_x, m_x) and (l_y, m_y) are the direction cosines for the lines *OX* and *OY* respectively as shown in Fig. 6.2.2.

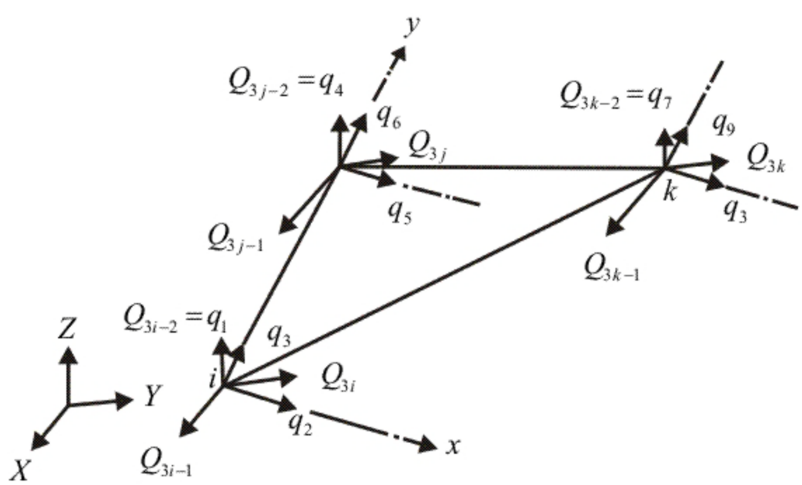


Fig. 6.2.2 Local and global coordinate system

6.2.2 Rectangular Plate Bending Element

A rectangular plate bending element is shown in Fig. 6.2.3. It has four corner nodes with three degrees of freedom (w, θ_x, θ_y) at each node. Hence, a polynomial with 12 independent terms for defining *w(x,y)* is necessary.

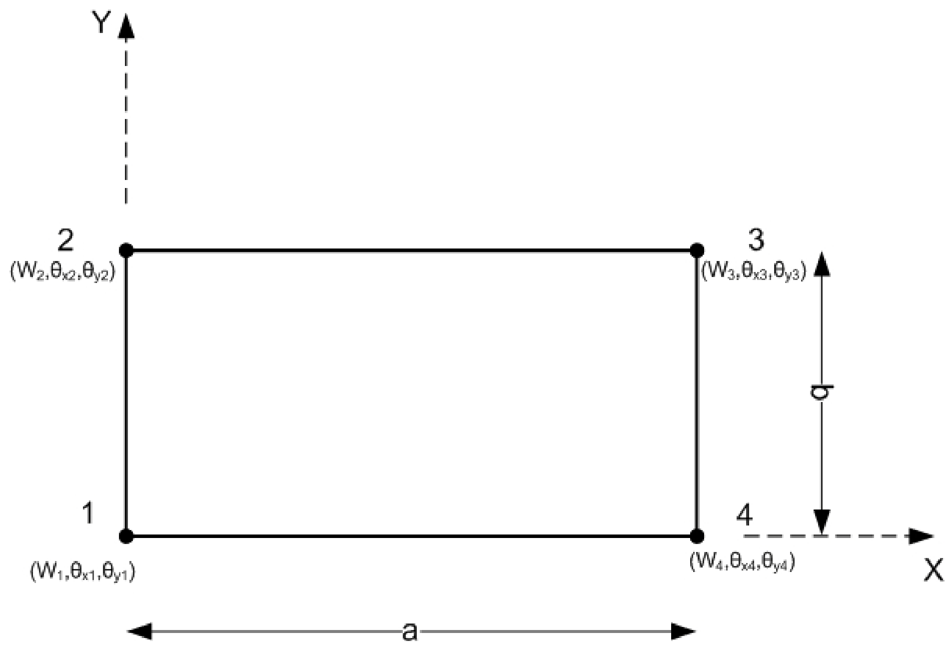


Fig 6.2.3 Rectangular plate bending element

Considering Pascal triangle, and in order to maintain geometric isotropy the following displacement function is chosen for finite element formulation.

$$\begin{aligned} w(x,y) = {} & \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + \alpha_6 x^3 + \alpha_7 x^2 y \\ & + \alpha_8 xy^2 + \alpha_9 y^3 + \alpha_{10} x^3 y + \alpha_{11} xy^3 \end{aligned} \tag{6.2.18}$$

Hence Corresponding values for $(\theta_x, \theta_y)$are,

$$\theta_x = \frac{\partial w}{\partial y} = \alpha_2 + \alpha_4 x + 2\alpha_5 y + \alpha_7 x^2 + 2\alpha_8 xy + 3\alpha_9 y^2 + \alpha_{10} x^3 + 3\alpha_{11} xy^2 \tag{6.2.19}$$

$$\theta_y = -\frac{\partial w}{\partial x} = -\alpha_1 - 2\alpha_3 x - \alpha_4 y - 3\alpha_6 x^2 - 2\alpha_7 xy - \alpha_8 y^2 - 3\alpha_{10} x^2 y - \alpha_{11} y^3 \tag{6.2.20}$$

The above can be expressed in matrix form

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2 y & xy^2 & y^3 & x^3 y & xy^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -2xy & -y^2 & 0 & -3x^2 y & -y^3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{11} \end{Bmatrix}$$

(6.2.21)

In a similar procedure to three node plate bending element the values of {α} can be found from the following relatios.

$$\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \\ w_4 \\ \theta_{x4} \\ \theta_{y4} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 \\ 0 & -1 & 0 & 0 & -b & 0 & 0 & 0 & -b^2 & 0 & 0 & -b^3 \\ 1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & ab^3 \\ 0 & 0 & 1 & 0 & a & 2b & 0 & a^2 & 2ab & 3b^2 & a^3 & 3ab^3 \\ 0 & -1 & 0 & -2a & -b & 0 & -3a^2 & -2ab & -b^2 & 0 & -3a^2b & -b^3 \\ 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 \\ 0 & -1 & 0 & -2a & 0 & 0 & -3a^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{Bmatrix} \tag{6.2.22}$$

Thus,

$$\{\alpha\} = [\Phi]^{-1}\{d\} \tag{6.2.23}$$

Further,

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ \frac{2\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{6.2.24}$$

Where,

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= 2\alpha_3 + 6\alpha_6 x + 2\alpha_7 y + 6\alpha_{10} xy \\ \frac{\partial^2 w}{\partial y^2} &= 2\alpha_5 + 2\alpha_8 x + 6\alpha_9 y + 6\alpha_{11} xy \\ \frac{\partial^2 w}{\partial x \partial y} &= \alpha_4 + 2\alpha_7 x + 2\alpha_8 y + 3\alpha_{10} x^2 + 3\alpha_{11} y^2 \end{aligned} \tag{6.2.25}$$

Thus, putting values of eq. (6.2.25) in eq. (6.2.24), the following relation is obtained.

$$\begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{11} \end{Bmatrix} \tag{6.2.26}$$

Or,

$$\{\chi\} = [B][\Phi^{-1}]\{d\} \tag{6.2.27}$$

Further in a similar method to triangular plate bending element we can estimate the stiffness matrix for rectangular plate bending element as,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \frac{Et^3}{12\left(1-\upsilon^2\right)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} \tag{6.2.28}$$

Or

$$\{M\} = [D]\{\chi\} = [D][B][\Phi]^{-1}\{d\} \tag{6.2.29}$$

The bending strain energy stored is

$$U = \frac{1}{2}\int_0^a \int_0^b [\chi]^T \{M\} dxdy = \frac{1}{2}\int_0^a \int_0^b \{d\}^T [\phi^{-1}]^T [B]^T [D][B][\phi^{-1}]\{d\} dxdy \tag{6.2.30}$$

Hence, the force vector will become

$$\{F\} = \frac{\partial U}{\partial\{d\}} = [\phi^{-1}]^T \int_0^a \int_0^b [B]^T [D][B] dxdy\, [\phi^{-1}]\{d\} = [k]\{d\} \tag{6.2.31}$$

Where, $[k]$ is the stiffness matrix given by

$$[k] = [\phi^{-1}]^T \int_0^a \int_0^b [B]^T [D][B] dxdy\, [\phi^{-1}] \qquad (6.2.30)$$

The stiffness matrix can be evaluated from the above expression. However, the stiffness matrix also can be formulated in terms of natural coordinate system using interpolation functions. In such case, the numerical integration needs to be carried out using Gauss Quadrature rule. Thus, after finding nodal displacement, the stresses will be obtained at the Gauss points which need to extrapolate to their corresponding nodes of the elements. By the use of stress smoothening technique, the various nodal stresses in the plate structure can be determined.

Lecture 3: Finite Element Analysis of Thick Plate

6.3.1 Introduction

Finite element formulation of the thick plate will be similar to that of thin plate. The difference will be the additional inclusion of energy due to shear deformation. Therefore, the moment curvature relation derived in first lecture of this module for thick plate theory will be the basis of finite element formulation. The relation is rewritten in the below for easy reference to follow the finite element implementation.

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} \frac{Et^3}{12(1-\mu^2)}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \frac{Et}{2(1+\mu)}\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_x \\ \gamma_y \end{Bmatrix} \tag{6.3.1}$$

Or,

$$\begin{Bmatrix} \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [D] & [0] \\ [0] & [D_s] \end{bmatrix} \begin{Bmatrix} \{\chi\} \\ \{\gamma\} \end{Bmatrix} \tag{6.3.2}$$

The above relation is comparable to stress-strain relations.

$$\{\sigma\}_P = [C]_P \{\varepsilon\}_P \tag{6.3.3}$$

Where,

$$\{\varepsilon\}_P = \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \\ \gamma_x \\ \gamma_y \end{Bmatrix} = [B]\{d_i\} \tag{6.3.4}$$

Where $[B]$ is the strain displacement matrix and $\{d_i\}$ is the nodal displacement vector. Thus, combining eqs. (6.3.3) and (6.3.4), the following expression is obtained.

$$\{\sigma\}_P = [C]_P [B]\{d_i\} \tag{6.3.5}$$

6.3.2 Strain Displacement Relation

Let consider a four node isoparametric element for the thick plate bending analysis purpose. The variation of displacement w and rotations, θ_x and θ_y within the element are expressed in the form of nodal values.

$$\begin{aligned} w &= \sum_{i=1}^{4} N_i w_i \\ \theta_x &= \sum_{i=1}^{4} N_i \theta_{xi} \\ \theta_y &= \sum_{i=1}^{4} N_i \theta_{yi} \end{aligned} \tag{6.3.6}$$

Where, the shape function for the four node element is expressed as,

$$N_i = \frac{1}{4}\left(1+\xi\,\xi_i\right)\left(1+\eta\,\eta_i\right) \tag{6.3.7}$$

Here, ξ_i *and* η_i are the local coordinates ξ *and* η of the i^{th} node. Using eq. (6.3.6), eq. (6.3.4) can be rewritten as

$$\begin{aligned}
\chi_x &= \sum_{i=1}^{4}\theta_{yi}\frac{\partial N_i}{\partial x}\\
\chi_y &= \sum_{i=1}^{4}-\theta_{xi}\frac{\partial N_i}{\partial y}\\
\chi_{xy} &= \sum_{i=1}^{4}\theta_{yi}\frac{\partial N_i}{\partial y}-\sum_{i=1}^{4}\theta_{xi}\frac{\partial N_i}{\partial x}\\
\gamma_x &= \sum_{i=1}^{4}w_i\frac{\partial N_i}{\partial x}+\sum_{i=1}^{4}\theta_{yi}N_i\\
\gamma_y &= \sum_{i=1}^{4}w_i\frac{\partial N_i}{\partial y}+\sum_{i=1}^{4}\theta_{xi}N_i
\end{aligned} \tag{6.3.8}$$

The above can be expressed in matrix form as follows:

$$\{\varepsilon\}_P = \begin{Bmatrix}\chi_x\\ \chi_y\\ \chi_{xy}\\ \gamma_x\\ \gamma_y\end{Bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x}\\ 0 & -\frac{\partial N_i}{\partial y} & 0\\ 0 & -\frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y}\\ \frac{\partial N_i}{\partial x} & 0 & N_i\\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix}\{d_i\} \tag{6.3.9}$$

Here, for a four node quadrilateral element, the nodal displacement vector $\{d_i\}$ will become

$$\{d_i\} = \left\{w_1,\theta_{x1},\theta_{y1},w_2,\theta_{x2},\theta_{y2},w_3,\theta_{x3},\theta_{y3},w_4,\theta_{x4},\theta_{y4}\right\}^T \tag{6.3.10}$$

Thus, the strain-displacement relationship matrix will be

$$[B]=\left[\begin{array}{ccc|ccc|ccc|ccc} 0 & 0 & \frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & 0 & 0 & \frac{\partial N_3}{\partial x} & 0 & 0 & \frac{\partial N_4}{\partial x} \\ 0 & -\frac{\partial N_1}{\partial y} & 0 & 0 & -\frac{\partial N_2}{\partial y} & 0 & 0 & -\frac{\partial N_3}{\partial y} & 0 & 0 & -\frac{\partial N_4}{\partial y} & 0 \\ 0 & -\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & 0 & -\frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & 0 & -\frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} & 0 & -\frac{\partial N_4}{\partial x} & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial x} & 0 & N_1 & \frac{\partial N_2}{\partial x} & 0 & N_2 & \frac{\partial N_3}{\partial x} & 0 & N_3 & \frac{\partial N_4}{\partial x} & 0 & N_4 \\ \frac{\partial N_1}{\partial y} & -N_1 & 0 & \frac{\partial N_2}{\partial y} & -N_2 & 0 & \frac{\partial N_3}{\partial y} & -N_3 & 0 & \frac{\partial N_4}{\partial y} & -N_4 & 0 \end{array}\right] \tag{6.3.11}$$

Or,

$$[B]=\left[[B_1]_{(5\times3)} \middle| [B_2]_{(5\times3)} \middle| [B_3]_{(5\times3)} \middle| [B_4]_{(5\times3)}\right] \tag{6.3.12}$$

Now, eq. (6.3.5) can be expressed using above relation as

$$\{\sigma\}_P=[C]_P[B]\{d_i\}=[C]_P\sum_{i=1}^{4}[B]\{d_i\} \tag{6.3.13}$$

Or,

$$[C]_P[B]=\left[[C_PB_1]_{(5\times3)} \middle| [C_PB_2]_{(5\times3)} \middle| [C_PB_3]_{(5\times3)} \middle| [C_PB_4]_{(5\times3)}\right] \tag{6.3.14}$$

Considering the i^{th} sub-matrix of the above equation,

$$[CB_i]=\frac{Et}{12(1+\mu)}\begin{bmatrix} 0 & \frac{-\mu t^2}{1-\mu}\left(\frac{\partial N_i}{\partial y}\right) & \frac{t^2}{1-\mu}\left(\frac{\partial N_i}{\partial x}\right) \\ 0 & \frac{-t^2}{1-\mu}\left(\frac{\partial N_i}{\partial x}\right) & \frac{\upsilon t^2}{1-\mu}\left(\frac{\partial N_i}{\partial y}\right) \\ 0 & \frac{-t^2}{2}\left(\frac{\partial N_i}{\partial x}\right) & \frac{t^2}{2}\left(\frac{\partial N_i}{\partial y}\right) \\ 6\alpha\left(\frac{\partial N_i}{\partial x}\right) & 0 & 6\alpha N_i \\ 6\alpha\left(\frac{\partial N_i}{\partial y}\right) & -6\alpha N_i & 0 \end{bmatrix} \tag{6.3.15}$$

The bending and shear terms form above equation are separated and written as

$$\left[CB_i\right]=\frac{Et}{12(1+\mu)}\left[\begin{bmatrix} 0 & \frac{-\mu t^2}{1-\mu}\left(\frac{\partial N_i}{\partial y}\right) & \frac{t^2}{1-\mu}\left(\frac{\partial N_i}{\partial x}\right) \\ 0 & \frac{-t^2}{1-\mu}\left(\frac{\partial N_i}{\partial x}\right) & \frac{\mu t^2}{1-\mu}\left(\frac{\partial N_i}{\partial y}\right) \\ 0 & \frac{-t^2}{2}\left(\frac{\partial N_i}{\partial x}\right) & \frac{t^2}{2}\left(\frac{\partial N_i}{\partial y}\right) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+6\alpha\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial N_i}{\partial x} & 0 & N_i \\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix}\right]$$

(6.3.16)

The above expression can be written in compact form as

$$\left[CB_i\right]=\left[CB_i\right]_b+\left[CB_i\right]_s \qquad (6.3.17)$$

Here, the contributions of bending and shear terms to stress displacement matrix is denoted as $[CB_i]_b$ and $[CB_i]_s$ respectively. Generally, the contribution due to bending, $[CB_i]_b$ in eq. (6.3.17) is evaluated considering 2×2 Gauss points where as the shear contribution $[CB_i]_s$ is evaluated considering 1×1 Gauss point.

6.3.3 Element Stiffness Matrix

The expression for element stiffness matrix is

$$[k]=\int\int_A [B]^T \; [C]_p \; [B]dx\, dy \qquad (6.3.18)$$

In natural coordinate system, the stiffness matrix is expressed as

$$[k]=\int\int_A [B]^T \; [C]_p \; [B] \; |J| d\xi \, d\eta \qquad (6.3.19)$$

Using value of $[C]_P$ form eq. 6.3.1 and $[B_i]$ from 6.3.11, the product of $[B]^T[C]_p[B]$ is evaluated as

$$[\bar{k}] = \left[[B]^T [C]_p [B]\right] = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ 0 & -\frac{\partial N_i}{\partial y} & -\frac{\partial N_i}{\partial x} & 0 & -N_i \\ \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_i}{\partial y} & N_i & 0 \end{bmatrix}_{i=1,2,3,4}$$

$$\times \begin{bmatrix} \frac{Et^3}{l^2(1-\upsilon)} \begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \frac{Et\alpha}{2(1+\upsilon)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & -\frac{\partial N_i}{\partial y} & 0 \\ 0 & -\frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial x} & 0 & N_i \\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix}_{i=1,2,3,4}$$

Or in short, (6.2.20)

$$[\bar{k}] = [B]^T [C]_p [B] = \begin{bmatrix} [\bar{k}_{11}] & [\bar{k}_{12}] & [\bar{k}_{13}] & [\bar{k}_{14}] \\ [\bar{k}_{21}] & [\bar{k}_{22}] & [\bar{k}_{23}] & [\bar{k}_{24}] \\ [\bar{k}_{33}] & [\bar{k}_{32}] & [\bar{k}_{33}] & [\bar{k}_{34}] \\ [\bar{k}_{41}] & [\bar{k}_{42}] & [\bar{k}_{43}] & [\bar{k}_{44}] \end{bmatrix} \tag{6.2.21}$$

Where,

$$[\bar{k}_{ij}] = \frac{Et}{12(1+\mu)} \times \begin{bmatrix} 6\alpha\left[\frac{\partial N_i}{\partial x}\frac{\partial N_i}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_i}{\partial y}\right] & -6\alpha\frac{\partial N_i}{\partial y}N_j & 6\alpha\frac{\partial N_i}{\partial y}N_j \\ -6\alpha N_i\frac{\partial N_i}{\partial y} & \left(\begin{array}{l} \frac{t^2}{1-\mu}\left[\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right] \\ +\frac{t^2}{2}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right] + 6\alpha N_i N_j \end{array}\right) & \left(\begin{array}{l} -\frac{\mu t^2}{1-\upsilon}\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\right) \\ -\frac{t^2}{2}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\right) \end{array}\right) \\ -6\alpha N_i\frac{\partial N_i}{\partial x} & \left(\begin{array}{l} -\frac{\mu t^2}{1-\mu}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\right] \\ -\frac{t^2}{2}\left[\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\right] \end{array}\right) & \left(\begin{array}{l} \frac{t^2}{1-\mu}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right) \\ +\frac{t^2}{2}\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) \end{array}\right) \end{bmatrix}$$

(6.3.22)

By separating the bending and shear terms from above equation,

$$\left[\bar{k}_{ij}\right]=\frac{Et}{1\ (12+\mu)}\times\begin{bmatrix} 0 & 0 & 0 \\ 0 & \left(\frac{t^2}{1-\mu}\left[\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right]+\frac{t^2}{2}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right]\right) & \left(-\frac{\mu t^2}{1-\mu}\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\right)-\frac{t^2}{2}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\right)\right) \\ 0 & \left(-\frac{\mu t^2}{1-\mu}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial y}\right]-\frac{t^2}{2}\left[\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial x}\right]\right) & \left(\frac{t^2}{1-\mu}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\right)+\frac{t^2}{2}\left(\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\right) \end{bmatrix}$$

$$+6\alpha\begin{bmatrix} \left[\frac{\partial N_i}{\partial x}\frac{\partial N_i}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_i}{\partial y}\right] & -\frac{\partial N_i}{\partial y}N_j & \frac{\partial N_i}{\partial x}N_j \\ N_i\frac{\partial N_i}{\partial y} & N_iN_j & 0 \\ \frac{\partial N_i}{\partial x}N_j & 0 & N_iN_j \end{bmatrix}$$

(6.3.23)

Thus, the matrix $\left[\bar{k}\right]$ can now be written as the sum of bending and shear contributions

$$\left[\bar{k}\right]=\left[\bar{k}\right]_b+\left[\bar{k}\right]_s \tag{6.2.24}$$

Or,

$$\left[\bar{k}\right]=\begin{bmatrix} [\bar{k}_{11}]_b & [\bar{k}_{12}]_b & [\bar{k}_{13}]_b & [\bar{k}_{14}]_b \\ [\bar{k}_{21}]_b & [\bar{k}_{22}]_b & [\bar{k}_{23}]_b & [\bar{k}_{24}]_b \\ [\bar{k}_{33}]_b & [\bar{k}_{32}]_b & [\bar{k}_{33}]_b & [\bar{k}_{34}]_b \\ [\bar{k}_{41}]_b & [\bar{k}_{42}]_b & [\bar{k}_{43}]_b & [\bar{k}_{44}]_b \end{bmatrix}+\begin{bmatrix} [\bar{k}_{11}]_s & [\bar{k}_{12}]_s & [\bar{k}_{13}]_s & [\bar{k}_{14}]_s \\ [\bar{k}_{21}]_s & [\bar{k}_{22}]_s & [\bar{k}_{23}]_s & [\bar{k}_{24}]_s \\ [\bar{k}_{33}]_s & [\bar{k}_{32}]_s & [\bar{k}_{33}]_s & [\bar{k}_{34}]_s \\ [\bar{k}_{41}]_s & [\bar{k}_{42}]_s & [\bar{k}_{43}]_s & [\bar{k}_{44}]_s \end{bmatrix} \tag{6.2.25}$$

The stiffness matrix $\left[k\right]$ can be evaluated from the following expression by substituting $\left[\bar{k}\right]$ for $[B]^T[C]_p[B]$ in eq. (6.3.19) and is given as

$$\left[k\right]=\int_{-1}^{+1}\int_{-1}^{+1}\left[\bar{k}\right]\left|J\right|d\xi\ d\eta \tag{6.2.26}$$

Here, $\left|J\right|$ is the determinate of the Jacobian matrix. The Gauss Quadrature integration rule is used to compute the stiffness matrix [k].

6.3.4 Nodal Load Vector

Considering a uniformly distributed load *q* on the plate, the equivalent nodal load vector can be calculated for finite element analysis from the flowing expression.

$$\{Q_i\} = \begin{Bmatrix} F_x \\ M_x \\ M_x \end{Bmatrix} = \int_A N_i \begin{Bmatrix} q \\ 0 \\ 0 \end{Bmatrix} dA = \int_{-1}^{1}\int_{-1}^{1} N_i \begin{Bmatrix} q \\ 0 \\ 0 \end{Bmatrix} |J|\, d\xi\, d\eta \qquad (6.3.27)$$

Using Gauss Quadrature integration rule the above expression can be evaluated as,

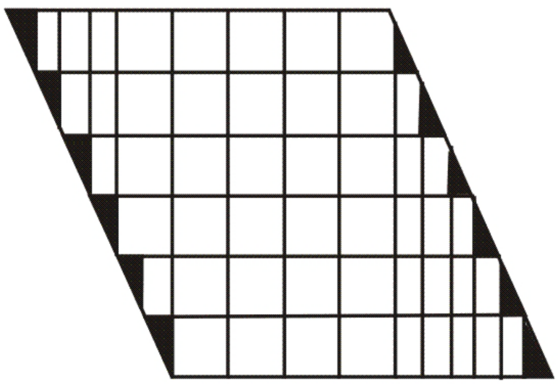
$$\{Q\} = q\sum_{i=1}^{n}\sum_{i=1}^{n} w_i w_j\, |J| \begin{Bmatrix} N_i \\ 0 \\ 0 \end{Bmatrix}_{i=1,2,3,4} \qquad (6.3.28)$$

The nodal load vectors from each element are assembled to find the global load vector at all the nodes.

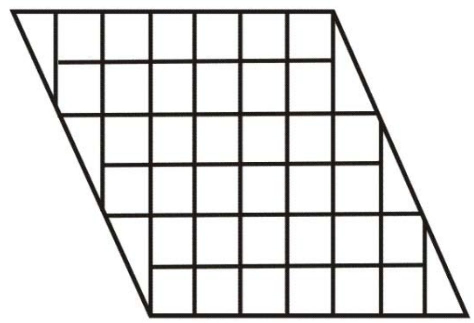
Lecture 4: Finite Element Analysis of Skew Plate

6.4.1 Introduction

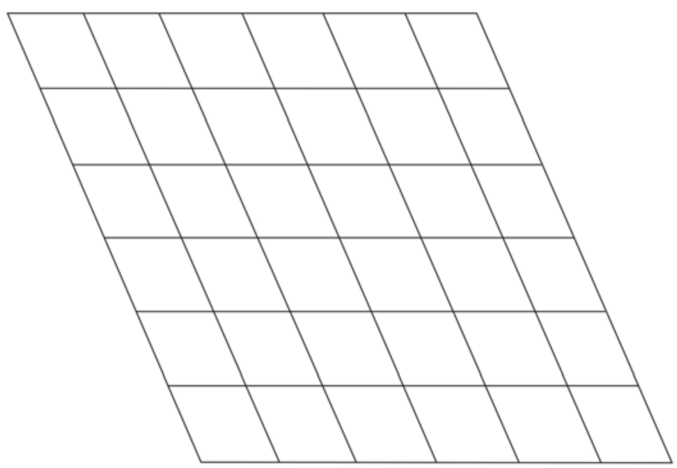
Skew plates often find its application in civil, aerospace, naval, mechanical engineering structures. Particularly in civil engineering fields they are mostly used in construction of bridges for dealing complex alignment requirements. Analytical solutions are available for few simple problems. However, several alternatives are also available for analyzing such complex problems by finite element methods. Commonly used three discretization methods for skew plates are shown in Fig 6.4.1.



(a) Discretization using rectangular plate elements



(b) Discretization using combination of rectangular and triangular plate elements



(c) Discretization using skew plate element

Fig 6.4.1 Discretization of a skew plate

If the skew plate is discretized using only rectangular plate elements, the area of continuum excluded from the finite element model may be adequate to provide incorrect results. Another method is to use combination of rectangular and triangular elements. However, such analysis will be complex and it may not provide best solution in terms of accuracy as, different order of polynomials is used to represent the field variables for different types of elements. Another alternative exist using skew element in place of rectangular element.

6.4.2 Finite Element Analysis of Skew Plate

Let consider a skew plate of dimension *"2a"* and *"2b"* as shown in Fig. 6.4.2. Let the skew angle of the element be "ϕ". It is possible for the parallelogram shown in Fig. 6.4.3 to map the coordinate from orthogonal global coordinate system to a skew local coordinate system. If the local coordinates are represented in the form of ξ, η, then the relationship can is represented as,

$$x = \xi + \eta\cos\phi, \qquad y = \eta\sin\phi \tag{6.4.1}$$

Hence,

$$\eta = y\cos ec\phi, \qquad \xi = x - y\cot\phi \tag{6.4.2}$$

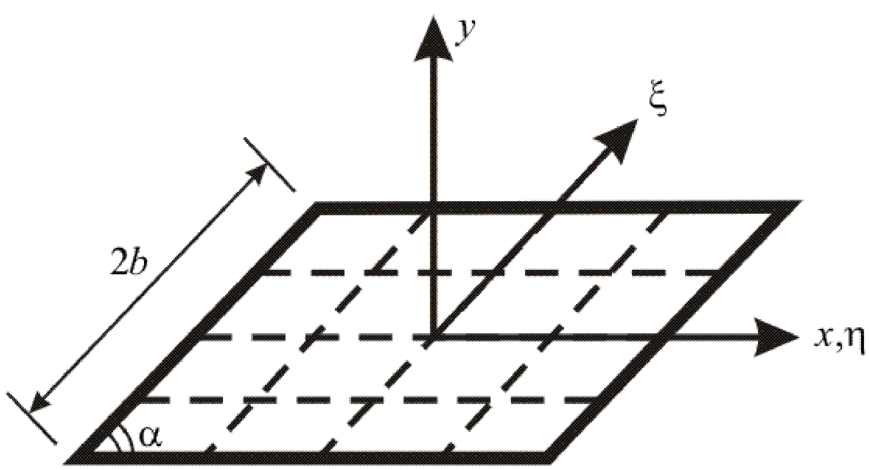


Fig. 6.4.2 Skew plate in global coordinate system

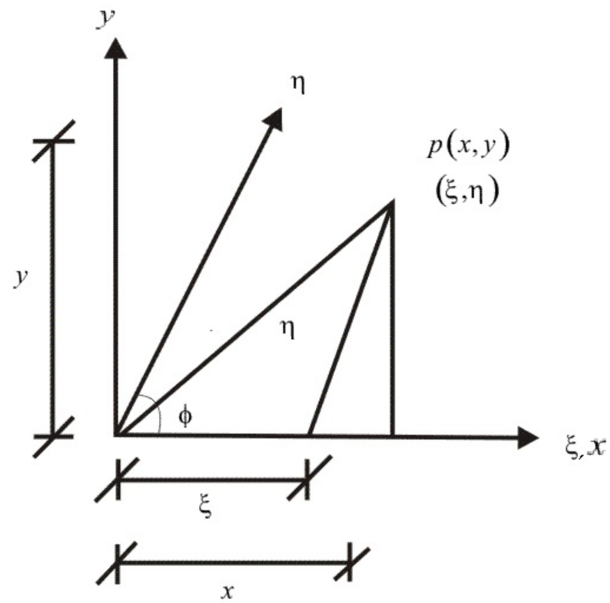


Fig. 6.4.3 Point “P” in global and local coordinate system

It is important to note that the terms *(ξ, η)* in the above equations represent the absolute coordinate of the point “P” in skew coordinate system, not the natural coordinates. Since the above element has four corner nodes and each node have three degrees of freedom present, a polynomial with minimum 12 independent terms are necessary for defining the displacement function *w(x,y).* Considering Pascal triangle, and in order to maintain geometric isotropy, the displacement function may be considered as follows:

$$\begin{aligned} \mathrm{w} &= \alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi^2 + \alpha_4\xi\eta + \alpha_5\eta^2 + \alpha_6\xi^3 + \alpha_7\xi^2\eta \\ &\quad + \alpha_8\xi\eta^2 + \alpha_9\eta^3 + \alpha_{10}\xi^3\eta + \alpha_{11}\xi\eta^3 \end{aligned} \tag{6.4.3}$$

Hence corresponding values for $(\theta_x, \theta_y)$are,

$$\theta_\xi = -\frac{\partial \mathrm{w}}{\partial \eta} = -\left(\alpha_2 + \alpha_4\xi + 2\alpha_5\eta + \alpha_7\xi^2 + 2\alpha_8\xi\eta + 3\alpha_9\eta^2 + \alpha_{10}\xi^3 + 3\alpha_{11}\xi\eta^2\right) \tag{6.4.4}$$

$$\theta_\eta = \frac{\partial \mathrm{w}}{\partial \xi} = \alpha_1 + 2\alpha_3\zeta + \alpha_4\eta + 3\alpha_6\xi^2 + 2\alpha_7\xi\eta + \alpha_8\eta^2 + 3\alpha_{10}\xi^2\eta + \alpha_{11}\eta^3 \tag{6.4.5}$$

Or, in matrix form,

$$\begin{Bmatrix} \mathrm{w} \\ \theta_\xi \\ \theta_\eta \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^3 & \xi^2\eta & \xi\eta^2 & \eta^3 & \xi^3\eta & \xi\eta^3 \\ 0 & 0 & -1 & 0 & -\xi & -2\eta & 0 & -\xi^2 & -2\xi\eta & -3\eta^2 & -\xi^3 & -3\xi\eta^2 \\ 0 & 1 & 0 & 2\xi & \eta & 0 & 3\xi^2 & 2\xi\eta & \eta^2 & 0 & 3\xi^2\eta & \eta^3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{11} \end{Bmatrix} \tag{6.4.6}$$

Or,

$$\{d\} = [\Phi]\{\alpha\} \tag{6.4.7}$$

The value of [α] can be determined using value of $(w, \theta_\xi, \theta_\eta)$ at four nodes as

$$\begin{Bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ w_3 \\ \theta_{x3} \\ \theta_{y3} \\ w_4 \\ \theta_{x4} \\ \theta_{y4} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 \\ 0 & -1 & 0 & 0 & -b & 0 & 0 & 0 & -b^2 & 0 & 0 & -b^3 \\ 1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & ab^3 \\ 0 & 0 & 1 & 0 & a & 2b & 0 & a^2 & 2ab & 3b^2 & a^3 & 3ab^3 \\ 0 & -1 & 0 & -2a & -b & 0 & -3a^2 & -2ab & -b^2 & 0 & -3a^2b & -b^3 \\ 1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 \\ 0 & -1 & 0 & -2a & 0 & 0 & -3a^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \end{Bmatrix} \tag{6.4.8}$$

Or,

$$\{\alpha\} = [\Phi]^{-1}\{d_i\} \tag{6.4.9}$$

Further, considering eq. (6.4.2)

$$\left.\begin{aligned} \frac{\partial\eta}{\partial x} &= 0, & \frac{\partial\eta}{\partial y} &= \operatorname{cosec}\phi \\ \frac{\partial\xi}{\partial x} &= 1, & \frac{\partial\xi}{\partial y} &= -\cot\phi \end{aligned}\right\} \tag{6.4.10}$$

Again,

$$\begin{pmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{pmatrix} = \begin{bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ \frac{2\partial^2 w}{\partial x \partial y} \end{bmatrix} \tag{6.4.11}$$

The values in right hand side of the eq. (6.4.11) can be calculated by using chain rule as,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \xi}.\frac{\partial \xi}{\partial x} + \frac{\partial w}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial w}{\partial \xi} \tag{6.4.12}$$

Therefore,

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial \xi^2} \tag{6.4.13}$$

Similarly,

$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial w}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=-\cos\phi\frac{\partial w}{\partial \xi}+\mathrm{cosec}\phi\frac{\partial w}{\partial \eta}$$

Thus, further derivation provides

$$\frac{\partial^2 w}{\partial y^2}=\mathrm{cosec}^2\phi\frac{\partial^2 w}{\partial \eta^2}+\cot^2\phi\frac{\partial^2 w}{\partial \xi^2}-2\cos\phi\ \mathrm{cosec}\phi\frac{\partial^2 w}{\partial \xi\partial \eta} \qquad (6.4.14)$$

And,

$$\frac{\partial^2 w}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)=-\cot\phi\frac{\partial^2 w}{\partial \xi^2}+\mathrm{cosec}^2\phi\frac{\partial^2 w}{\partial \xi\partial \eta} \qquad (6.4.15)$$

Hence eq. (6.4.11) is converted to

$$\begin{Bmatrix}-\dfrac{\partial^2 w}{\partial x^2}\\ -\dfrac{\partial^2 w}{\partial y^2}\\ \dfrac{2\partial^2 w}{\partial x\partial y}\end{Bmatrix}=\begin{bmatrix}1 & 0 & 0\\ \cot^2\phi & \mathrm{cosec}^2\phi & \text{-}\cot\phi\,\mathrm{cosec}\phi\\ \text{-}2\cot\phi & 0 & \mathrm{cosec}^2\phi\end{bmatrix}\begin{Bmatrix}-\dfrac{\partial^2 w}{\partial \xi^2}\\ -\dfrac{\partial^2 w}{\partial \eta^2}\\ \dfrac{2\partial^2 w}{\partial \xi\partial \eta}\end{Bmatrix} \qquad (6.4.16)$$

Or,

$$\{\chi_{x,y}\}=[H(\phi)]\{\chi_{\xi,\eta}\} \qquad (6.4.17)$$

Further, by partial differentiation of eq. (6.4.3),

$$\left.\begin{aligned}-\frac{\partial^2 w}{\partial \xi^2}&=-2\alpha_3-6\alpha_6\xi-2\alpha_7\eta-6\alpha_{10}\xi\eta\\ -\frac{\partial^2 w}{\partial \eta^2}&=-2\alpha_5-2\alpha_8\xi-6\alpha_9\eta-6\alpha_{11}\xi\eta\\ 2\frac{\partial^2 w}{\partial \xi\partial \eta}&=4\alpha_7\xi+4\alpha_8\eta+6\alpha_{10}\xi^2+6\alpha_{11}\eta^2\end{aligned}\right\} \qquad (6.4.18)$$

Or, in matrix form,

$$\begin{Bmatrix} -\dfrac{\partial^2 w}{\partial \xi^2} \\ -\dfrac{\partial^2 w}{\partial \eta^2} \\ \dfrac{2\partial^2 w}{\partial \xi \partial \eta} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6\xi & -2\eta & 0 & 0 & -6\xi\eta & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2\xi & -6\eta & 0 & -6\xi\eta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\xi & 4\eta & 0 & 6\xi^2 & 6\eta^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \vdots \\ \alpha_{11} \end{Bmatrix}$$

(6.4.19)

Or,

$$\{\chi_{\xi,\eta}\} = [B]\{\alpha\} = [B][\Phi^{-1}]\{d_i\}$$

Or,

$$\{\chi_{x,y}\} = [H(\phi)][B][\Phi^{-1}]\{d_i\} \tag{6.4.20}$$

Again,

$$\{M_{x,y}\} = [D]\{\chi_{x,y}\} \tag{6.4.21}$$

Where [D] for plane stress condition is

$$[D] = \frac{Eh^3}{12(1-\mu^2)} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \dfrac{1-\mu}{2} \end{bmatrix} \tag{6.4.22}$$

Using eq. (6.4.20) in eq. (6.4.21),

$$\{M_{x,y}\} = [D][H(\phi)][B][A^{-1}]\{d\} \tag{6.4.23}$$

The expression for bending strain energy stored,

$$U = \frac{1}{2}\int_0^a \int_0^b \{\chi_{x,y}\}^T \{M_{x,y}\} dxdy \tag{6.4.24}$$

Hence force vector,

$$\begin{aligned} \frac{\partial U}{\partial \{d\}} &= \int_0^a \int_0^b [\Phi^{-1}]^T [B]^T [H(\phi)]^T [D][H(\phi)][B][\Phi^{-1}]\{d\} dxdy \\ &= \int_0^a \int_0^b [\Phi^{-1}]^T [B]^T [H(\phi)]^T [D][H(\phi)][B][\Phi^{-1}] |J| d\xi d\eta \{d\} \end{aligned} \tag{6.4.25}$$

Where,

$$[J]=\left[\frac{\partial(x,y)}{\partial(\xi,\eta)}\right]=\begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}1 & 0\\ \cos\phi & \sin\phi\end{bmatrix}=\sin\phi \qquad (6.4.26)$$

From expression in eq. (6.4.25)

$$\{F\}=\frac{\partial U}{\partial\{d\}}=[k]\{d\} \qquad (6.4.27)$$

Hence,

$$[k]=\sin\Omega\cdot\left[\Phi^{-1}\right]^{T}\int_{0}^{a}\int_{0}^{b}\left[B(\xi,\eta)\right]^{T}\left[H(\phi)\right]^{T}[D]\left[H(\phi)\right]\left[B(\xi,\eta)\right]d\xi d\eta\left[\Phi^{-1}\right] \qquad (6.4.28)$$

Thus, the element stiffness matrix of a skew element for plate bending analysis can be evaluated from the above expression using Gauss Quadrature numerical integration.

Lecture 5: Introduction to Finite Strip Method

6.5.1 Introduction

The finite strip method (FSM) was first developed by Y. K. Cheung in 1968. This is an efficient tool for analyzing structures with regular geometric platform and simple boundary conditions. If the structure is regular, the whole structure can be idealized as an assembly of 2D strips or 3D prisms. Thus the geometry of the structure needs to be constant along one or two coordinate directions so that the width of the strips or the cross-section of the prisms does not change. Therefore, the finite strip method can reduce three and two-dimensional problems to two and one-dimensional problems respectively. The major advantages of this method are (i) reduction of computation time, (ii) small amount of input (iii) easy to develop the computer code etc. However, this method will not be suitable for irregular geometry, material properties and boundary conditions.

6.5.2 Finite Strip Method

To understand the finite strip method, let consider a rectangular plate with x and y axes in the plane of the plate and axis z in the thickness direction as shown in Fig. 6.5.1. The corresponding displacement components of the plate are denoted as u, v and w.

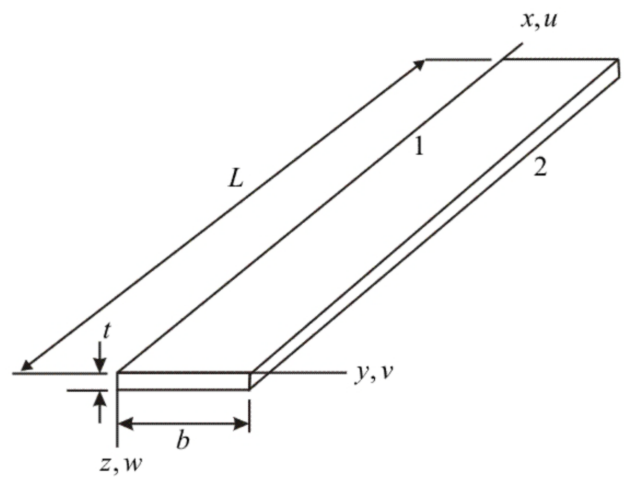


Fig. 6.5.1 Finite strip in a plate

The strips are assumed to be connected to each other along a discreet number of nodal lines that coincide with the longitudinal boundaries of the strip. The general form of the displacement function in two dimensions for a typical strip is given by

$$w = w(x, y) = \sum_{m=1}^{n} f_m(y) X_m(x) \qquad (6.5.1)$$

Here, the functions $f_m(y)$ are polynomials and the functions $X_m(x)$ are trigonometric terms that satisfy the end conditions in the x direction. The functions $X_m(x)$ can be taken as basic functions (mode shapes) of the beam vibration equation.

$$\frac{d^4X}{dy^4} - \frac{\mu^4}{L^4}Y = 0 \tag{6.5.2}$$

Here L is the length of beam strip and μ is a parameter related to material, frequency and geometric properties. The general solution of the above equation will become

$$X_m(x) = C_1 \sin\left(\frac{n\pi x}{L}\right) + C_2 \cos\left(\frac{n\pi x}{L}\right) + C_3 \sinh\left(\frac{n\pi x}{L}\right) + C_4 \cosh\left(\frac{n\pi x}{L}\right) \tag{6.5.3}$$

Four conditions at the boundaries are necessary to determine the coefficients C_1 to C_4 in the above expression.

6.5.2.1 Boundary conditions

According to different end conditions eq. (6.5.3) can be solved. Solution of the above equation is evaluated for few boundary conditions in the below.

(a) Both end simply supported

For simply supported end, following conditions will arise:

(i) At one end (say at x =0) displacement and moment will be zero: $x(0) = x''(0) = 0$

(ii) At other end (at x =L) displacement and moment will be zero: $x(L) = x''(L) = 0$

Thus, considering above boundary conditions, eq. (6.5.3) yields to the following mode shape function:

$$X_m(x) = \sum_{n=1}^{m} sin\frac{n\pi x}{l} = sin\left(\frac{\mu_m \pi x}{l}\right) \quad \left(\text{Where, } \mu_m = \pi, 2\pi.......\text{upto } n^{\text{th}} \text{ term}\right) \tag{6.5.4}$$

Since the functions *Xm* are mode shapes, they are orthogonal and therefore, they satisfy the following relations:

$$\int_0^L X_m(x) X_n(x) dx = 0 \qquad for\ m \neq n \tag{6.5.5}$$

And

$$\int_0^L X''(x)_m X''(x)_n dx = 0 \qquad for\ m \neq n \tag{6.5.6}$$

The orthogonal properties of $Xm(x)$ result in structural matrices with narrow bandwidths and thus minimizing computational time and storage. Using relation in eq. (6.5.4), eq. (6.5.1) can now be written as

$$w = w(x,y) = \sum_{n=1}^{m} f_m(y).\sin\left(\frac{n\pi x}{L}\right) \tag{6.5.7}$$

(b) Both end fixed supported

In case of fixed supported end at both the side, the following boundary conditions will be adopted:

(i) At one end (say at x =0) displacement and slope will be zero: $x(0) = x'(0) = 0$

(ii) At other end (at x =L) displacement and slope will be zero: $x(L) = x'(L) = 0$

For the above boundary conditions, eq. (6.5.3) yields to the following:

$$X_m(x) = sin\left(\frac{\mu_m x}{L}\right) - sin\,h\left(\frac{\mu_m y}{L}\right) - \alpha_m\left[cos\left(\frac{\mu_m y}{L}\right) - cos\,h\left(\frac{\mu_m y}{L}\right)\right] \tag{6.5.8}$$

Where $\left(\mu_m = 4.73, 7.8532, 10.996 .... \frac{2m+1}{2}\pi\right)$ and $\alpha_m = \frac{\sin\mu_m - \sinh\mu_m}{\cos\mu_m - \cosh\mu_m}$

(c) One end simply supported, other end fixed

(i) At simply supported end (say at x =0) displacement and moment will be zero: $x(0) = x''(0) = 0$

(ii) At fixed end (say at x =L) displacement and slope will be zero: $x(L) = x'(L) = 0$

Thus, the solution of eq. (6.5.3) will become

$$X_m(x) = \sin\left(\frac{\mu_m x}{l}\right) - \alpha_m \sinh\left(\frac{\mu_m x}{l}\right) \qquad (6.5.9)$$

where, $\mu_m = 3.9266, 7.0685, 10.2102 .... \frac{4m+1}{4}\pi$ and $\alpha_m = \frac{\sin\mu_m}{\sinh\mu_m}$

(d) Both end free

If both the end of the strip element is free, the following boundary conditions will be assumed:

(i) At one end (say at x =0) moment and shear will become zero: $x''(0) = x'''(0) = 0$

(ii) At other end (at x =L) moment and shear will become zero: $x''(L) = x'''(L) = 0$

Thus, for the above end conditions, eq. (6.5.3) yields to the following:

$$X_1 = 1, \mu_1 = 0 \;\&\; X_2 = 1 - \frac{2x}{L}, \mu_2 = 1$$

$$X_m(x) = \sin\left(\frac{\mu_m x}{l}\right) + \sinh\frac{\mu_m x}{l} - \alpha_m\left(\cos\frac{\mu_m x}{l} + \cosh\frac{\mu_m x}{l}\right) \qquad (6.5.10)$$

Where, $\alpha_m = \frac{\sin\mu_m - \sinh\mu_m}{\cos\mu_m - \cosh\mu_m}$ and $\mu_m = 4.73, 7.8532, 10.996 .... \frac{2m-3}{2}\pi$, for $m = 3, 4, - - - \propto$

6.5.3 Finite Element Formulation

In this section, finite element solution for a finite strip will be evaluated considering simply supported conditions at both the end. As a result, the functions $f_m(Y)$ in eq. (6.5.7) can be expressed for the bending problem as

$$f_m(Y) = w(y) = \alpha_0 + \alpha_1\, y + \alpha_2\, y^2 + \alpha_3\, y^3 \qquad (6.5.11)$$

Applying boundary conditions of the strip plate of width b, the following relations will be obtained.

$$\begin{Bmatrix} w_0 \\ \theta_{x0} \\ w_0 \\ \theta_{x0} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \qquad (6.5.12)$$

Thus, the nodal displacement can be written in short as

$$\{d\} = [A]\{\alpha\} \qquad (6.5.13)$$

Thus, the unknown coefficient α are obtained from the following relations.

$$\{\alpha\} = [A]^{-1}\{d\} \qquad (6.5.14)$$

The formulation of the finite strip method is similar to that of the finite element method. For example, for a strip subjected to bending, the moment curvature relation will become

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & 2D_{xy} \end{bmatrix} \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{6.5.15}$$

Where *Mx* , *My* and *Mxy* are moments per unit length and [*D*] is the elasticity matrix. From eq. (6.5.7) the following expressions are evaluated to incorporate in the above equation.

$$\begin{aligned} -\frac{\partial^2 w}{\partial x^2} &= \sum_{n=1}^{m} w(y)\left(\frac{n\pi}{L}\right)^2 sin\frac{n\pi x}{L} \\ -\frac{\partial^2 w}{\partial y^2} &= -\sum_{n=1}^{m} -(2\alpha_2 + 6\alpha_3 y) sin\frac{n\pi x}{L} \\ 2\frac{\partial^2 w}{\partial x \partial y} &= \sum_{n=1}^{m} 2(\alpha_1 + 2\alpha_2 y + 3\alpha_3 y^2)\left(\frac{n\pi}{L}\right) cos\frac{n\pi x}{L} \end{aligned} \tag{6.5.16}$$

The above expression is written in matrix form in the below

$$\begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \begin{Bmatrix} \sum w(y) N^2 sinNx \\ -\sum -(2\alpha_2 + 6\alpha_3 y) sinNx \\ \sum 2(\alpha_1 + 2\alpha_2 y + 3\alpha_3 y^2) N cosNx \end{Bmatrix} \tag{6.5.17}$$

Here, $\frac{n\pi}{L}$ is denoted as *N*. Rearranging the above expression, one can find the following.

$$\begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \begin{bmatrix} \sin Nx & 0 & 0 \\ 0 & \sin Nx & 0 \\ 0 & 0 & \cos Nx \end{bmatrix} \begin{Bmatrix} w(y) N^2 \\ -(2\alpha_2 + 6\alpha_3 y) \\ 2(\alpha_1 + 2\alpha_2 y + 3\alpha_3 y^2) N \end{Bmatrix}$$

$$= \begin{bmatrix} \sin Nx & 0 & 0 \\ 0 & \sin Nx & 0 \\ 0 & 0 & \cos Nx \end{bmatrix} \begin{bmatrix} N^2 & N^2 y & N^2 y^2 & N^2 y^3 \\ 0 & 0 & -2 & -6y \\ 0 & 2N & 4Ny & 6Ny^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{6.5.18}$$

Thus, in short, the curvature and moment equation will become

$$\{\chi\} = [H(x)][B(y)]\{\alpha\} = [H(x)][B(y)][A]^{-1}\{d\} \tag{6.5.19}$$

$$\{M\} = [D]\{\chi\} = [D][H(x)][B(y)][A]^{-1}\{d\} \tag{6.5.20}$$

Now, the strain energy for the bending element can be written similar to plate bending formulation.

$$U = \frac{1}{2}\int_0^L \int_0^b \{\chi\}^T \{M\}\, dx\, dy = \frac{1}{2}\int_0^L \int_0^b \{\alpha\}^T [A^{-1}]^T [B(y)]^T [H(x)]^T [D][H(x)][B(y)][A]^{-1}\{d\}\, dx\, dy \tag{6.5.21}$$

Thus the force vector can be derived as

$$\{F\} = \frac{\partial U}{\partial\{d\}} = \int_0^L \int_0^b [A^{-1}]^T [B(y)]^T [H(x)]^T [D][H(x)][B(y)][A]^{-1}\, dx\, dy\, \{d\} = [k]\{d\} \tag{6.5.22}$$

Thus, the stiffness matrix of a strip element can be obtained from the following expression.

$$\begin{aligned} [k] &= \int_0^L \int_0^b [A^{-1}]^T [B(y)]^T [H(x)]^T [D][H(x)][B(y)][A]^{-1}\, dx\, dy \\ &= [A^{-1}]^T \int_0^L \int_0^b [B(y)]^T [H(x)]^T [D][H(x)][B(y)]\, dx\, dy\, [A]^{-1} \end{aligned} \tag{6.5.23}$$

The stiffness matrix $[k]$ can be simplified by integrating the term $\int_0^L [H(x)]^T [D][H(x)]\, dx$ as follows.

$$\begin{aligned} &\int_0^L [H(x)]^T [D][H(x)]\, dx \\ &= \int_0^L \begin{bmatrix} \sin Nx & 0 & 0 \\ 0 & \sin Nx & 0 \\ 0 & 0 & \cos Nx \end{bmatrix} \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & 2D_{xy} \end{bmatrix} \begin{bmatrix} \sin Nx & 0 & 0 \\ 0 & \sin Nx & 0 \\ 0 & 0 & \cos Nx \end{bmatrix} dx \\ &= \int_0^L \begin{bmatrix} D_x \sin Nx & D_1 \sin Nx & 0 \\ D_1 \sin Nx & D_y \sin Nx & 0 \\ 0 & 0 & 2D_{xy} \cos Nx \end{bmatrix} \begin{bmatrix} \sin Nx & 0 & 0 \\ 0 & \sin Nx & 0 \\ 0 & 0 & \cos Nx \end{bmatrix} dx \\ &= \int_0^L \begin{bmatrix} D_x \sin^2 Nx & D_1 \sin^2 Nx & 0 \\ D_1 \sin^2 Nx & D_y \sin^2 Nx & 0 \\ 0 & 0 & 2D_{xy} \cos^2 Nx \end{bmatrix} dx \end{aligned} \tag{6.5.24}$$

Here, the terms D_x, D_y and D_{xy} are constant and not varied with x or y. Following integrations are carried out to simplify the above expression further.

Now $\int_0^L sin^2 Nx\, dx = \int_0^L \left(\frac{1-cos2Nx}{2}\right) dx = \frac{1}{2}\left[x - \frac{sin2Nx}{2N}\right]_0^L = \frac{1}{2}\left[L - \frac{sin2NL}{2N}\right]$

Putting, $N = \frac{n\pi}{L}$ finally $\int_0^L sin^2 Nx\, dx$ will become $\frac{1}{2}\left[L - \frac{sin2\frac{n\pi}{L}L}{2\frac{n\pi}{L}}\right] = \frac{1}{2}L$

Similarly; $\int_0^L cos^2 Nx\, dx = \int_0^L \left(\frac{1+cos2Nx}{2}\right) dx = \frac{1}{2}\left[x + \frac{sin2Nx}{2N}\right]_0^L = \frac{1}{2}L$

Thus,

$$\int_0^L [H(x)]^T [D][H(x)]\, dx = \frac{L}{2} \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & 2D_{xy} \end{bmatrix} = \frac{L}{2}[D] \tag{6.5.25}$$

Using eq. (6.5.25), the expression for stiffness matrix $[k]$ in eq. (6.5.23) is simplified as follows.

$$\begin{aligned} [k] &= [A^{-1}]^T \int_0^b [B(y)]^T \frac{L}{2} [D][B(y)]\, dy\, [A^{-1}] \\ &= \frac{L}{2} [A^{-1}]^T \int_0^b \begin{bmatrix} N^2 & 0 & 0 \\ N^2 y & 0 & 2N \\ N^2 y^2 & -2 & 4Ny \\ N^2 y^3 & -6y & 6Ny^2 \end{bmatrix} \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & 2D_{xy} \end{bmatrix} \begin{bmatrix} N^2 & N^2 y & N^2 y^2 & N^2 y^3 \\ 0 & 0 & -2 & -6y \\ 0 & 2N & 4Ny & 6Ny^2 \end{bmatrix} dy [A]^{-1} \end{aligned}$$

$$= \frac{L}{2}[A^{-1}]^T \int_o^b \begin{bmatrix} N^2D_x & N^2D_1 & 0 \\ N^2yD_x & N^2yD_1 & 4ND_{xy} \\ N^2y^2D_x - 2D_1 & N^2y^2D_1 - 2D_y & 8D_{xy}Ny \\ N^2y^3D_x - 6yD_1 & N^2y^3D_1 - 6yD_y & 12Ny^2D_{xy} \end{bmatrix} \begin{bmatrix} N^2 & N^2y & N^2y^2 & N^2y^3 \\ 0 & 0 & -2 & -6y \\ 0 & 2N & 4Ny & 6Ny^2 \end{bmatrix} dy[A]^{-1}$$

$$= \frac{L}{2}[A^{-1}]^T \int_0^b \begin{bmatrix} (N^4D_x) & (N^4yD_x) & (N^4y^2D_x - 2N^2D_1) & (N^4y^3D_x - 6N^2yD_1) \\ (N^4yD_x) & \left(N^4y^2D_x + 8N^2D_{xy}\right) & \left(N^4y^3D_x - 2N^2yD_1 + 16N^2yD_{xy}\right) & \left(N^4y^4D_x - 6N^2y^2D_1 + 24N^2y^2D_{xy}\right) \\ (N^4y^2D_x - 2N^2D_1) & \left(N^4y^3D_x - 2N^2yD_1 + 8N^2yD_{xy}\right) & \begin{pmatrix} N^4y^4D_x - 4N^2y^2D_1 \\ +4D_y + 32N^2y^2D_{xy} \end{pmatrix} & \begin{pmatrix} N^4y^5D_x - 2D_1N^2y^3 - 6N^2y^3D_1 \\ -12yD_y + 48N^2y^3D_{xy} \end{pmatrix} \\ (N^4y^3D_x - 6N^2yD_1) & \left(N^4y^4D_x - 6N^2y^2D_1 + 24N^2y^2D_{xy}\right) & \begin{pmatrix} N^4y^5D_x - 6N^2y^3D_1 \\ -2N^2y^3D_1 + 12yD_y \\ +48N^2y^3D_{xy} \end{pmatrix} & \begin{pmatrix} N^4y^6D_x - 6N^2y^4D_1 \\ -6N^2y^4D_1 + 36y^2D_y \\ +72N^2y^4D_{xy} \end{pmatrix} \end{bmatrix} dy[A]^{-1} \quad (6.5.26)$$

Thus, by putting the assumed shape function, the stiffness matrix of a strip element can be evaluated numerically using Gaussian Quadrature or other numerical integration methods.

Lecture 6: Finite Element Analysis of Shell

6.6.1 Introduction

A shell is a curved surface, which by virtue of their shape can withstand both membrane and bending forces. A shell structure can take higher loads if, membrane stresses are predominant, which is primarily caused due to in-plane forces (plane stress condition). However, localized bending stresses will appear near load concentrations or geometric discontinuities. The shells are analogous to cable or arch structure depending on whether the shell resists tensile or, compressive stresses respectively. Few advantages using shell elements are given below.

1. Higher load carrying capacity
2. Lesser thickness and hence lesser dead load
3. Lesser support requirement
4. Larger useful space
5. Higher aesthetic value.

The example of shell structures includes large-span roof, cooling towers, piping system, pressure vessel, aircraft fuselage, rockets, water tank, arch dams, and many more. Even in the field of biomechanics, shell elements are used for analysis of skull, Crustaceans shape, red blood cells, etc.

6.6.2 Classification of Shells

Shell may be classified with several alternatives which are presented in Fig 6.6.1.

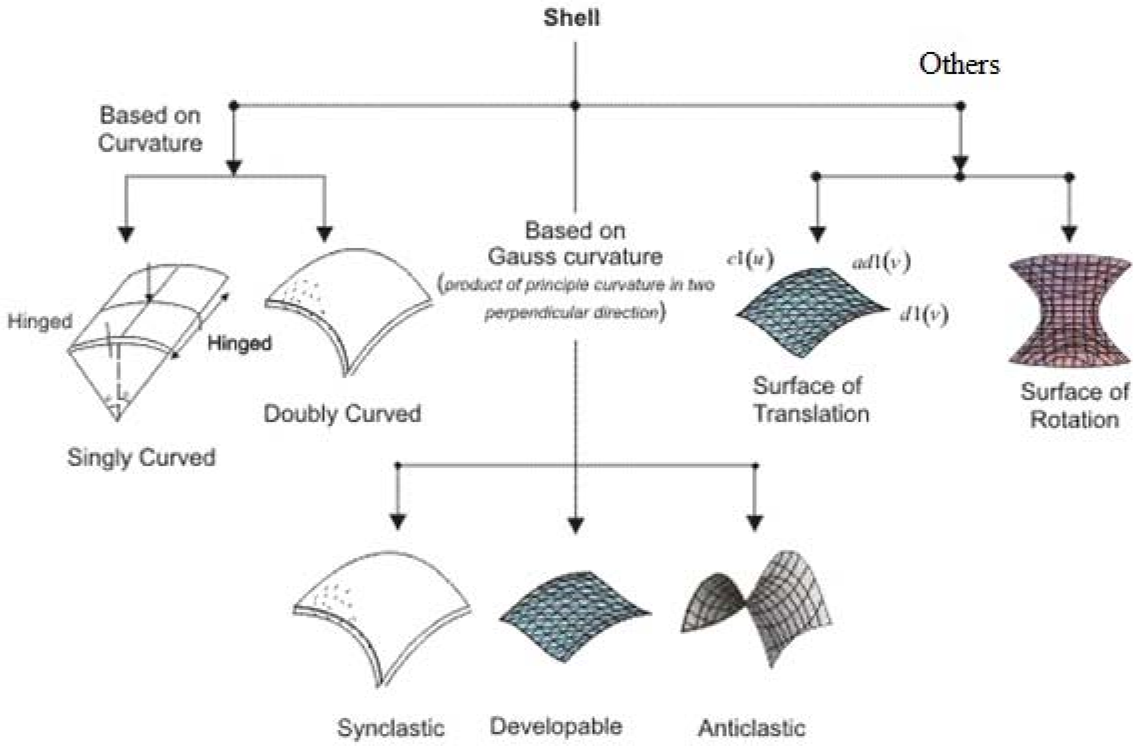


Fig 6.6.1 Classification of shells

Depending upon deflection in transverse direction due to transverse shear force per unit length, the shell can be classified into structurally thin or thick shell. Further, depending upon the thickness of the shell in comparison to the radii of curvature of the mid surface, the shell is referred to as geometrically thin or thick shell. Typically, if thickness to radii of curvature is less than 0.05, then the shell can be assumed as a thin shell. For most of the engineering application the thickness of shell remains within 0.001 to 0.05 and treated as thin shell.

6.6.3 Assumptions for Thin Shell Theory

Thin shell theories are basically based on Love-Kirchoff assumptions as follows.

1. As the shell deforms, the normal to the un-deformed middle surface remain straight and normal to the deformed middle surface undergo no extension. i.e., all strain components in the direction of the normal to the middle surface is zero.
2. The transverse normal stress is neglected.

Thus, above assumptions reduce the three dimensional problems into two dimensional.

6.6.4 Overview of Shell Finite Elements

Many approaches exist for deriving shell finite elements, such as, flat shell element, curved shell element, solid shell element and degenerated shell element. These are discussed briefly bellow.

(a) Flat shell element

The geometry of these types of elements is assumed as flat. The curved geometry of shell is obtained by assembling number of flat elements. These elements are based on combination of membrane element and bending element that enforced Kirchoff's hypothesis. It is important to note that the coupling of membrane and bending effects due to curvature of the shell is absent in the interior of the individual elements.

(b) Curved shell element

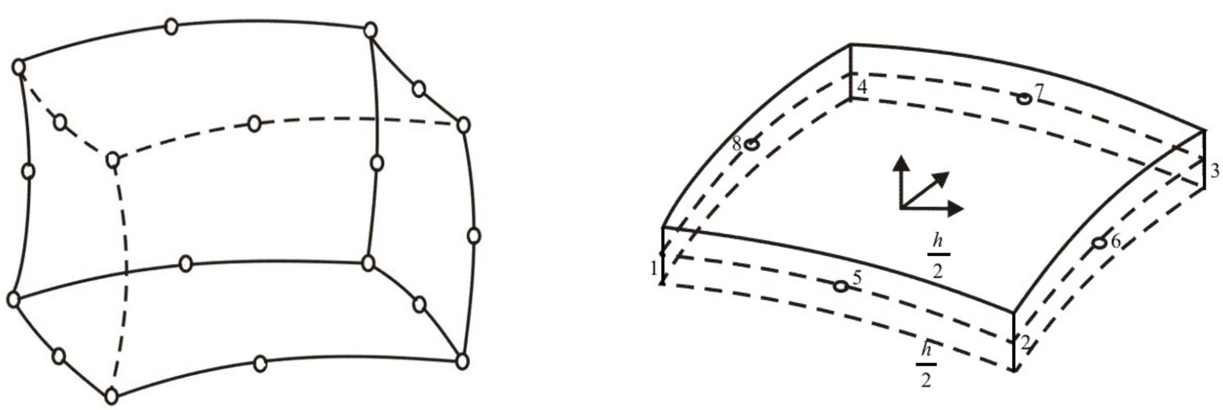
Curved shell elements are symmetrical about an axis of rotation. As in case of axisymmetric plate elements, membrane forces for these elements are represented with respected to meridian direction as(u, N_z, M_θ) and in circumferential directions as(w, N_θ, M_z). However, the difficulties associated with these elements includes, difficulty in describing geometry and achieving inter-elemental compatibility. Also, the satisfaction of rigid body modes of behaviour is acute in curved shell elements.

(c) Solid shell element

Though, use of 3D solid element is another option for analysis of shell structure, dealing with too many degrees of freedom makes it uneconomic in terms of computation time. Further, due to small thickness of shell element, the strain normal to the mid surface is associated with very large stiffness coefficients and thus makes the equations ill conditioned.

(d) Degenerated shell elements

Here, elements are derived by degenerating a 3D solid element into a shell surface element, by deleting the intermediate nodes in the thickness direction and then by projecting the nodes on each surface to the mid surface as shown in Fig. 6.6.2.



(a) 3D solid element (b) Degenerated Shell element

6.6.2 Degeneration of 3D element

This approach has the advantage of being independent of any particular shell theory. This approach can be used to formulate a general shell element for geometric and material nonlinear analysis. Such element has been employed very successfully when used with 9 or, in particular, 16 nodes. However, the 16-node element is quite expensive in computation. In a degenerated shell model, the numbers of unknowns present are five per node (three mid-surface displacements and two director rotations). Moderately thick shells can be analysed using such elements. However, selective and reduced integration techniques are necessary to use due to shear locking effects in case of thin shells. The assumptions for degenerated shell are similar to the Reissner-Mindlin assumptions.

6.6.5 Finite Element Formulation of a Degenerated Shell

Let consider a degenerated shell element, obtained by degenerating 3D solid element. The degenerated shell element as shown in Fig 6.6.2(b) has eight nodes, for which the analysis is carried out. Let (ξ, η) are the natural coordinates in the mid surface. And ς is the natural coordinate along thickness direction. The shape functions of a two dimensional eight node isoparametric element are:

$$\begin{aligned}
N_1 &= \frac{(1-\xi)(1-\eta)(-\xi-\eta-1)}{4} & N_5 &= \frac{(1+\xi)(1-\xi)(1-\eta)}{2} \\
N_2 &= \frac{(1+\xi)(1-\eta)(\xi-\eta-1)}{4} & N_6 &= \frac{(1+\xi)(1+\eta)(1-\eta)}{2} \\
N_3 &= \frac{(1+\xi)(1+\eta)(\xi+\eta-1)}{4} & N_7 &= \frac{(1+\xi)(1-\xi)(1+\eta)}{2} \\
N_4 &= \frac{(1-\xi)(1+\eta)(-\xi+\eta-1)}{4} & N_8 &= \frac{(1-\xi)(1+\eta)(1-\eta)}{2}
\end{aligned} \tag{6.6.1}$$

The position of any point inside the shell element can be written in terms of nodal coordinates as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta) \left\{ \frac{1+\varsigma}{2} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} + \frac{1-\varsigma}{2} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \right\} \tag{6.6.2}$$

Since, ς is assumed to be normal to the mid surface, the above expression can be rewritten in terms of a vector connecting the upper and lower points of shell as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta) \left\{ \frac{1}{2} \left\{ \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} + \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \right\} + \frac{\varsigma}{2} \left\{ \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} - \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \right\} \right\}$$

Or,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta) \left\{ \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \frac{\varsigma}{2} V_{3i} \right\} \tag{6.6.3}$$

Where,

$$\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} = \frac{1}{2} \left\{ \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} + \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \right\} \text{ and, } V_{3i} = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} - \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom} \tag{6.6.4}$$

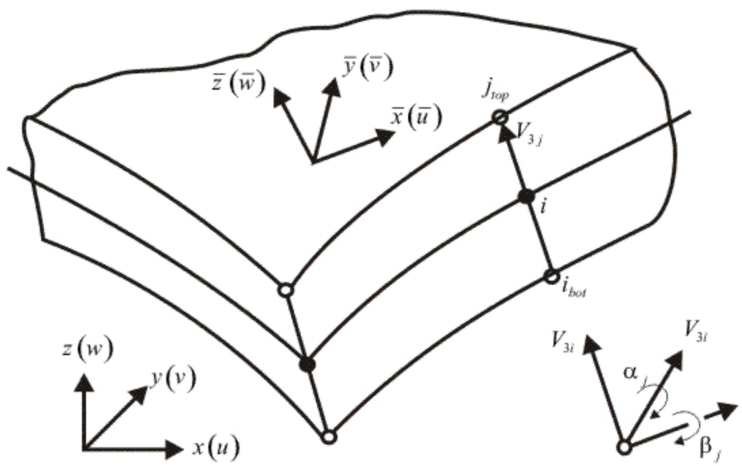


Fig. 6.6.3 Local and global coordinates

For small thickness, the vector $\boldsymbol{V_{3i}}$ can be represented as a unit vector $t_i\boldsymbol{v_{3i}}$:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta) \left\{ \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \frac{\varsigma}{2} t_i v_{3i} \right\} \tag{6.6.5}$$

Where, t_i is the thickness of shell at i^{th} node. In a similar way, the displacement at any point of the shell element can be expressed in terms of three displacements and two rotation components about two orthogonal directions normal to nodal load vector $\boldsymbol{V_{3i}}$ as,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta) \left\{ \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \frac{\varsigma t_i}{2} \begin{bmatrix} v_{1i} & -v_{2i} \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} \right\} \tag{6.6.6}$$

Where, (α_i, β_i) are the rotations of two unit vectors v_{1i} & v_{2i} about two orthogonal directions normal to nodal load vector $\boldsymbol{V_{3i}}$. The values of v_{1i} and v_{2i} can be calculated in following way:

The coordinate vector of the point to which a normal direction is to be constructed may be defined as

$$x = x\hat{i} + y\hat{j} + z\hat{k} \tag{6.6.7}$$

In which, $\hat{i}$, $\hat{j}$, $\hat{k}$ are three (orthogonal) base vectors. Then, V_{1i} is the cross product of $\hat{i}$ & V_{3i} as shown below.

$$V_{1i} = \hat{i} \times V_{3i} \ \ \& \ \ V_{2i} = V_{3i} \times V_{1i} \tag{6.6.8}$$

and,

$$v_{1i} = {V_{1i}}\big/{\left|V_{1i}\right|} \ \ \& \ \ v_{2i} = {V_{2i}}\big/{\left|V_{2i}\right|} \tag{6.6.9}$$

6.6.5.1 Jacobian matrix

The Jacobian matrix for eight node shell element can be expressed as,

$$[J] = \begin{bmatrix} \sum_{i=1}^{8}\left(x_i + tx_i^*\right)\frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8}\left(y_i + ty_i^*\right)\frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8}\left(z_i + tz_i^*\right)\frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{8}\left(x_i + tx_i^*\right)\frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{8}\left(y_i + ty_i^*\right)\frac{\partial N_i}{\partial \eta} & \sum_{i=1}^{8}\left(z_i + tz_i^*\right)\frac{\partial N_i}{\partial \eta} \\ \sum_{i=1}^{8} Nx_i^* & \sum_{i=1}^{8} Ny_i^* & \sum_{i=1}^{8} Nz_i^* \end{bmatrix} \tag{6.6.10}$$

6.6.5.2 Strain displacement matrix

The relationship between strain and displacement is described by

$$\{\varepsilon\} = [B]\{d\} \tag{6.6.11}$$

Where, the displacement vector will become:

$$\{d\}^T = \{u_1 \quad v_1 \quad w_1 \quad v_{11}v_{21} \quad \cdots \quad u_8 \quad v_8 \quad w_8 \quad v_{18}v_{28}\} \tag{6.6.12}$$

And the strain components will be

$$[\varepsilon] = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{Bmatrix} \tag{6.6.13}$$

Using eq. (6.6.6) in eq. (6.6.13) and then differentiating w.r.t. (ξ, η, ς) the strain displacement matrix will be obtained as

$$\begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\ \frac{\partial u}{\partial \varsigma} & \frac{\partial v}{\partial \varsigma} & \frac{\partial w}{\partial \varsigma} \end{bmatrix} = \sum_{i=1}^{8} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ 0 \end{Bmatrix} \begin{bmatrix} u_i & v_i & w_i \end{bmatrix} - \sum_{i=1}^{8} \frac{t_i v_{2i}}{2} \begin{Bmatrix} \varsigma \frac{\partial N_i}{\partial \xi} \\ \varsigma \frac{\partial N_i}{\partial \eta} \\ N_i \end{Bmatrix} \times \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}_i^T + \sum_{i=1}^{8} \frac{t_i v_{1i}}{2} \begin{Bmatrix} \varsigma \frac{\partial N_i}{\partial \xi} \\ \varsigma \frac{\partial N_i}{\partial \eta} \\ N_i \end{Bmatrix} \times \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}_i^T \tag{6.6.14}$$

6.6.5.3 Stress strain relation

The stress strain relationship is given by

$$\{\sigma\} = [D]\{\varepsilon\} \tag{6.6.15}$$

Using eq. (6.6.11) in eq. (6.6.15) one can find the following relation.

$$\{\sigma\} = [D][B]\{d\} \tag{6.6.16}$$

Where, the stress strain relationship matrix is represented by

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\alpha(1-\mu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha(1-\mu)}{2} \end{bmatrix} \tag{6.6.17}$$

The value of shear correction factor α is considered generally as 5/6. The above constitutive matrix can be split into two parts ([D_b] and [D_s])for adoption of different numerical integration schemes for bending and shear contributions to the stiffness matrix.

$$[D] = \begin{bmatrix} [D_b] & \vdots & [0] \\ \cdots & \cdots & \cdots \\ [0] & \vdots & [D_s] \end{bmatrix} \tag{6.6.18}$$

Thus,

$$[D_b] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \tag{6.6.19}$$

and

$$[D_s] = \frac{E\alpha}{2(1+\mu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6.6.20}$$

It may be important to note that the constitutive relation expressed in eq. (6.6.19) is same as for the case of plane stress formulation. Also, eq. (6.6.20) with a multiplication of thickness h is similar to the terms corresponds to shear force in case of plate bending problem.

6.6.5.4 Element stiffness matrix

Finally, the stiffness matrix for the shell element can be computed from the expression

$$[k] = \iiint [B]^T [D][B]\, d\Omega \tag{6.6.21}$$

However, it is convenient to divide the elemental stiffness matrix into two parts: (i) bending and membrane effect and (ii) transverse shear effects. This will facilitate the use of appropriate order of numerical integration of each part. Thus,

$$[k] = [k]_b + [k]_s \tag{6.6.22}$$

Where, contribution due to bending and membrane effects to stiffness is denoted as $[k]_b$ and transverse shear contribution to stiffness is denoted as $[k]_s$ and expressed in the following form.

$$[k]_b = \iiint [B]_b^T [D]_b [B]_b\, d\Omega \text{ and } [k]_s = \iiint [B]_s^T [D]_s [B]_s\, d\Omega \tag{6.6.23}$$

Numerical procedure will be used to evaluate the stiffness matrix. A 2 ×2 Gauss Quadrature can be used to evaluate the integral of $[k]_b$ and one point Gauss Quadrature may be used to integrate $[k]_s$ to avoid shear locking effect.