

Lecture 1: Constant Strain Triangle

The triangular elements with different numbers of nodes are used for solving two dimensional solid members. The linear triangular element was the first type of element developed for the finite element analysis of 2D solids. However, it is observed that the linear triangular element is less accurate compared to linear quadrilateral elements. But the triangular element is still a very useful element for its adaptivity to complex geometry. These are used if the geometry of the 2D model is complex in nature. Constant strain triangle (CST) is the simplest element to develop mathematically. In CST, strain inside the element has no variation (Ref. module 3, lecture 2) and hence element size should be small enough to obtain accurate results. As indicated earlier, the displacement is expressed in two orthogonal directions in case of 2D solid elements. Thus the displacement field can be written as

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} \quad (5.1.1)$$

Here, u and v are the displacements parallel to x and y directions respectively.

5.1.1 Element Stiffness Matrix for CST

A typical triangular element assumed to represent a subdomain of a plane body under plane stress/strain condition is represented in Fig. 5.1.1. The displacement (u , v) of any point P is represented in terms of nodal displacements

$$\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 \\ v &= N_1 v_1 + N_2 v_2 + N_3 v_3 \end{aligned} \quad (5.1.2)$$

Where, N_1 , N_2 , N_3 are the shape functions as described in module 3, lecture 2.

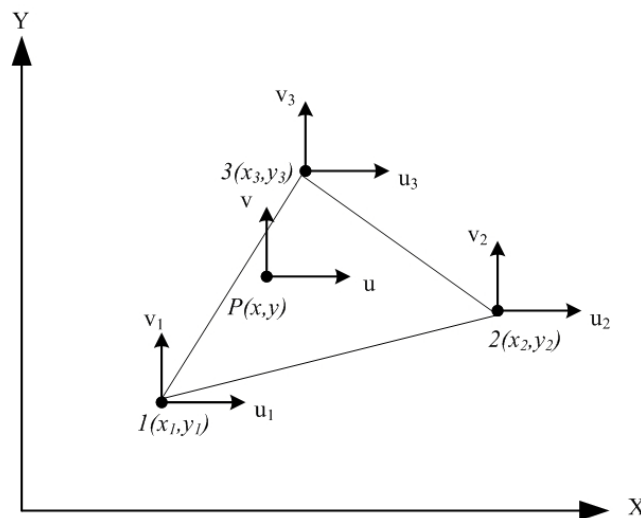


Fig. 5.1.1 Linear triangular element for plane stress/strain

The strain-displacement relationship for two dimensional plane stress/strain problem can be simplified in the following form from three dimensional cases (eq.1.3.9 to 1.3.14).

$$\begin{aligned}
 \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \\
 \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\
 \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right]
 \end{aligned} \tag{5.1.3}$$

In case of small amplitude of displacement, one can ignore the nonlinear term of the above equation and will reach the following expression.

$$\begin{aligned}
 \varepsilon_x &= \frac{\partial u}{\partial x} \\
 \varepsilon_y &= \frac{\partial v}{\partial y} \\
 \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}
 \end{aligned} \tag{5.1.4}$$

Hence the element strain components can be represented as,

$$\varepsilon = \begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3 \\ \varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} v_3 \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3 \end{cases}$$

Or,

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{5.1.5}$$

$$\text{Or, } \varepsilon = [B] \{d\} \tag{5.1.6}$$

In the above equation [B] is called as strain displacement relationship matrix. The shape functions for the 3 node triangular element in Cartesian coordinate is represented as,

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\ \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \\ \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \end{Bmatrix}$$

Or,

$$\begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2A}[\alpha_1 + \beta_1x + \gamma_1y] \\ \frac{1}{2A}[\alpha_2 + \beta_2x + \gamma_2y] \\ \frac{1}{2A}[\alpha_3 + \beta_3x + \gamma_3y] \end{Bmatrix} \quad (5.1.7)$$

Where,

$$\begin{aligned} \alpha_1 &= (x_2y_3 - x_3y_2), & \alpha_2 &= (x_3y_1 - x_1y_3), & \alpha_3 &= (x_1y_2 - x_2y_1), \\ \beta_1 &= (y_2 - y_3), & \beta_2 &= (y_3 - y_1), & \beta_3 &= (y_1 - y_2), \\ \gamma_1 &= (x_3 - x_2), & \gamma_2 &= (x_2 - x_1), & \gamma_3 &= (x_2 - x_1), \end{aligned} \quad (5.1.8)$$

Hence the required partial derivatives of shape functions are,

$$\begin{aligned} \frac{\partial N_1}{\partial x} &= \frac{\beta_1}{2A}, & \frac{\partial N_2}{\partial x} &= \frac{\beta_2}{2A}, & \frac{\partial N_3}{\partial x} &= \frac{\beta_3}{2A}, \\ \frac{\partial N_1}{\partial y} &= \frac{\gamma_1}{2A}, & \frac{\partial N_2}{\partial y} &= \frac{\gamma_2}{2A}, & \frac{\partial N_3}{\partial y} &= \frac{\gamma_3}{2A}, \end{aligned}$$

Hence the value of [B] becomes:

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$

$$\text{Or, } [B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \quad (5.1.9)$$

According to Variational principle described in module 2, lecture 1, the stiffness matrix is represented as,

$$[k] = \iiint_{\Omega} [B]^T [D][B] d\Omega \quad (5.1.10)$$

Since, [B] and [D] are constant matrices; the above expression can be expressed as

$$[k] = [B]^T [D][B] \iiint_V d \quad \forall [B]^T [D][B]V \quad (5.1.11)$$

For a constant thickness (t), the volume of the element will become $A.t$. Hence the above equation becomes,

$$[k] = [B]^T [D][B] At \quad (5.1.12)$$

For plane stress condition, [D] matrix will become:

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \quad (5.1.13)$$

Therefore, for a plane stress problem, the element stiffness matrix becomes,

$$[k] = \frac{Et}{4A(1-\mu^2)} \begin{bmatrix} \beta_1 & 0 & \gamma_1 \\ \beta_2 & 0 & \gamma_2 \\ \beta_3 & 0 & \gamma_3 \\ 0 & \gamma_1 & \beta_1 \\ 0 & \gamma_2 & \beta_2 \\ 0 & \gamma_3 & \beta_3 \end{bmatrix} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix} \quad (5.1.14)$$

Or,

$$[k] = \frac{Et}{4A(1-\mu^2)} \begin{bmatrix} \beta_1^2 + C\gamma_1^2 & \beta_1\beta_2 + C\gamma_1\gamma_2 & \beta_1\beta_3 + C\gamma_1\gamma_3 & \frac{(1+\mu)}{2}\beta_1\gamma_1 & \mu\beta_1\gamma_2 + C\beta_2\gamma_1 & \mu\beta_1\gamma_3 + C\beta_3\gamma_1 \\ & \beta_2^2 + C\gamma_2^2 & \beta_2\beta_3 + C\gamma_2\gamma_3 & \mu\beta_2\gamma_1 + C\beta_1\gamma_2 & \frac{(1+\mu)}{2}\beta_2\gamma_2 & \mu\beta_2\gamma_3 + C\beta_3\gamma_2 \\ & & \beta_3^2 + C\gamma_3^2 & \mu\beta_3\gamma_1 + C\beta_1\gamma_3 & \mu\beta_3\gamma_2 + C\beta_2\gamma_3 & \frac{(1+\mu)}{2}\beta_3\gamma_3 \\ & & & \gamma_1^2 + C\beta_1^2 & \gamma_1\gamma_2 + C\beta_1\beta_2 & \gamma_1\gamma_3 + C\beta_1\beta_3 \\ & & & & \gamma_2^2 + C\beta_2^2 & \gamma_2\gamma_3 + C\beta_2\beta_3 \\ & & & & & \gamma_3^2 + C\beta_3^2 \end{bmatrix} \quad (5.1.15)$$

Where, $C = \frac{(1-\mu)}{2}$

Similarly for plane strain condition, [D] matrix is equal to,

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \quad (5.1.16)$$

Hence the element stiffness matrix will become:

$$[k] = \frac{Et}{2A(1+\mu)} \begin{bmatrix} M\beta_1^2 + \gamma_1^2 & M\beta_1\beta_2 + \gamma_1\gamma_2 & M\beta_1\beta_3 + \gamma_1\gamma_3 & (\mu+1)\beta_1\gamma_1 & \mu\beta_1\gamma_2 + \beta_2\gamma_1 & \mu\beta_1\gamma_3 + \beta_3\gamma_1 \\ & M\beta_2^2 + \gamma_2^2 & M\beta_2\beta_3 + \gamma_2\gamma_3 & \mu\beta_2\gamma_1 + \beta_1\gamma_2 & (\mu+1)\beta_2\gamma_2 & \mu\beta_2\gamma_3 + \beta_3\gamma_2 \\ & & M\beta_3^2 + \gamma_3^2 & \mu\beta_3\gamma_1 + \beta_1\gamma_3 & \mu\beta_3\gamma_2 + \beta_2\gamma_3 & (\mu+1)\beta_3\gamma_3 \\ & & & M\gamma_1^2 + \beta_1^2 & M\gamma_1\gamma_2 + \beta_1\beta_2 & M\gamma_1\gamma_3 + \beta_1\beta_3 \\ & & & & M\gamma_2^2 + \beta_2^2 & M\gamma_2\gamma_3 + \beta_2\beta_3 \\ & & & & & M\gamma_3^2 + \beta_3^2 \end{bmatrix} \quad (5.1.17)$$

Sym.

Where $M = (1-\mu)$

5.1.2 Nodal Load Vector for CST

From the principle of virtual work,

$$\int_{\Omega} \delta \{\epsilon\}^T \{\sigma\} d\Omega = \int_{\Gamma} \delta \{u\}^T \{F_{\Gamma}\} d\Gamma + \int_{\Omega} \delta \{u\}^T \{F_{\Omega}\} d\Omega \quad (5.1.18)$$

Where, F_{Γ} , and F_{Ω} are the surface and body forces respectively. Using the relationship between stress-strain and strain displacement, one can derive the following expressions:

$$\{\sigma\} = [D][B]\{d\}, \quad \delta\{\epsilon\} = [B]\delta\{d\} \quad \text{and} \quad \delta\{u\} = [N]\delta\{d\} \quad (5.1.19)$$

Hence eq. (5.1.18) can be rewritten as,

$$\int_{\Omega} \delta \{d\}^T [B]^T [D] [B] \{d\} d\Omega = \int_{\Gamma} \delta \{d\}^T [N^s]^T \{F_{\Gamma}\} d\Gamma + \int_{\Omega} \delta \{d\}^T [N]^T \{F_{\Omega}\} d\Omega \quad (5.1.20)$$

$$\text{Or,} \quad \int_{\Omega} [B]^T [D] [B] \{d\} d\Omega = \int_{\Gamma} [N^s]^T \{F_{\Gamma}\} d\Gamma + \int_{\Omega} [N]^T \{F_{\Omega}\} d\Omega \quad (5.1.21)$$

Here, $[N^s]$ is the shape function along the boundary where forces are prescribed. Eq.(5.1.21) is equivalent to $[k]\{d\} = \{F\}$, and thus, the nodal load vector becomes

$$\{F\} = \int_{\Gamma} [N^s]^T \{F_{\Gamma}\} d\Gamma + \int_{\Omega} [N]^T \{F_{\Omega}\} d\Omega \quad (5.1.22)$$

For a constant thickness of the triangular element eq.(5.1.22) can be rewritten as

$$\{F\} = t \int_{S} [N^s]^T \{F_{\Gamma}\} ds + t \int_{A} [N]^T \{F_{\Omega}\} dA \quad (5.1.23)$$

For the a three node triangular two dimensional element, one can represent F_{Ω} and F_{Γ} as,

$$\{F_{\Omega}\} = \begin{Bmatrix} F_{\Omega x} \\ F_{\Omega y} \end{Bmatrix} \text{ and } \{F_{\Gamma}\} = \begin{Bmatrix} F_{\Gamma x} \\ F_{\Gamma y} \end{Bmatrix}$$

For example, in case of gravity load on CST element, $\{F_{\Omega}\} = \begin{Bmatrix} F_{\Omega x} \\ F_{\Omega y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix}$

For this case, the shape functions in terms of area coordinates are:

$$[N] = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix} \quad (5.1.24)$$

As a result, the force vector on the element considering only gravity load, will become,

$$\{F\} = t \int_A \begin{bmatrix} L_1 & 0 \\ L_2 & 0 \\ L_3 & 0 \\ 0 & L_1 \\ 0 & L_2 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} 0 \\ -\rho g \end{Bmatrix} dA = t \int_A \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -L_1 \rho g \\ -L_2 \rho g \\ -L_3 \rho g \end{Bmatrix} dA = -\rho g t \int_A \begin{Bmatrix} 0 \\ 0 \\ 0 \\ L_1 \\ L_2 \\ L_3 \end{Bmatrix} dA \quad (5.1.25)$$

The integration in terms of area coordinate is given by,

$$\int_A L_1^p L_2^q L_3^r dA = \frac{p! q! r!}{(p+q+r+2)!} 2A \quad (5.1.26)$$

Thus, the nodal load vector will finally become

$$\{F\} = -\rho g t \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{1!0!0!}{(1+0+0+2)!} 2A \\ \frac{0!1!0!}{(0+1+0+2)!} 2A \\ \frac{0!0!1!}{(0+0+1+2)!} 2A \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\frac{\rho g A t}{3} \\ -\frac{\rho g A t}{3} \\ -\frac{\rho g A t}{3} \end{Bmatrix} = -\frac{\rho g A t}{3} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \quad (5.1.27)$$

Lecture 2: Linear Strain Triangle

5.2.1 Element Stiffness Matrix for LST

In case of CST, it is observed that the strain within the element remains constant. Though, these elements are able to provide enough information about displacement pattern of the element, but it is unable to provide adequate information about stress inside an element. This limitation will be significant enough in regions of high strain gradients. The use of a higher order triangular element called Linear Strain Triangle (LST) significantly improves the results at these areas as the strain inside the element is varying. The LST element has six nodes (Fig. 5.2.1) and hence, twelve degrees of freedom. Thus the displacement function can be chosen as follows.

$$\begin{aligned} u &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 \\ v &= a_6 + a_7 x + a_8 y + a_9 x^2 + a_{10} xy + a_{11} y^2 \end{aligned} \quad (5.2.1)$$

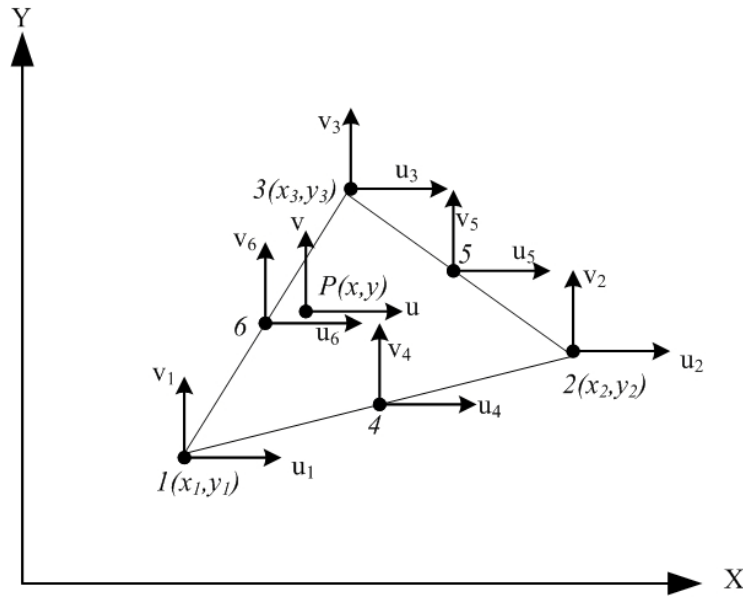


Fig. 5.2.1 Linear strain triangle element

Therefore, the element strain matrix is obtained as

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = \alpha_1 + 2\alpha_3 x + \alpha_4 y \\ \epsilon_y &= \frac{\partial v}{\partial y} = \alpha_8 + \alpha_{10} x + 2\alpha_{11} y \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = (\alpha_2 + \alpha_7) + (\alpha_4 + 2\alpha_9) x + (2\alpha_5 + \alpha_{10}) y \end{aligned} \quad (5.2.2)$$

In the area coordinate system as discussed in module 3, lecture 3 we can write the shape function for the six node triangular element as

$$\begin{aligned} N_1 &= L_1(2L_1 - 1) & N_2 &= L_2(2L_2 - 1) & N_3 &= L_3(2L_3 - 1) \\ N_4 &= 4L_1L_2 & N_5 &= 4L_2L_3 & N_6 &= 4L_3L_1 \end{aligned} \quad (5.2.3)$$

The displacement (u, v) of any point within the element can be represented in terms of their nodal displacements with the use of interpolation function.

$$\begin{aligned} u &= \sum_{i=1}^6 N_i u_i \\ v &= \sum_{i=1}^6 N_i v_i \end{aligned} \quad (5.2.4)$$

Using eq.(5.2.4) we can rewrite eq.(5.2.2) as,

$$\varepsilon = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

Or,

$$\varepsilon = [B]\{d\} \quad (5.2.5)$$

Where,

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \end{bmatrix} \quad (5.2.6)$$

Using Chain rule,

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_1}{\partial L_1} \cdot \frac{\partial L_1}{\partial x} + \frac{\partial N_1}{\partial L_2} \cdot \frac{\partial L_2}{\partial x} + \frac{\partial N_1}{\partial L_3} \cdot \frac{\partial L_3}{\partial x}$$

As discussed in module 3, lecture 1, we can write the above expression as,

$$\frac{\partial N_1}{\partial x} = \frac{b_1}{2A} \cdot \frac{\partial N_1}{\partial L_1} + \frac{b_2}{2A} \cdot \frac{\partial N_1}{\partial L_2} + \frac{b_3}{2A} \cdot \frac{\partial N_1}{\partial L_3}$$

$$\frac{\partial N_1}{\partial x} = \frac{b_1}{2A} \cdot (4L_1 - 1)$$

Similarly we can evaluate expressions for other terms and can be written as,

$$\begin{aligned} \frac{\partial N_1}{\partial x} &= \frac{b_1}{2A} \cdot (4L_1 - 1) & \frac{\partial N_2}{\partial x} &= \frac{b_2}{2A} \cdot (4L_2 - 1) & \frac{\partial N_3}{\partial x} &= \frac{b_3}{2A} \cdot (4L_3 - 1) \\ \frac{\partial N_4}{\partial x} &= 4(L_2 b_1 + L_1 b_2) & \frac{\partial N_5}{\partial x} &= 4(L_3 b_2 + L_2 b_3) & \frac{\partial N_6}{\partial x} &= 4(L_1 b_3 + L_3 b_1) \end{aligned}$$

And,

$$\begin{aligned} \frac{\partial N_1}{\partial y} &= \frac{a_1}{2A} \cdot (4L_1 - 1) & \frac{\partial N_2}{\partial y} &= \frac{a_2}{2A} \cdot (4L_2 - 1) & \frac{\partial N_3}{\partial y} &= \frac{a_3}{2A} \cdot (4L_3 - 1) \\ \frac{\partial N_4}{\partial y} &= 4(L_2 a_1 + L_1 a_2) & \frac{\partial N_5}{\partial y} &= 4(L_3 a_2 + L_2 a_3) & \frac{\partial N_6}{\partial y} &= 4(L_1 a_3 + L_3 a_1) \end{aligned}$$

Where,

$$\begin{aligned} a_1 &= x_2 - x_3 & a_2 &= x_3 - x_1 & a_3 &= x_1 - x_2 \\ b_1 &= y_2 - y_3 & b_2 &= y_3 - y_1 & b_3 &= y_1 - y_2 \end{aligned}$$

The stiffness matrix of the element is represented by,

$$[k] = \iiint_{\Omega} [B]^T [D][B] d\Omega \quad (5.2.7)$$

The, [D] matrix is the constitutive matrix which will be taken according to plane stress or plane strain condition. The nodal strain and stress vectors are given by,

$$\{\varepsilon_n\} = \{\varepsilon_{x1} \quad \varepsilon_{x2} \quad \varepsilon_{x3} \quad \varepsilon_{y1} \quad \varepsilon_{y2} \quad \varepsilon_{y3} \quad \gamma_{xy1} \quad \gamma_{xy2} \quad \gamma_{xy3}\}^T \quad (5.2.9a)$$

$$\{\sigma_n\} = \{\sigma_{x1} \quad \sigma_{x2} \quad \sigma_{x3} \quad \sigma_{y1} \quad \sigma_{y2} \quad \sigma_{y3} \quad \tau_{xy1} \quad \tau_{xy2} \quad \tau_{xy3}\}^T \quad (5.2.9b)$$

$$\{\varepsilon_n\} = \begin{bmatrix} [B_{n1}] & [0] \\ [0] & [B_{n2}] \\ [B_{n2}] & [B_{n1}] \end{bmatrix} \{d\} \quad (5.2.10)$$

Referring to section 3.3.1, using proper values of area coordinates in [B] matrix, one can find

$$[B_{n1}] = \frac{1}{2A} \begin{bmatrix} 3b_1 & -b_2 & -b_3 & 4b_2 & 0 & 4b_3 \\ -b_1 & 3b_2 & -b_3 & 4b_1 & 4b_3 & 0 \\ -b_1 & -b_2 & 3b_3 & 0 & 4b_2 & 4b_1 \end{bmatrix} \quad (5.2.11a)$$

And,

$$[B_{n2}] = \frac{1}{2A} \begin{bmatrix} 3a_1 & -a_2 & -a_3 & 4a_2 & 0 & 4a_3 \\ -a_1 & 3a_2 & -a_3 & 4a_1 & 4a_3 & 0 \\ -a_1 & -a_2 & 3a_3 & 0 & 4a_2 & 4a_1 \end{bmatrix} \quad (5.2.11b)$$

Thus, the element stiffness can be evaluated by putting the values from eq. (5.2.11) in eq. (5.2.7).

5.2.2 Nodal Load Vector for LST

Similar to 3-node triangular element, the load will be lumped at each node which can be computed using the earlier expression,

$$\{F\} = \int_{\Gamma} [N^s]^T \{F_r\} d\Gamma + \int_{\Omega} [N]^T \{F_{\Omega}\} d\Omega \quad (5.2.12)$$

And for element with constant thickness,

$$\{F\} = t \int_s [N^s]^T \{F_r\} ds + t \int_A [N]^T \{F_{\Omega}\} dA \quad (5.2.13)$$

5.2.3 Numerical Example using CST

Determine the displacements at the nodes for the following 2D solid continuum considering a constant thickness of 25 mm, Poisson's ratio, μ as 0.25 and modulus of elasticity E as 2×10^5 N/mm². The continuum is discretized with two CST plane stress elements.

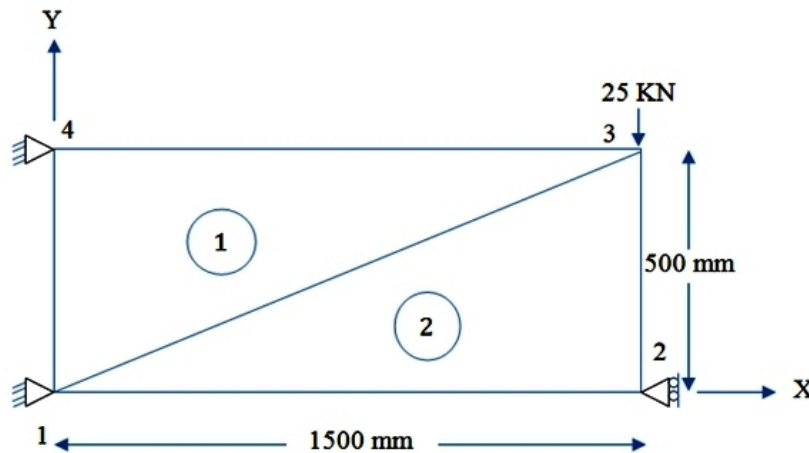


Fig. 5.2.2 Geometry and discretization of the continuum

The element 1 is connected with node 1, 3 and 4 and let assume its Cartesian coordinates are (x_1, y_1) , (x_3, y_3) and (x_4, y_4) respectively. If we consider nodes 1, 3 and 4 are similar to node 1, 2 and 3 in eq.(5.1.9) then the $[B]$ can be written as

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

By introducing values of β & γ discussed in previous lecture note, we can get value of $[B]$ as

$$[B] = \frac{1}{1500} \begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ -3 & 0 & 3 & 0 & 1 & -1 \end{bmatrix}$$

For plain stress problem, putting the values of E and μ one can find the following values.

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = \frac{4 \times 10^4}{3} \begin{bmatrix} 16 & 4 & 0 \\ 4 & 16 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Therefore the stiffness matrix for the element 1 will be

$$[k]_1 = tA[B]^T [D][B]$$

Putting values of t , A , $[B]$ & $[D]$ we will get,

$$[k]_1 = 4 \times 10^4 \times \begin{bmatrix} 750 & 0 & -750 & 0 & -250 & -250 \\ 0 & 222.2222 & -222.2222 & -166.6667 & 0 & 166.6667 \\ -750 & -222.2222 & 972.2222 & 166.6667 & 250 & -416.6667 \\ 0 & -166.6667 & 166.6667 & 2000 & 0 & -2000 \\ -250 & 0 & 250 & 0 & 83.3333 & -83.3333 \\ 250 & 166.6667 & -416.6667 & -2000 & -83.3333 & 2083.3333 \end{bmatrix}$$

Similarly element 2 is connected with nodes 1, 2 and 3 and global coordinates of these nodes are (x_1, y_1) , (x_2, y_2) and (x_3, y_3) respectively. For this element, by proceeding in a similar manner to element 1 we can calculate $[B]$ matrix as,

$$[B] = \frac{1}{1500} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & -3 & 3 & -1 & 1 & 0 \end{bmatrix}$$

Hence, the elemental stiffness matrix becomes,

$$[k]_2 = 4 \times 10^4 \times \begin{bmatrix} 222.2222 & -222.2222 & 0 & 0 & 166.6667 & -166.6667 \\ -222.2222 & 972.2222 & -750 & 250 & -416.6667 & 166.6667 \\ 0 & -750 & 750 & -250 & 250 & 0 \\ 0 & 250 & -250 & 83.3333 & -83.3333 & 0 \\ 166.6667 & -416.6667 & 250 & -83.3333 & 2083.3333 & -2000 \\ -166.6667 & 166.6667 & 0 & 0 & -2000 & 2000 \end{bmatrix}$$

By assembling the stiffness matrices into global stiffness matrix [K],

$$[K] = 4 \times 10^4 \times \begin{bmatrix} 972.2222 & -222.2222 & 0 & -750 & 0 & 166.6667 & -416.6667 & -250 \\ -222.2222 & 972.2222 & -750 & 0 & 250 & -416.6667 & 166.6667 & 0 \\ 0 & -750 & 972.2222 & -222.2222 & -416.6667 & 250 & 0 & 166.6667 \\ -750 & 0 & -222.2222 & 972.2222 & 166.6667 & 0 & 250 & -416.6667 \\ 0 & 250 & -416.6667 & 166.6667 & 2083.3333 & -83.3333 & 0 & -2000 \\ 166.6667 & -416.6667 & 250 & 0 & -83.3333 & 2083.3333 & -2000 & 0 \\ -416.6667 & 166.6667 & 0 & 250 & 0 & -2000 & 2083.3333 & -83.3333 \\ 250 & 0 & 166.6667 & -416.6667 & -2000 & 0 & -83.3333 & 2083.3333 \end{bmatrix}$$

Now, applying equation $[F] = [K]\{d\}$, the following expression can be written.

$$\begin{Bmatrix} F_{u1} \\ F_{u2} \\ F_{u3} \\ F_{u4} \\ F_{v1} \\ F_{v2} \\ F_{v3} \\ F_{v4} \end{Bmatrix} = 4 \times 10^4 \times \begin{bmatrix} 972.2222 & -222.2222 & 0 & -750 & 0 & 166.6667 & -416.6667 & -250 \\ -222.2222 & 972.2222 & -750 & 0 & 250 & -416.6667 & 166.6667 & 0 \\ 0 & -750 & 972.2222 & -222.2222 & -416.6667 & 250 & 0 & 166.6667 \\ -750 & 0 & -222.2222 & 972.2222 & 166.6667 & 0 & 250 & -416.6667 \\ 0 & 250 & -416.6667 & 166.6667 & 2083.3333 & -83.3333 & 0 & -2000 \\ 166.6667 & -416.6667 & 250 & 0 & -83.3333 & 2083.3333 & -2000 & 0 \\ -416.6667 & 166.6667 & 0 & 250 & 0 & -2000 & 2083.3333 & -83.3333 \\ 250 & 0 & 166.6667 & -416.6667 & -2000 & 0 & -83.3333 & 2083.3333 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

Putting boundary conditions $u_1 = v_1 = u_2 = u_4 = v_4 = 0$ and adopting elimination technique for applying boundary condition we get expression,

$$\begin{Bmatrix} 0 \\ 25000 \\ 0 \end{Bmatrix} = 4 \times 10^4 \times \begin{bmatrix} 972.2222 & 250 & 0 \\ 250 & 2083.3333 & 2000 \\ 0 & 2000 & 2083.3333 \end{bmatrix} \begin{Bmatrix} v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

Solving the above expression, the unknown nodal displacements may be obtained as follows.

$$v_2 = 25.96 \times 10^{-5} \text{ mm}, u_3 = -10.02 \times 10^{-5} \text{ mm and } v_3 = 96.92 \times 10^{-5} \text{ mm.}$$

Lecture 3: Rectangular Elements

The rectangular elements are widely used for solving two dimensional continuums. The main advantage of this type of element is the easy formulation and easy development of computer code. The element stiffness of such elements is derived here using the concept of isoparametric formulation.

5.3.1 Computation of Element Stiffness

In case of a four node rectangular element, the geometry and displacement field can be expressed in terms of their nodal values with the help of interpolation function. As the formulation will be isoparametric, the interpolation function will become same for expressing both the variables. Thus, coordinates and displacements at any point inside the element (Fig. 5.3.1) can be expressed as

$$\begin{aligned}x &= N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \\y &= N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4\end{aligned}\tag{5.3.1}$$

And

$$\begin{aligned}u &= N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 \\v &= N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4\end{aligned}\tag{5.3.2}$$

The above equations can be written in matrix form as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}\tag{5.3.3}$$

And

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \quad (5.3.4)$$

The shape function four node rectangular element is derived and shown in module 3, lecture 4. However the shape functions are reproduced here for easy reference for the derivation of the stiffness matrix.

$$N_i = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} = \begin{Bmatrix} \frac{(1-\xi)(1-\eta)}{4} \\ \frac{(1+\xi)(1-\eta)}{4} \\ \frac{(1+\xi)(1+\eta)}{4} \\ \frac{(1-\xi)(1+\eta)}{4} \end{Bmatrix} \quad (5.3.5)$$

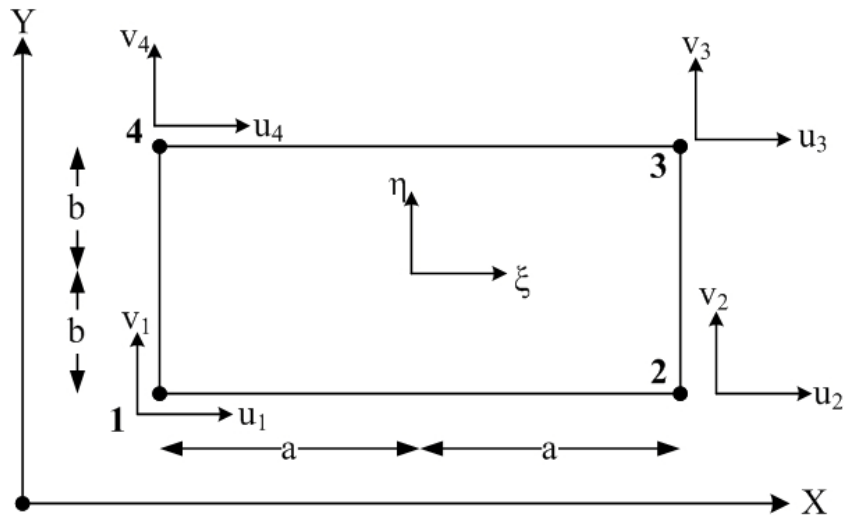


Fig. 5.3.1 Four node rectangular element

The strain-displacement relationship for two dimensional plane stress/strain problem can be simplified in the following form from three dimensional cases (eq.1.3.9 to 1.3.14).

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \\ \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right]\end{aligned}\tag{5.3.6}$$

In case of small amplitude of displacement, one can ignore the nonlinear term of the above equation and will reach the following expression.

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\end{aligned}\tag{5.3.7}$$

Using the shape function the above expression can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = [B] \{d\}\tag{5.3.8}$$

Here, [B] is known as strain displacement relationship matrix. The derivatives of the shape functions are calculated using the chain rule.

$$\begin{aligned}\frac{\partial N_i}{\partial \xi} &= \frac{\partial N_i}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial y}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} &= \frac{\partial N_i}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \cdot \frac{\partial y}{\partial \eta}\end{aligned}\tag{5.3.9}$$

Here, i is referred to number of nodes in an element and will be 4 in this case. Converting above expression in matrix form

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \quad (5.3.10)$$

The matrix $[J]$ is referred to Jacobian matrix which is discussed in Lecture 7, module 3. Using eq. (5.3.1) one can write

$$\frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4 \quad (5.3.11)$$

Putting the values of the nodal coordinates and shape functions of the four node element in the above equation the following relations will be obtained.

$$\frac{\partial x}{\partial \xi} = -\frac{1 \times (1 - \eta)}{4} \times 0 + \frac{1 \times (1 - \eta)}{4} \times a + \frac{1 \times (1 + \eta)}{4} \times a - \frac{1 \times (1 + \eta)}{4} \times 0 = \frac{a}{2} \quad (5.3.12a)$$

Similarly,

$$\frac{\partial y}{\partial \xi} = -\frac{1 \times (1 - \eta)}{4} \times 0 + \frac{1 \times (1 - \eta)}{4} \times 0 + \frac{1 \times (1 + \eta)}{4} \times b - \frac{1 \times (1 + \eta)}{4} \times b = 0 \quad (5.3.12b)$$

$$\frac{\partial x}{\partial \eta} = -\frac{1 \times (1 - \xi)}{4} \times 0 - \frac{1 \times (1 + \xi)}{4} \times a + \frac{1 \times (1 + \xi)}{4} \times a + \frac{1 \times (1 - \xi)}{4} \times 0 = 0 \quad (5.3.12c)$$

$$\frac{\partial y}{\partial \eta} = -\frac{1 \times (1 - \xi)}{4} \times 0 - \frac{1 \times (1 + \xi)}{4} \times 0 + \frac{1 \times (1 + \xi)}{4} \times b + \frac{1 \times (1 - \xi)}{4} \times b = \frac{b}{2} \quad (5.3.12d)$$

Substituting above values in Jacobian matrix the following relations will be obtained.

$$[J] = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{b}{2} \end{bmatrix} \text{ and } [J]^{-1} = \begin{bmatrix} \frac{2}{a} & 0 \\ 0 & \frac{2}{b} \end{bmatrix} \quad (5.3.13)$$

Thus, eq.(5.3.10) can be written as

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{2}{a} & 0 \\ 0 & \frac{2}{b} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{2}{a} \cdot \frac{\partial N_i}{\partial \xi} \\ \frac{2}{b} \cdot \frac{\partial N_i}{\partial \eta} \end{Bmatrix} \quad (5.3.14)$$

After derivation of the shape functions expressed in eq.(5.3.5), the following values will be obtained.

$$\begin{aligned} \frac{\partial N_1}{\partial x} &= \frac{2}{a} \cdot \frac{\partial N_1}{\partial \xi} = -\frac{(1-\eta)}{2a}; & \frac{\partial N_2}{\partial x} &= \frac{(1-\eta)}{2a}; & \frac{\partial N_3}{\partial x} &= \frac{(1+\eta)}{2a}; & \frac{\partial N_4}{\partial x} &= -\frac{(1+\eta)}{2a} \\ \frac{\partial N_1}{\partial y} &= \frac{2}{b} \cdot \frac{\partial N_1}{\partial \eta} = -\frac{(1-\xi)}{2b}; & \frac{\partial N_2}{\partial y} &= -\frac{(1+\xi)}{2b}; & \frac{\partial N_3}{\partial y} &= \frac{(1+\xi)}{2b}; & \frac{\partial N_4}{\partial y} &= \frac{(1-\xi)}{2b} \end{aligned} \quad (5.3.15)$$

So, the strain displacement relationship matrix, $[B]$ will become as follows.

$$[B] = \begin{bmatrix} -\frac{(1-\eta)}{2a} & 0 & \frac{(1-\eta)}{2a} & 0 & \frac{(1+\eta)}{2a} & 0 & -\frac{(1+\eta)}{2a} & 0 \\ 0 & -\frac{(1-\xi)}{2b} & 0 & -\frac{(1+\xi)}{2b} & 0 & \frac{(1+\xi)}{2b} & 0 & \frac{(1-\xi)}{2b} \\ -\frac{(1-\xi)}{2b} & -\frac{(1-\eta)}{2a} & -\frac{(1+\xi)}{2b} & \frac{(1-\eta)}{2a} & \frac{(1+\xi)}{2b} & \frac{(1+\eta)}{2a} & \frac{(1-\xi)}{2b} & -\frac{(1+\eta)}{2a} \end{bmatrix} \quad (5.3.16)$$

The element stiffness matrix will become

$$[k] = t \iint [B]^T [D][B] dx dy = t \iint [B]^T [D][B]|J| d\xi d\eta \quad (5.3.17)$$

It is seen that the above is expressed in terms of ξ and η and hence can be numerically integrated by the Gauss Quadrature rule. The stiffness matrix for each element can be found which needs to be globally assembled for getting the global stiffness matrix to obtain the solution. The stiffness matrix of higher order rectangular element can be derived in a similar fashion. For example, in case of eight node rectangle element, the size of $[B]$ matrix will become 16×3 which was 8×3 for four node element. Thus the size of element stiffness for eight node element will become 16×16 .

5.3.2 Computation of Nodal Loads

If a distributed load acts on a side of a four node rectangular element, the nodal load vector can be calculated the similar procedure as discussed in case of triangular element. If an element as shown below is subjected to a linearly varying intensities of load at its one side, then the magnitude of this at any point on the side can be expressed by its interpolation function as follows.

$$q_x = \begin{bmatrix} \frac{1-\eta}{2} & \frac{1+\eta}{2} \end{bmatrix} \begin{Bmatrix} q_{x2} \\ q_{x3} \end{Bmatrix} \quad (5.3.18)$$

Here, q_{x2} and q_{x3} are the force intensities per unit length at nodes 2 and 3 respectively. The load at nodes can be calculated from the following expression.

$$\{F_x\} = \int \{N_2^S\} q_x d\Gamma_1 \quad (5.3.19)$$

As $\xi=1$ along the side 2-3, the interpolation function will become

$$\{N_2^s\} = \begin{Bmatrix} \frac{(1-\xi)(1-\eta)}{4} \\ \frac{(1+\xi)(1-\eta)}{4} \\ \frac{(1+\xi)(1+\eta)}{4} \\ \frac{(1-\xi)(1+\eta)}{4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{(1-\eta)}{2} \\ \frac{(1+\eta)}{2} \\ 0 \end{Bmatrix} \quad (5.3.20)$$

If the element thickness is t , then $d\Gamma_1 = t \cdot dl$. Thus the eq.(5.3.19) can be replaced as

$$\{F_x\} = t \int_{-1}^{+1} \begin{Bmatrix} 0 \\ \frac{(1-\eta)}{2} \\ \frac{(1+\eta)}{2} \\ 0 \end{Bmatrix} \begin{bmatrix} \frac{1-\eta}{2} & \frac{1+\eta}{2} \end{bmatrix} \begin{Bmatrix} q_{x2} \\ q_{x3} \end{Bmatrix} dl \quad (5.3.21)$$

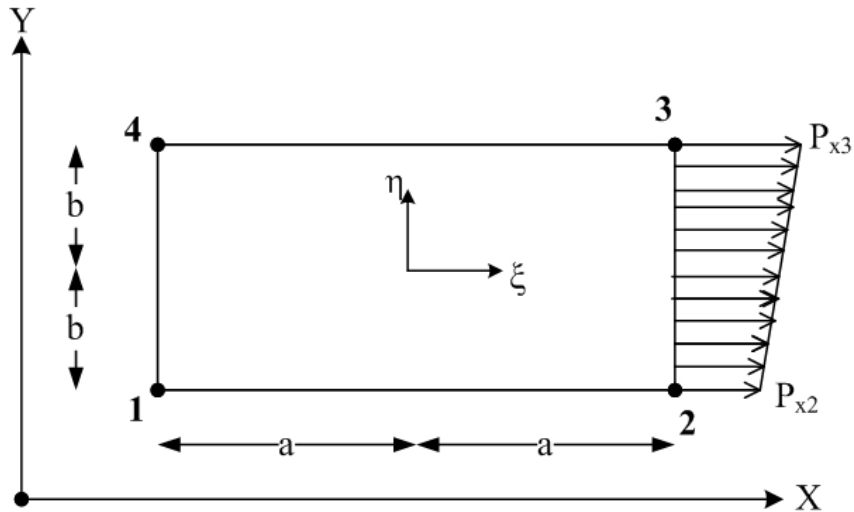


Fig. 5.3.2 Varying load on a four node element

After integrating the above expression, the nodal load vector along x direction will become as follows.

$$\{F_x\} = \frac{t}{3} \begin{Bmatrix} 0 \\ 2q_{x2} + q_{x3} \\ q_{x2} + 2q_{x3} \\ 0 \end{Bmatrix} \quad (5.3.22)$$

Lecture 4: Numerical Evaluation of Element Stiffness

Derivation of element stiffness for a four node rectangle element has been demonstrated in last lecture. The stiffness matrix of each element can be calculated easily by developing a suitable computer algorithm. To help students for developing their own computer code, a numerical example has been solved and demonstrated here.

5.4.1 Numerical Example

Calculate the stiffness matrix for the given four node rectangular element by the Gauss Quadrature integration rule using one point and two point formula assuming plane stress formulation. Consider, the thickness of element = 20 cm, $E=2 \times 10^3$ kN/cm² and $\mu=0$.

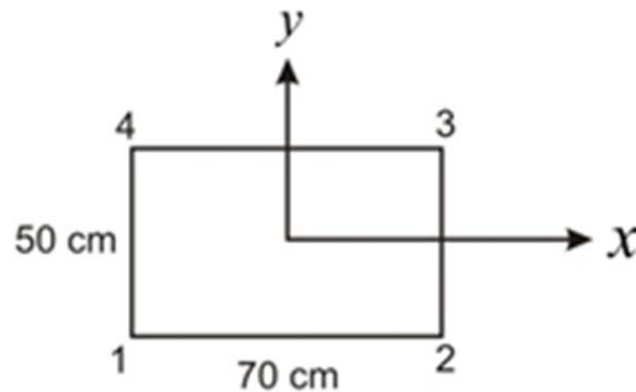


Fig. 5.4.1 Element Dimension

5.4.2 Evaluation of Stiffness using One Point Gauss Quadrature

For the calculation of stiffness matrix, first, 1×1 Gauss Quadrature integration procedure has been carried out. Thus, the natural coordinate of the sampling point will become 0,0 and weight will become 2.0 which is shown in the figure below.

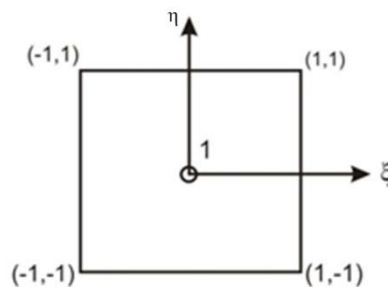


Fig. 5.4.2 Natural coordinates for one point Gauss Quadrature

For a four node quadrilateral element, the shape functions and their derivatives are as follows.

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}, \quad N_2 = \frac{(1+\xi)(1-\eta)}{4} \quad \text{and} \quad N_3 = \frac{(1+\xi)(1+\eta)}{4} \quad \text{and} \quad N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \xi} = \frac{-(1-\eta)}{4}; \quad \frac{\partial N_2}{\partial \xi} = \frac{(1-\eta)}{4}; \quad \frac{\partial N_3}{\partial \xi} = \frac{(1+\eta)}{4}; \quad \frac{\partial N_4}{\partial \xi} = \frac{-(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \eta} = \frac{-(1-\xi)}{4}; \quad \frac{\partial N_2}{\partial \eta} = \frac{-(1+\xi)}{4}; \quad \frac{\partial N_3}{\partial \eta} = \frac{(1+\xi)}{4}; \quad \frac{\partial N_4}{\partial \eta} = \frac{(1-\xi)}{4}$$

The Jacobian matrix can be found from the following relations.

$$|J| = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} \frac{-(1-\eta)}{4} & \frac{(1-\eta)}{4} & \frac{(1+\eta)}{4} & \frac{-(1+\eta)}{4} \\ \frac{-(1-\xi)}{4} & \frac{-(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Considering the sampling point, ($\xi=0$ and $\eta=0$), the value of the Jacobian, $[J]$ is

$$[J] = \begin{bmatrix} \frac{-1}{4} & \frac{+1}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 70 & 0 \\ 70 & 50 \\ 0 & 50 \end{bmatrix}$$

Thus, $[J] = \begin{bmatrix} 35 & 0 \\ 0 & 25 \end{bmatrix}$ and $[J]^{-1} = \begin{bmatrix} \frac{1}{35} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix}$; and $|J| = 875$

Now, the strain vector for the element will become

$$[\varepsilon] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times [B'] \{d\} = [B] \{d\}$$

$$[B'] = \begin{bmatrix} -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ \frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix} \text{ and } [B] = \begin{bmatrix} \frac{1}{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{25} \\ 0 & \frac{1}{25} & \frac{1}{35} & 0 \end{bmatrix} [B']$$

$$[B] = \begin{bmatrix} -7.143 & 0 & 7.143 & 0 & 7.143 & 0 & -7.143 & 0 \\ 0 & -10 & 0 & -10 & 0 & 10 & 0 & 10 \\ -10 & -7.143 & -10 & 7.143 & 10 & 7.143 & 10 & -7.143 \end{bmatrix}$$

For plane stress condition

$$[C] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = 2 \times 10^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$[C][B] = 2 \times 10^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \times$$

$$\begin{bmatrix} -7.143 & 0 & 7.143 & 0 & 7.143 & 0 & -7.143 & 0 \\ 0 & -10 & 0 & -10 & 0 & 10 & 0 & 10 \\ -10 & -7.143 & -10 & 7.143 & 10 & 7.143 & 10 & -7.143 \end{bmatrix}$$

Assume the values of gauss weight, $w = 2$, the stiffness matrix $[k]$ at this sampling point is

$[k] = tw [B]^T [C] [B] |J|$, Where t is thickness of the element. Thus,

$$[k] = 20 \times 2 \times 875 \times 2 \times 10^3 \times \begin{bmatrix} -7.143 & 0 & -10 \\ 0 & -10 & -7.143 \\ 7.143 & 0 & -10 \\ 0 & -10 & 7.143 \\ 7.143 & 0 & 10 \\ 0 & 10 & 7.143 \\ -7.143 & 0 & 4 \\ 0 & 10 & -7.143 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\times \begin{bmatrix} -7.143 & 0 & 7.143 & 0 & 7.143 & 0 & -7.143 & 0 \\ 0 & -10 & 0 & -10 & 0 & 10 & 0 & 10 \\ -10 & -7.143 & -10 & 7.143 & 10 & 7.143 & 10 & -7.143 \end{bmatrix}$$

$$= 10^3 \times \begin{bmatrix} 7.071 & 2.5 & -0.714 & -2.5 & -7.071 & -2.5 & 0.714 & 2.5 \\ 2.5 & 8.785 & 2.5 & 5.214 & -2.5 & -8.785 & 2.5 & -5.214 \\ -0.714 & 2.5 & 7.071 & -2.5 & 0.714 & -2.5 & -7.071 & 2.5 \\ -2.5 & 5.214 & -2.5 & 8.785 & 2.5 & -5.214 & 2.5 & -8.785 \\ -7.071 & -2.5 & 0.714 & 2.5 & 7.071 & 2.5 & -0.714 & -2.5 \\ -2.5 & -8.785 & -2.5 & -5.214 & 2.5 & 8.785 & 2.5 & 5.214 \\ 0.714 & -2.5 & -7.071 & 2.5 & -0.714 & 2.5 & 7.071 & -2.5 \\ 2.5 & -5.214 & 2.5 & -8.785 & -2.5 & 5.214 & -2.5 & 8.785 \end{bmatrix}$$

5.4.3 Evaluation of Stiffness using Two Point Gauss Quadrature

In this case, 2×2 Gauss Quadrature integration procedure has been carried out to calculate the stiffness matrix of the same element for a comparison. The natural coordinate of the sampling point is shown in the figure below.

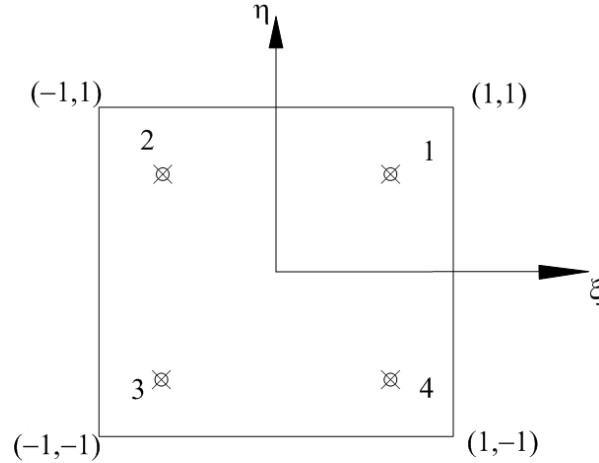


Fig. 5.4.3 Natural Coordinates for Two Points Gauss Quadrature

The natural co-ordinates of the sampling points for 2×2 Gauss Quadrature integration are

1	+0.57735	+0.57735
2	-0.57735	+0.57735
3	-0.57735	-0.57735
4	+0.57735	-0.57735

For a four node quadrilateral element, the shape functions and their derivatives are as follows.

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}, \quad N_2 = \frac{(1+\xi)(1-\eta)}{4}, \quad N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \xi} = \frac{-(1-\eta)}{4}; \quad \frac{\partial N_2}{\partial \xi} = \frac{(1-\eta)}{4}; \quad \frac{\partial N_3}{\partial \xi} = \frac{(1+\eta)}{4}; \quad \frac{\partial N_4}{\partial \xi} = \frac{-(1+\eta)}{4}$$

$$\frac{\partial N_1}{\partial \eta} = \frac{-(1-\xi)}{4}; \quad \frac{\partial N_2}{\partial \eta} = \frac{-(1+\xi)}{4}; \quad \frac{\partial N_3}{\partial \eta} = \frac{(1+\xi)}{4}; \quad \frac{\partial N_4}{\partial \eta} = \frac{(1-\xi)}{4}$$

The Jacobian matrix will be

$$|J| = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} \frac{-(1-\eta)}{4} & \frac{+(1-\eta)}{4} & \frac{+(1+\eta)}{4} & \frac{-(1+\eta)}{4} \\ \frac{-(1-\xi)}{4} & \frac{-(1+\xi)}{4} & \frac{+(1+\xi)}{4} & \frac{+(1-\xi)}{4} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

(a) At sampling point 1, ($\xi=0.57735$, $\eta=0.57735$)

The value of the Jacobian, $[J]$ at sampling point 1 will become

$$[J] = \begin{bmatrix} \frac{-(1-0.57735)}{4} & \frac{+(1-0.57735)}{4} & \frac{+(1+0.57735)}{4} & \frac{-(1+0.57735)}{4} \\ \frac{-(1-0.57735)}{4} & \frac{-(1+0.57735)}{4} & \frac{+(1+0.57735)}{4} & \frac{+(1-0.57735)}{4} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 70 & 0 \\ 70 & 50 \\ 0 & 50 \end{bmatrix}$$

$$\text{Thus, } [J] = \begin{bmatrix} 35 & 0 \\ 0 & 25 \end{bmatrix} \text{ and } [J]^{-1} = \begin{bmatrix} \frac{1}{35} & 0 \\ 0 & \frac{1}{25} \end{bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix}; \text{ Thus } |J| = 875$$

$$[\varepsilon] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \times [B']\{d\} = [B]\{d\}$$

$$[B'] = \begin{bmatrix} \frac{1-0.57735}{4} & 0 & \frac{1-0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & -\frac{1+0.57735}{4} & 0 \\ \frac{1-0.57735}{4} & 0 & -\frac{1+0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & \frac{1-0.57735}{4} & 0 \\ 0 & -\frac{1-0.57735}{4} & 0 & \frac{1-0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & -\frac{1+0.57735}{4} \\ 0 & -\frac{1-0.57735}{4} & 0 & -\frac{1+0.57735}{4} & 0 & \frac{1+0.57735}{4} & 0 & \frac{1-0.57735}{4} \end{bmatrix}$$

$$[B'] = \begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 & 0 \\ -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 & 0 \\ 0 & -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 \\ 0 & -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 \end{bmatrix}$$

Thus,

$$[B] = \begin{bmatrix} \frac{1}{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{25} \\ 0 & \frac{1}{25} & \frac{1}{35} & 0 \end{bmatrix} \times \begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 & 0 \\ -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 & 0 \\ 0 & -0.1057 & 0 & 0.1057 & 0 & 0.3943 & 0 & -0.3943 \\ 0 & -0.1057 & 0 & -0.3943 & 0 & 0.3943 & 0 & 0.1057 \end{bmatrix}$$

$$[B] = \begin{bmatrix} -0.003 & 0 & 0.003 & 0 & 0.0113 & 0 & -0.0113 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0042 & -0.003 & -0.0158 & 0.003 & 0.0158 & 0.0113 & 0.0042 & -0.0113 \end{bmatrix}$$

For plane stress condition

$$[C] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = 2 \times 10^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$[C][B] = 2 \times 10^3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.003 & 0 & 0.003 & 0 & 0.0113 & 0 & -0.0113 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0042 & -0.003 & -0.0158 & 0.003 & 0.0158 & 0.0113 & 0.0042 & -0.0113 \end{bmatrix}$$

The values of gauss weights are $w_i=w_j=1.0$. Therefore, the stiffness matrix $[k]$ at this sampling point is $[k] = tw_iw_j[B]_{ij}^T[C]_{ij}[B]_{ij} |J|$, where t is thickness of the element. Thus at sampling point 1,

$$[k_1] = 20 \times 1 \times 1 \times 875 \times 2 \times 10^3 \times \begin{bmatrix} -0.003 & 0 & -0.0042 \\ 0 & -0.0042 & -0.003 \\ 0.003 & 0 & -0.0158 \\ 0 & -0.0158 & 0.003 \\ 0.0113 & 0 & 0.0158 \\ 0 & 0.0158 & 0.0113 \\ -0.0113 & 0 & 0.0042 \\ 0 & 0.0042 & -0.0113 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.003 & 0 & 0.003 & 0 & 0.0113 & 0 & -0.0113 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0042 & -0.003 & -0.0158 & 0.003 & 0.0158 & 0.0113 & 0.0042 & -0.0113 \end{bmatrix}$$

$$[k_1] = 10^4 \times \begin{bmatrix} 0.0632 & 0.0223 & 0.0848 & -0.0223 & -0.2358 & -0.0834 & 0.0878 & 0.0834 \\ & 0.0785 & 0.0834 & 0.2174 & -0.0834 & -0.2929 & -0.0223 & -0.0030 \\ & & 0.4672 & -0.0834 & -0.3162 & -0.3109 & -0.2358 & 0.3109 \\ & & & 0.8866 & 0.0834 & -0.8111 & 0.0223 & -0.2929 \\ & & & & 0.8795 & 0.3109 & -0.3275 & -0.3109 \\ & & & & & & 1.0927 & 0.0834 & 0.0113 \\ & & & & & & & 0.4755 & -0.0834 \\ & & & & & & & & 0.2847 \end{bmatrix}$$

(b) At sampling point 2, ($\xi = -0.57735$, $\eta = 0.57735$)

The value of the Jacobian, $[J]$ at sampling point 2 can be calculated in a similar way and finally the strain-displacement relationship matrix and then the stiffness matrix $[k_2]$ can be evaluated and is shown below.

$$[k_2] = 20 \times 1 \times 1 \times 875 \times 2 \times 10^3 \times \begin{bmatrix} -0.0113 & 0 & -0.0042 \\ 0 & -0.0042 & -0.0113 \\ 0.0113 & 0 & -0.0158 \\ 0 & -0.0158 & 0.0113 \\ 0.003 & 0 & 0.0158 \\ 0 & 0.0158 & 0.003 \\ -0.003 & 0 & 0.0042 \\ 0 & 0.0042 & -0.0030 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.0113 & 0 & 0.0113 & 0 & 0.0030 & 0 & -0.0030 & 0 \\ 0 & -0.0042 & 0 & -0.0158 & 0 & 0.0158 & 0 & 0.0042 \\ -0.0021 & -0.0056 & -0.0079 & 0.0056 & 0.0079 & 0.0015 & 0.0021 & -0.0015 \end{bmatrix}$$

$$[k_2] = 10^4 \times \begin{bmatrix} 0.4756 & 0.0833 & -0.3276 & -0.0833 & -0.2357 & -0.0223 & 0.0878 & 0.0223 \\ & 0.2847 & 0.3110 & 0.0112 & -0.3110 & -0.2929 & -0.0833 & -0.0030 \\ & & 0.8797 & -0.3110 & -0.3164 & -0.0833 & -0.2357 & 0.0833 \\ & & & 1.0930 & 0.3110 & -0.8113 & 0.0833 & -0.2929 \\ & & & & 0.4673 & 0.0833 & 0.0848 & -0.0833 \\ & & & & & & 0.8868 & 0.0223 & 0.2174 \\ & & & & & & & 0.0632 & -0.0223 \\ & & & & & & & & 0.0785 \end{bmatrix}$$

(c) At sampling point 3, ($\xi=-0.57735$, $\eta=-0.57735$)

The value of the strain-displacement relationship matrix and then the stiffness matrix $[k_3]$ can be evaluated and is shown below.

$$[k_3] = 20 \times 1 \times 1 \times 875 \times 2 \times 10^3 \times \begin{bmatrix} -0.0113 & 0 & -0.0158 \\ 0 & -0.0158 & -0.0113 \\ 0.0113 & 0 & -0.0042 \\ 0 & -0.0042 & 0.0113 \\ 0.0030 & 0 & 0.0042 \\ 0 & 0.0042 & 0.0030 \\ -0.0030 & 0 & 0.0158 \\ 0 & 0.0158 & -0.0030 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.0030 & 0 & 0.0113 & 0 & 0.0030 & 0 & -0.0030 & 0 \\ 0 & -0.0158 & 0 & -0.0042 & 0 & 0.0042 & 0 & 0.0158 \\ -0.0079 & -0.0056 & -0.0021 & 0.0056 & 0.0021 & 0.0015 & 0.0079 & -0.0015 \end{bmatrix}$$

$$[k_3] = 10^4 \times \begin{bmatrix} 0.8797 & 0.3110 & -0.3276 & -0.3110 & -0.2357 & -0.0833 & -0.3164 & 0.0833 \\ & 1.0930 & 0.0833 & 0.0112 & -0.0833 & -0.2929 & -0.3110 & -0.8113 \\ & & 0.4756 & -0.0833 & 0.0878 & -0.0223 & -0.2357 & 0.0223 \\ & & & 0.2847 & 0.0833 & -0.0030 & 0.3110 & -0.2929 \\ & & & & 0.0632 & 0.0223 & 0.0848 & -0.0223 \\ & & sym & & & 0.0785 & 0.0833 & 0.2174 \\ & & & & & & 0.4673 & -0.0833 \\ & & & & & & & 0.8868 \end{bmatrix}$$

(d) At sampling point 4, ($\xi=0.57735$, $\eta=-0.57735$)

The value of the strain-displacement relationship matrix and then the stiffness matrix $[k_3]$ can be evaluated and is shown below.

$$[k_4] = 20 \times 1 \times 1 \times 875 \times 2 \times 10^3 \times \begin{bmatrix} -0.0030 & 0 & -0.0158 \\ 0 & -0.0158 & -0.0030 \\ 0.0030 & 0 & -0.0042 \\ 0 & -0.0042 & 0.0030 \\ 0.0113 & 0 & 0.0042 \\ 0 & 0.0042 & 0.0113 \\ -0.0113 & 0 & 0.0158 \\ 0 & 0.0158 & -0.0113 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.0030 & 0 & 0.0030 & 0 & 0.0113 & 0 & -0.0113 & 0 \\ 0 & -0.0158 & 0 & -0.0042 & 0 & 0.0042 & 0 & 0.0158 \\ -0.0079 & -0.0015 & -0.0021 & 0.0015 & 0.0021 & 0.0056 & 0.0079 & -0.0056 \end{bmatrix}$$

$$[k_4] = 10^4 \times \begin{bmatrix} 0.4673 & 0.0833 & 0.0848 & -0.0833 & -0.2357 & -0.3110 & -0.3164 & 0.3110 \\ & 0.8868 & 0.0223 & 0.2174 & -0.0223 & -0.2929 & -0.0833 & -0.8113 \\ & & 0.0632 & -0.0223 & 0.0878 & -0.0833 & -0.2357 & 0.0833 \\ & & & 0.0785 & 0.0223 & -0.0030 & 0.0833 & -0.2929 \\ & & & & 0.4756 & 0.0833 & -0.3276 & -0.0833 \\ & & sym & & & 0.2847 & 0.3110 & 0.0112 \\ & & & & & & 0.8797 & -0.3110 \\ & & & & & & & 1.0930 \end{bmatrix}$$

The stiffness matrix of the element can be computed as the sum of the values at the four sampling points: $[k] = [k_1] + [k_2] + [k_3] + [k_4]$. Thus, the final value of the stiffness matrix will become

$$[k] = 10^4 \times \begin{bmatrix} 1.8857 & 0.5000 & -0.4857 & -0.5000 & -0.9429 & -0.5000 & -0.4571 & 0.3110 \\ & 2.3429 & 0.5000 & 0.4571 & -0.5000 & -1.1714 & -0.5000 & -1.6286 \\ & & 1.8857 & -0.5000 & -0.4571 & -0.5000 & -0.9429 & 0.5000 \\ & & & 2.3429 & -0.5000 & -1.6286 & 0.5000 & -1.1714 \\ & & & & 1.8857 & 0.5000 & -0.4857 & -0.5000 \\ & & sym & & & 2.3429 & 0.5000 & 0.4571 \\ & & & & & & 1.8857 & -0.5000 \\ & & & & & & & 2.3429 \end{bmatrix}$$

Lecture 5: Computation of Stresses, Geometric Nonlinearity and Static Condensation

5.5.1 Computation of Stresses

After solving the static equation of $\{F\} = [K]\{d\}$, the nodal displacement $\{d\}$ can be obtained in global coordinate system. The element nodal displacement $\{\bar{d}\}$ can then be calculated from the nodal connectivity of the element. Using strain-displacement relation and then stress-strain relation, the stress at the element level are derived.

$$\{\bar{\sigma}\} = [D][B]\{d\} \quad (5.5.1)$$

Here, $\{\bar{\sigma}\}$ is the stress at the Gauss point of the element as the sampling points for the integration has been considered as Gauss points. Here, $[D]$ is the constitutive matrix, $[B]$ is the strain displacement matrix of the element. As a result these stresses at Gauss points need to extrapolate to the corresponding nodes of the element. It is well established that 2×2 Gauss integration points are the optimal sampling points for two dimensional isoparametric elements. The 'local stress smoothing' is a technique that can be used to extrapolate stresses computed at Gauss points to nodal points. The stresses are computed at four Gauss points (I, II, III and IV) of an 8 node element as shown in Fig. 5.5.1. For example, at point III, $r = s = 1$ and $\xi = \eta = 1/\sqrt{3}$.

Therefore the factor of proportionality is $\sqrt{3}$; *i.e.*,

$$r = \xi\sqrt{3} \text{ and } s = \eta\sqrt{3} \quad (5.5.2)$$

Stresses at any point P in the element are found by the usual shape function as

$$\sigma_P = \sum N'_{di} \sigma_i \text{ for } i = 1, 2, 3, 4 \quad (3.5.3)$$

In the above equation, σ_P is σ_x, σ_y and τ_{xy} at point P . N'_{di} are the bilinear shape functions written in terms of r and s rather than ξ and η as

$$N'_{di} = \frac{1}{4}(1 \pm r)(1 \pm s) \quad (5.5.4)$$

N'_{di} are evaluated at r and s coordinates of point P . Let the point P coincides with the corner 1. To calculate stress σ_{xI} at corner 1 from σ_x values at the four Gauss points, substitution of r and s into the shape functions will give

$$\sigma_{xI} = 1.8666\sigma_{xI} - 0.500\sigma_{xII} + 0.134\sigma_{xIII} - 0.500\sigma_{xIV} \quad (5.5.5)$$

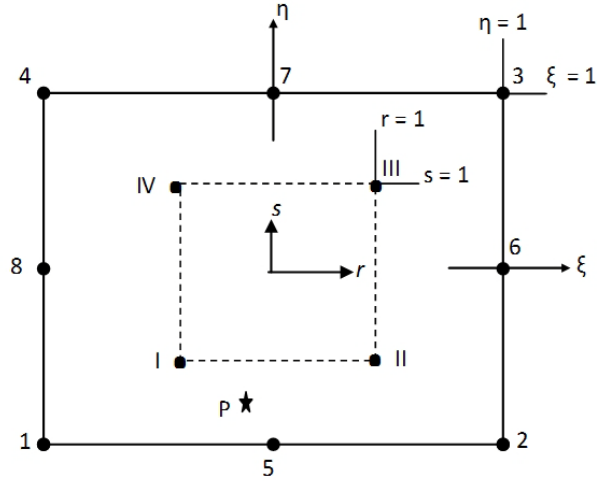


Fig. 5.5.1 Natural coordinate systems used in extrapolation of stresses from Gauss points

The resultant extrapolation matrix thus obtained may be written as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \\ \sigma_8 \end{bmatrix} = \begin{bmatrix} (1+\sqrt{3}/2) & -0.5 & (1-\sqrt{3}/2) & -0.5 \\ -0.5 & (1+\sqrt{3}/2) & -0.5 & (1-\sqrt{3}/2) \\ (1-\sqrt{3}/2) & -0.5 & (1+\sqrt{3}/2) & -0.5 \\ -0.5 & (1-\sqrt{3}/2) & -0.5 & (1+\sqrt{3}/2) \\ (1+\sqrt{3})/4 & (1+\sqrt{3})/4 & (1-\sqrt{3})/4 & (1-\sqrt{3})/4 \\ (1-\sqrt{3})/4 & (1+\sqrt{3})/4 & (1+\sqrt{3})/4 & (1-\sqrt{3})/4 \\ (1-\sqrt{3})/4 & (1-\sqrt{3})/4 & (1+\sqrt{3})/4 & (1+\sqrt{3})/4 \\ (1+\sqrt{3})/4 & (1-\sqrt{3})/4 & (1-\sqrt{3})/4 & (1+\sqrt{3})/4 \end{bmatrix} \begin{Bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{III} \\ \sigma_{IV} \end{Bmatrix} \quad (5.5.6)$$

Here, $\sigma_1, \sigma_2, \dots, \sigma_8$ are the smoothed nodal values and $\sigma_I \dots \sigma_{IV}$ are the stresses at the Gauss points. Smoothed nodal stress values for four node rectangular element can be also be evaluated in a similar fashion. The relation between the stresses at Gauss points and nodal point for four node element will be

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix} = \begin{bmatrix} 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} \\ 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{III} \\ \sigma_{IV} \end{Bmatrix} \quad (5.5.7)$$

The stress at particular node joining with more than one element will have different magnitude as calculated from adjacent elements (Fig. 5.5.2(a)). The stress resultants are then modified by finding the average of resultants of all elements meeting at a common node. A typical stress distribution for adjacent elements is shown in Fig. 5.5.2(b) after stress smoothing.

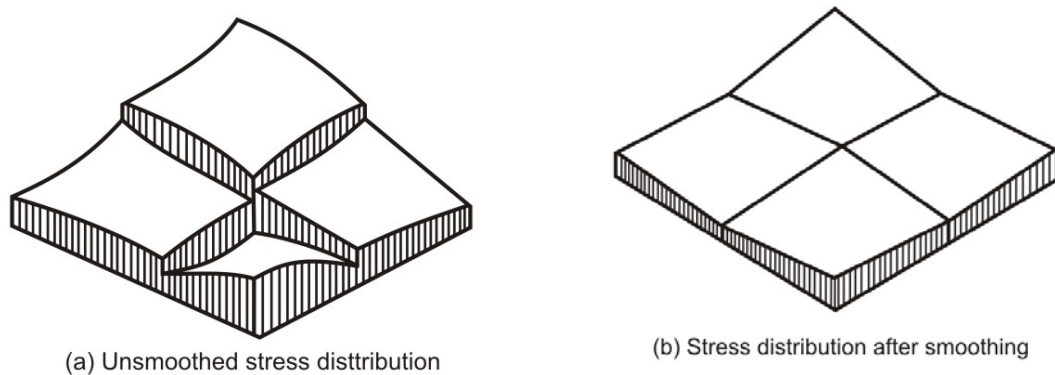


Fig. 5.5.2 Stress smoothing at common node

5.5.2 Geometric Nonlinearity

As discussed earlier, nonlinear analysis is mainly of two types: (i) Geometric nonlinearity and (ii) Material nonlinearity. For geometric nonlinearity consideration, the relation between strain and displacement is of utmost importance in the finite element formulation for stress analysis problems. In case of plane stress/strain problem, the nonlinear term of the strain expression are dropped for the sake of simplicity in the analysis. However, for large displacement problems, the nonlinear strain term plays a vital role to obtain accurate response. The generalized strain-displacement relations for the two-dimensional plane stress/strain problems are rewritten here to derive the nonlinear solution.

$$\begin{aligned}
 \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right] \\
 \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \\
 \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \left[\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right]
 \end{aligned}
 \tag{5.5.8}$$

The displacements at any point inside the node are expressed in terms of their nodal displacements. Thus,

$$u = \sum_{i=1}^n N_i u_i = [N_i] \{u_i\} \quad \text{and} \quad v = \sum_{i=1}^n N_i v_i = [N_i] \{v_i\}
 \tag{5.5.9}$$

Therefore,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial [N_i]}{\partial x} \{u_i\} = [B_1] \{u_i\} \\
\frac{\partial v}{\partial x} &= [B_1] \{v_i\} \\
\frac{\partial u}{\partial y} &= [B_2] \{u_i\} \\
\frac{\partial v}{\partial y} &= [B_2] \{v_i\}
\end{aligned}
\tag{5.5.10}$$

Here, $[B_1]$ and $[B_2]$ are the derivative of the shape function $[N_i]$ with respect to x and y respectively. The vectors $\{u_i\}$ and $\{v_i\}$ represent the nodal displacements vectors in x and y directions respectively. The vector of strains at any point inside an element, $\{\varepsilon\}$ may be expressed in terms of nodal displacement as

$$\{\varepsilon\} = [B] \{d\}
\tag{5.5.11}$$

where $[B]$ is the strain displacement matrix. $\{d\}$ is the nodal displacement vector and may be expressed as

$$\{d\} = \begin{Bmatrix} \{u_i\} \\ \{v_i\} \end{Bmatrix}
\tag{5.5.12}$$

The matrix $[B]$ may be expressed with two components as

$$[B] = [B_l] + [B_{nl}]
\tag{5.5.13}$$

where, $[B_l]$ and $[B_{nl}]$ are the linear and nonlinear part of the strain-displacement matrix respectively and are expressed as follows:

$$[B_l] = \begin{bmatrix} [B_1] & [0] \\ [0] & [B_2] \\ [B_2] & [B_1] \end{bmatrix}
\tag{5.5.14}$$

and

$$[B_{nl}] = \begin{bmatrix} \frac{1}{2} \{\bar{u}\}^T [B_1]^T [B_1] & \frac{1}{2} \{\bar{v}\}^T [B_1]^T [B_1] \\ \frac{1}{2} \{\bar{u}\}^T [B_2]^T [B_2] & \frac{1}{2} \{\bar{v}\}^T [B_2]^T [B_2] \\ \{\bar{u}\}^T [B_2]^T [B_1] & \{\bar{v}\}^T [B_2]^T [B_1] \end{bmatrix} \quad (5.5.15)$$

5.5.2.1 Steps to include effect of geometrical nonlinearity

The nonlinear geometric effect of the structure at a particular instant of time can be obtained by performing the following steps.

1. Calculation of displacement $\{d\}_1$ considering linear part of strain matrix $[B_l]$.
2. Evaluation of nonlinear part of the strain matrix $[B_{nl}]$ (eq5.5.15) adopting $\{d\}_1$ from previous step.
3. Evaluation of total strain matrix $[B] = [B_l] + [B_{nl}]$.
4. Calculation of displacement $\{d\}_2$ considering both linear and nonlinear part of strain matrix $[B]$.
5. Repetition of steps 2 to 4 with $\{d\}_2$, from which modified displacement, $\{d\}_3$ are obtained.
6. Step 5 is carried out until the displacements for two consecutive iteration converge *i.e.*,

$$\left| \frac{\{d\}_j - \{d\}_{j-1}}{\{d\}_j} \right| < \varepsilon$$

Where ε is any pre-assigned small value and j is the number of iterations.

5.5.3 Static Condensation

The higher order Lagrangian elements (i.e., nine node, sixteen node rectangular element) contain number of internal nodes. This is necessary sometimes for the completeness of the desired polynomial used in displacement function for derivation of interpolation function. These internal nodes are not connected to the adjoining elements in the assemblage (Fig. 5.5.3).

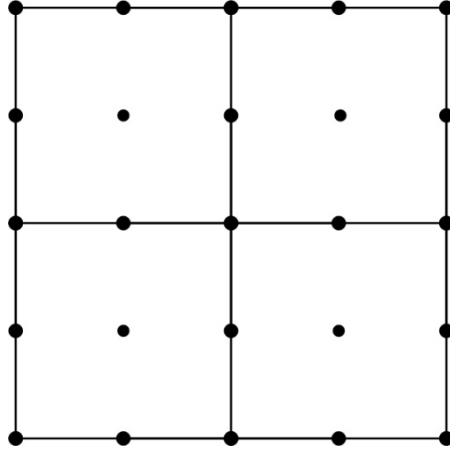


Fig. 5.5.3 Internal nodes of Nine node elements

Thus, the displacements of these nodes are not required to formulate overall equilibrium equations of the structure. This limits the usefulness of these elements. A technique known as “static condensation” can be used to suppress the degrees of freedom associated with the internal nodes in the final computation. The technique of static condensation is explained below. The equilibrium equation for a system are expressed in the finite element form as

$$\{F\} = [K]\{d\} \quad (5.5.16)$$

Where, $\{F\}$, $[K]$ and $\{d\}$ are the load vector, stiffness matrix and displacement vector for the entire structure. The above equation can be rearranged by separating the relevant terms corresponding to internal and external nodes of the elements.

$$\begin{Bmatrix} \{F_i\} \\ \{F_e\} \end{Bmatrix} = \begin{bmatrix} [K_{ii}] & [K_{ie}] \\ [K_{ei}] & [K_{ee}] \end{bmatrix} \begin{Bmatrix} \{d_i\} \\ \{d_e\} \end{Bmatrix} \quad (5.5.17)$$

Here, $\{d_i\}$ and $\{d_e\}$ are the displacement vectors corresponding to internal and external nodes respectively. Similarly, $\{F_i\}$ and $\{F_e\}$ are force vectors corresponding to internal and external nodes. Now, the above expression can be written in the following form separately.

$$\{F_i\} = [K_{ii}]\{d_i\} + [K_{ie}]\{d_e\} \quad (5.5.18)$$

$$\{F_e\} = [K_{ei}]\{d_i\} + [K_{ee}]\{d_e\} \quad (5.5.19)$$

The stiffness matrix and nodal load vector corresponds to the internal nodes can be separated out. For this, eq.(5.5.18) can be rewritten as

$$\{d_i\} = [K_{ii}]^{-1}\{F_i\} - [K_{ii}]^{-1}[K_{ie}]\{d_e\} \quad (5.5.20)$$

Substituting the value of $\{d_i\}$ obtained from the above equation in eq.(5.5.19), the following expression will be obtained.

$$\{F_e\} = [K_{ei}] \left\{ [K_{ii}]^{-1} \{F_i\} - [K_{ii}]^{-1} [K_{ie}] \{d_e\} \right\} + [K_{ee}] \{d_e\} \quad (5.5.21)$$

Here, the equations are reduced to a form involving only the external nodes of the elements. The above reduced substructure equations are assembled to achieve the overall equations involving only the boundary unknowns. Thus the above equation can be rewritten as

$$\left\{ \{F_e\} - [K_{ei}] [K_{ii}]^{-1} \{F_i\} \right\} = \left[[K_{ee}] - [K_{ei}] [K_{ii}]^{-1} [K_{ie}] \right] \{d_e\} \quad (5.5.22)$$

Or

$$\{F_c\} = [K_c] \{d_e\} \quad (5.5.23)$$

Where, $\{F_c\} = \{F_e\} - [K_{ei}] [K_{ii}]^{-1} \{F_i\}$ and $[K_c] = [K_{ee}] - [K_{ei}] [K_{ii}]^{-1} [K_{ie}]$. Here, $[K_c]$ is called condensed or reduced stiffness matrix and $\{F_c\}$ is the condensed or effective nodal load vector corresponding to external nodes of the elements. In this process, the size of the matrix for inversion will be comparatively small. The unknown displacements of the exterior nodes, $\{d_e\}$ can be obtained by inverting the matrix $[K_c]$ in eq.(5.5.23). Once, the values of $\{d_e\}$ are obtained, the displacements of internal nodes $\{d_i\}$ can be found from eq.(5.5.20).

Lecture 6: Axisymmetric Element

5.6.1 Introduction

Many three-dimensional problems show symmetry about an axis of rotation. If the problem geometry is symmetric about an axis and the loading and boundary conditions are symmetric about the same axis, the problem is said to be axisymmetric. Such three-dimensional problems can be solved using two-dimensional finite elements. The axisymmetric problems are most conveniently defined by polar coordinate system with coordinates (r, θ, z) as shown in Fig. 5.6.1. Thus, for axisymmetric analysis, following conditions are to be satisfied.

1. The domain should have an axis of symmetry and is considered as z axis.
2. The loadings on the domain has to be symmetric about the axis of revolution, thus they are independent of circumferential coordinate θ .
3. The boundary condition and material properties are symmetric about the same axis and will be independent of circumferential coordinate.

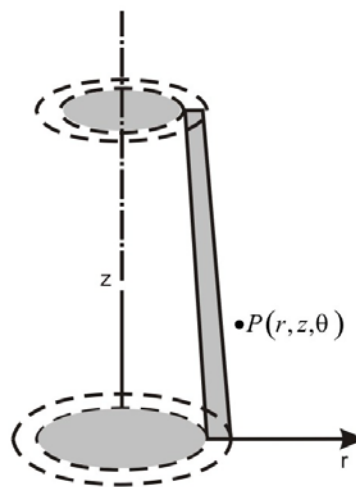


Fig. 5.6.1 Cylindrical coordinates

Axisymmetric solids are of total symmetry about the axis of revolution (i.e., z -axis), the field variables, such as the stress and deformation is independent of rotational angle θ . Therefore, the field variables can be defined as a function of (r, z) and hence the problem becomes a two dimensional problem similar to those of plane stress/strain problems. Axisymmetric problems includes, circular cylinder loaded with uniform external or internal pressure, circular water tank, pressure vessels, chimney, boiler, circular footing resting on soil mass, etc.

5.6.2 Relation between Strain and Displacement

An axisymmetric problem is readily described in cylindrical polar coordinate system: r, z and θ . Here, θ measures the angle between the plane containing the point and the axis of the coordinate

system. At $\theta = 0$, the radial and axial coordinates coincide with the global Cartesian X and Y coordinates. Fig. 5.6.2 shows a cylindrical coordinate system and the definition of the position vectors. Let \hat{r} , \hat{z} and $\hat{\theta}$ be unit vectors in the radial, axial, and circumferential directions at a point in the cylindrical coordinate system.

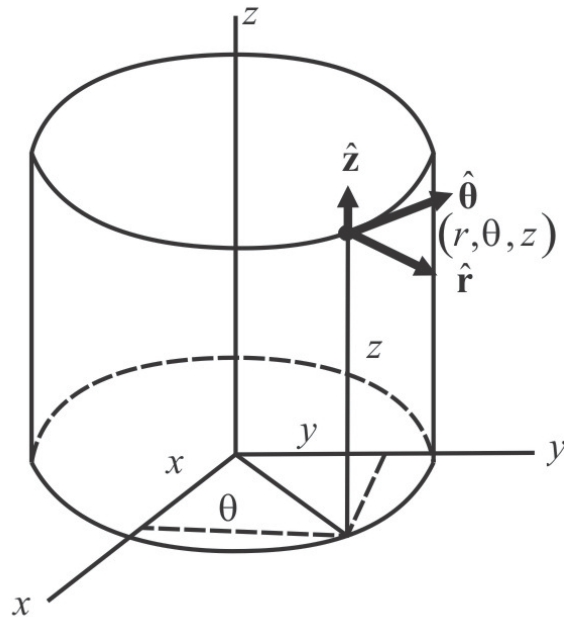
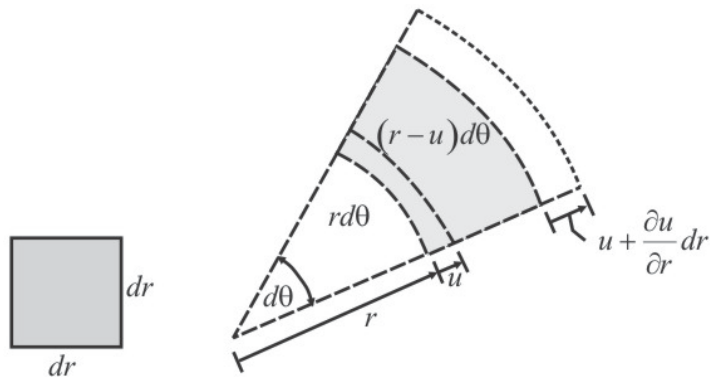


Fig. 5.6.2 Cylindrical Coordinate System

If the loading consists of radial and axial components that are independent of θ and the material is either isotropic or orthotropic and the material properties are independent of θ , the displacement at any point will only have radial (u_r) and axial (u_z) components. The only stress components that will be nonzero are σ_{rr} , σ_{zz} , $\sigma_{\theta\theta}$ and τ_{rz} .



(a) Element in r-z plane

(b) Element in r- θ plane

Fig. 5.6.3 Deformation of the axisymmetric element

A differential element of the body in the r - z plane is shown in Fig. 5.6.3(a). The element undergoes deformation in the radial direction. Therefore, it initiates increase in circumference and associated circumferential strain. Let denote the radial displacement as u , the circumferential displacement as v , and the axial displacement as w . Dashed line represents the deformed positions of the body in Fig. 5.6.3(b). The radial strain can be calculated from the above diagram as

$$\varepsilon_r = \frac{1}{dr} \left(dr + \frac{\partial u}{\partial r} dr \right) = \frac{\partial u}{\partial r} \quad (5.6.1)$$

Since the rz plane is effectively the same as a rectangular coordinate system, the axial strain will become

$$\varepsilon_z = \frac{1}{dz} \left(dz + \frac{\partial w}{\partial z} dz \right) = \frac{\partial w}{\partial z} \quad (5.6.2)$$

Considering the original arc length versus the deformed arc length, the differential element undergoes an expansion in the circumferential direction. Before deformation, let the arc length is assumed as $ds = r d\theta$. After deformation, the arc length will become $ds = (r+u) d\theta$. Thus, the tangential strain will be

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r} \quad (5.6.3)$$

Similarly, the shear strain will be

$$\begin{aligned} \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ \gamma_{r\theta} &= 0 \text{ and } \gamma_{z\theta} = 0 \end{aligned} \quad (5.6.4)$$

Thus, there are four strain components present in this case and is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix} \quad (5.6.5)$$

5.6.3 Relation between Stress and Strain

The stress strain relation for axisymmetric case can be derived from the three dimensional constitutive relations. We know the stress-strain relation for a three-dimensional solid is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \nu_{xy} \\ \nu_{yz} \\ \nu_{zx} \end{Bmatrix} \quad (5.6.6)$$

The stresses acting on a differential volume of an axisymmetric solid under axisymmetric loading is shown in Fig. 5.6.4.

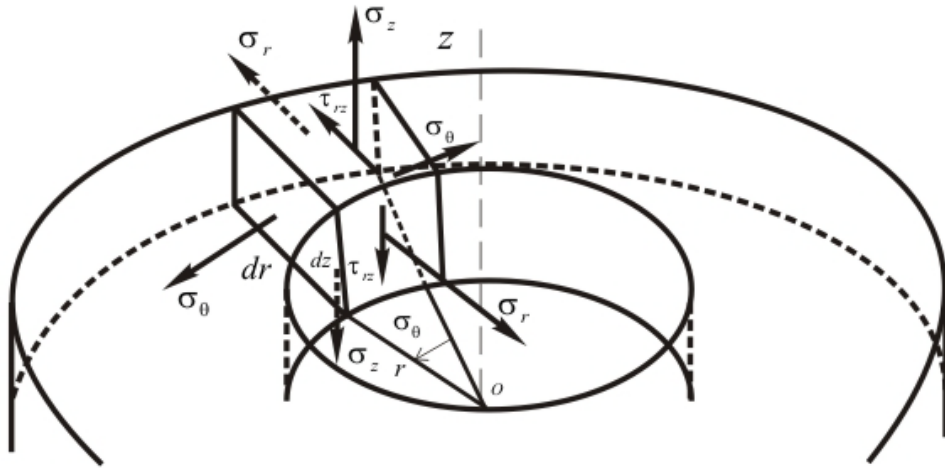


Fig. 5.6.4 Stresses acting on a differential volume

Now, comparing the stress-strain components present in the axisymmetric case, the stress-strain relation can be expressed from the above expression as follows

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \nu_{rz} \end{Bmatrix} \quad (5.6.7)$$

Thus, the constitutive matrix [D] for the axisymmetric elastic solid will be

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 \\ \mu & 1-\mu & \mu & 0 \\ \mu & \mu & 1-\mu & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix} \quad (5.6.8)$$

5.6.4 Axisymmetric Shell Element

A cylindrical liquid storage container like structures (Fig. 5.6.5) may be idealized using axisymmetric shell element for the finite element analysis. It may be noted that the liquid in the container may be idealized with two dimensional axisymmetric elements. Let us consider the radius, height and, thickness of the circular tank are R , H and h respectively.

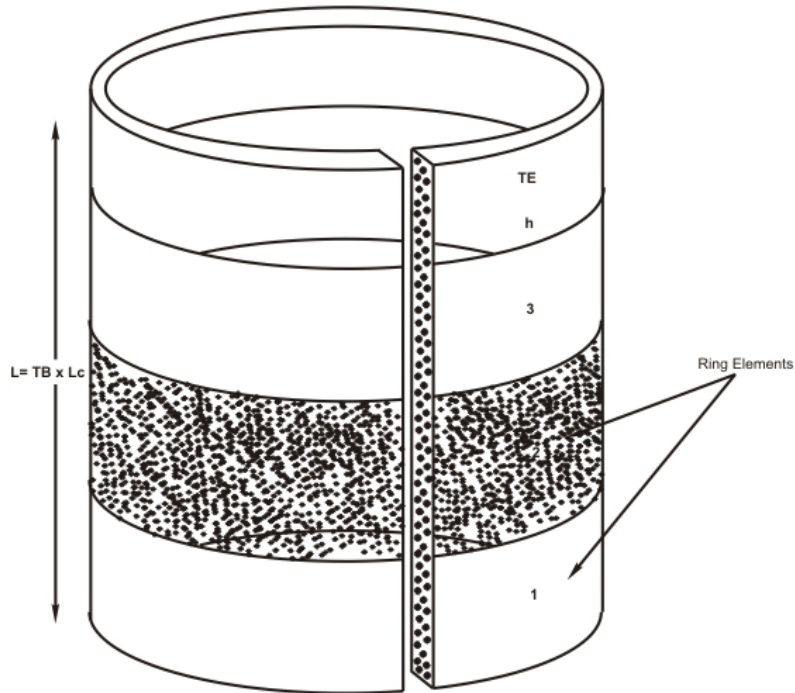


Fig. 5.6.5 Thin wall cylindrical container

The strain energy of the axisymmetric shell element (Fig. 5.6.6) including the effect of both stretching and bending are expressed as

$$U = \frac{1}{2} \int_0^L (N_y \varepsilon_y + M \chi_y) 2\pi R dy \quad (5.6.9)$$

Here, N_y and N_θ are the membrane force resultants and M_y is the bending moment resultant. The shell is assumed to be linearly elastic, homogeneous and isotropic. Thus the force and moment resultants can be expressed in terms of the mid-surface change in curvature χ_y as follows.

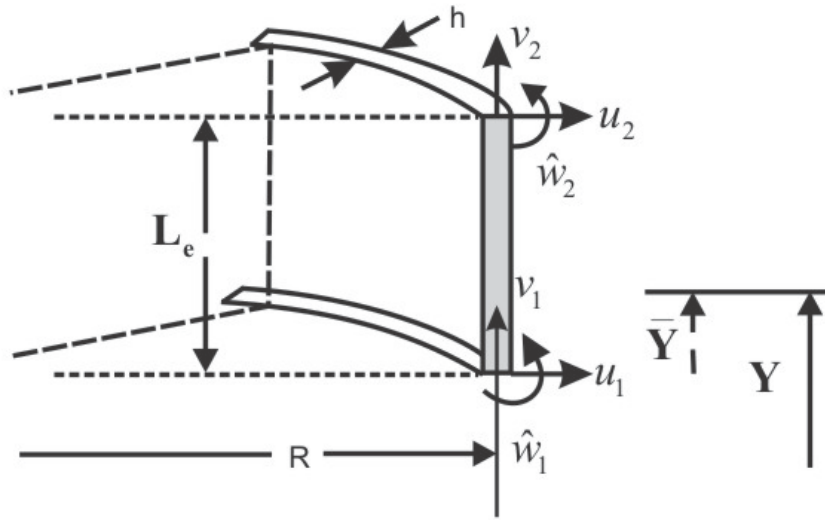


Fig 5.6.6 Axisymmetric plate element

Here, the strain-displacement relation is given by

$$\{\sigma\} = [D]\{\varepsilon\} \quad (5.6.10)$$

In which,

$$\{\sigma\} = \begin{Bmatrix} N_y \\ N_\theta \\ M_y \end{Bmatrix}, \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_y \\ \varepsilon_\theta \\ \chi_y \end{Bmatrix} \quad \text{and} \quad [D] = \frac{Eh}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{h^2}{12} \end{bmatrix} \quad (5.6.11)$$

The generalized strain vector can be expressed in terms of the displacement vectors as follows.

$$\{\varepsilon\} = [B]\{d\} \quad (5.6.12)$$

Where,

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} \text{ and } [B] = \begin{bmatrix} 0 & \frac{\partial}{\partial y} \\ \frac{1}{R} & 0 \\ -\frac{\partial^2}{\partial y^2} & 0 \end{bmatrix} \quad (5.6.13)$$

Here, u and v are the displacement components in two perpendicular directions. With the use of stress and strain vectors, the potential energy expression are written in terms of displacement vectors as

$$U = \frac{1}{2} \int_0^H \left(\{d\}^T [B]^T [D] [B] \{d\} \right) dy \quad (5.6.14)$$

Thus, the element stiffness are derived as

$$[k] = 2\pi R \int_0^H [B]^T [D] [B] dy \quad (5.6.15)$$

Similarly, neglecting the rotary inertia, the kinetic energy can be expressed as

$$T = \frac{1}{2} \int_0^H \left(\{\dot{d}\}^T [N]^T m [N] \{\dot{d}\} \right) dy \quad (5.6.16)$$

Where, m denotes the mass of the shell element per unit area and $\{\dot{d}\}$ represents the velocity vector. Thus, the element mass matrix is given by

$$[M] = 2\pi R m \int_0^{L_c} [N]^T [N] dy \quad (5.6.17)$$

Lecture 7: Finite Element Formulation of Axisymmetric Element

Finite element formulation for the axisymmetric problem will be similar to that of the two dimensional solid elements. As the field variables, such as the stress and strain is independent of rotational angle θ , circumferential displacement will not appear. Thus, the displacement field variables are expressed as

$$u(r, z) = \sum_{i=1}^n N_i(r, z) u_i \quad (5.7.1)$$

$$w(r, z) = \sum_{i=1}^n N_i(r, z) w_i$$

Here, u_i and w_i represent radial and axial displacements respectively at nodes. $N_i(r, z)$ are the shape functions. As the geometry and field variables are independent of rotational angle θ , the interpolation function $N_i(r, z)$ can be expressed similar to 2-dimensional problems by replacing the x and y terms with r and z terms respectively.

5.7.1 Stiffness Matrix of a Triangular Element

Fig. 5.7.1 shows the cylindrical coordinates of a three node triangular element. Hence the analysis of the axisymmetric element can be approached in a similar way as the CST element. Thus the field variables of such an element can be expressed as

$$\begin{aligned} u &= \alpha_0 + \alpha_1 r + \alpha_2 z \\ w &= \alpha_3 + \alpha_4 r + \alpha_5 z \end{aligned} \quad (5.7.2)$$

Or,

$$\{d\} = [\phi]\{\alpha\} \quad (5.7.3)$$

Where,

$$\{d\} = \begin{Bmatrix} u \\ w \end{Bmatrix}, [\phi] = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \text{ and } \{\alpha\}^T = \{\alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5\}$$

Using end conditions,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i & 0 & 0 & 0 \\ 1 & r_j & z_j & 0 & 0 & 0 \\ 1 & r_k & z_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r_i & z_i \\ 0 & 0 & 0 & 1 & r_j & z_j \\ 0 & 0 & 0 & 1 & r_k & z_k \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \quad (5.7.4)$$

Or,

$$\begin{aligned} \{\bar{d}\} &= [A]\{\alpha\} \\ \Rightarrow \{\alpha\} &= [A]^{-1}\{\bar{d}\} \end{aligned} \quad (5.7.5)$$

Here $\{\bar{d}\}$ are the nodal displacement vectors.

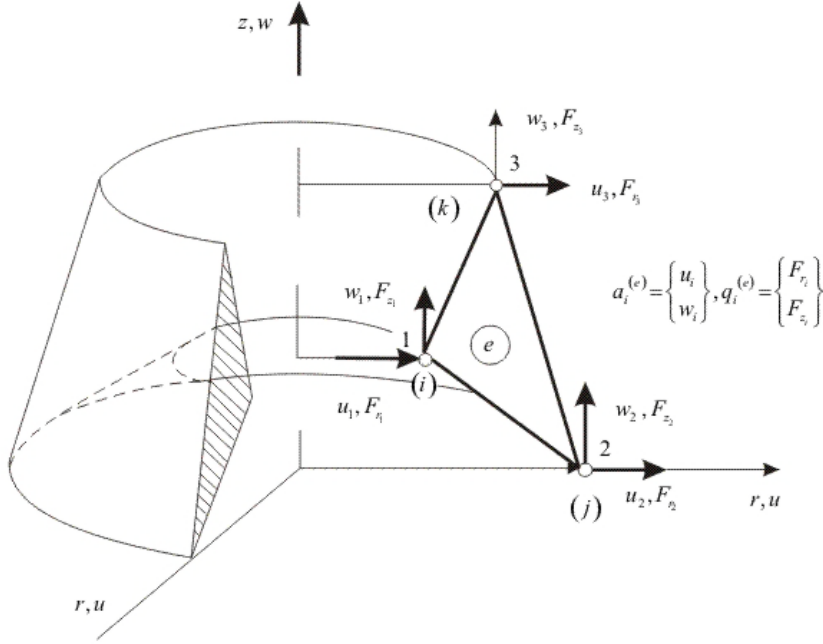


Fig. 5.7.1 Axisymmetric three node triangle in cylindrical coordinates

Putting above values in eq.(5.7.3), the following relations will be obtained.

$$\{d\} = [\phi][A]^{-1} \{\bar{d}\} = [N]\{\bar{d}\} \quad (5.7.6)$$

Or,

$$\{d\} = \begin{Bmatrix} u \\ w \end{Bmatrix} = \begin{bmatrix} N_i & N_j & N_k & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & N_j & N_k \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ z_1 \\ z_2 \\ z_3 \end{Bmatrix} \quad (5.7.7)$$

Using a similar approach as in case of CST elements, the three shape functions $[N_1, N_2, N_3]$ can be assumed as,

$$N_1(r, z) = \frac{1}{2A} [(r_2 z_3 - r_3 z_2) + (z_2 - z_3)r + (r_3 - r_2)z]$$

$$N_2(r, z) = \frac{1}{2A} [(r_3 z_1 - r_1 z_3) + (z_3 - z_1)r + (r_1 - r_3)z]$$

$$N_3(r, z) = \frac{1}{2A} [(r_1 z_2 - r_2 z_1) + (z_1 - z_2)r + (r_2 - r_1)z]$$

Or,

$$\begin{aligned}
 N_i(r, z) &= \frac{1}{2A}(\alpha_i + r\beta_i + z\gamma_i) \\
 N_j(r, z) &= \frac{1}{2A}(\alpha_j + r\beta_j + z\gamma_j) \\
 N_k(r, z) &= \frac{1}{2A}(\alpha_k + r\beta_k + z\gamma_k)
 \end{aligned} \tag{5.7.8}$$

Where,

$$\begin{aligned}
 \alpha_i &= r_j z_k - r_k z_j & \alpha_j &= r_k z_i - r_i z_k & \alpha_k &= r_i z_j - r_j z_i \\
 \beta_i &= z_j - z_k & \beta_j &= z_k - z_i & \beta_k &= z_i - z_j \\
 \gamma_i &= r_k - r_j & \gamma_j &= r_i - r_k & \gamma_k &= r_j - r_i
 \end{aligned} \tag{5.7.9}$$

$$2A = \frac{1}{2}(r_i z_j + r_j z_k + r_k z_i - r_i z_k - r_j z_i - r_k z_j)$$

Putting the value of {u,w} in eq. (5.7.7) from eq. (5.6.5),

$$\{\varepsilon\} = \begin{bmatrix} \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial r} & 0 & 0 & 0 \\ \frac{N_i}{r} & \frac{N_j}{r} & \frac{N_k}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_i}{\partial r} & \frac{\partial N_j}{\partial r} & \frac{\partial N_k}{\partial r} \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ z_1 \\ z_2 \\ z_3 \end{Bmatrix} = [B]\{\bar{d}\} \tag{5.7.10}$$

Thus, the strain displacement matrix can be expressed as,

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_i & \beta_j & \beta_k & 0 & 0 & 0 \\ \frac{N_i}{r} & \frac{N_j}{r} & \frac{N_k}{r} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_i & \gamma_j & \gamma_k \\ \gamma_i & \gamma_j & \gamma_k & \beta_i & \beta_j & \beta_k \end{bmatrix} \tag{5.7.11}$$

Where, $r = \frac{r_i + r_j + r_k}{3}$. Thus the stiffness matrix will become

$$[k] = \iiint [B]^T [D][B] d\Omega$$

$$\text{Or, } [k] = \int \int \int_0^{2\pi} [B]^T [D][B] r d\theta dA = 2\pi \int \int [B]^T [D][B] r dr dz \tag{5.7.12}$$

Since, the term $[B]$ is dependent of 'r' terms; the term $[B]^T [D][B]$ cannot be taken out of integration. Yet, a reasonably accurate solution can be obtained by evaluating the $[B]$ (denoted as $[\underline{B}]$) matrix at the centroid.

$$\text{Hence, } [k] = 2\pi r [\underline{B}]^T [D][\underline{B}] \int \int dr dz$$

Or,

$$[k] \approx [\underline{B}]^T [D][\underline{B}] 2\pi r A \quad (5.7.13)$$

5.7.2 Stiffness Matrix of a Quadrilateral Element

The strain-displacement relation for axisymmetric problem derived earlier (eq.(5.6.5)) can be rewritten as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{u}{r} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \\ u \end{Bmatrix} \quad (5.7.14)$$

Applying chain rule of differentiation equation we get,

$$\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \\ \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \\ u \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \\ u \end{Bmatrix} \quad (5.7.15)$$

Hence, the strain components are calculated as

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \\ u \end{Bmatrix}$$

Or,

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \\ u \end{Bmatrix} \quad (5.7.16)$$

With the use of interpolation function and nodal displacements, $\left(\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial w}{\partial \xi}, \frac{\partial w}{\partial \eta}\right)$ can be expressed for a four node quadrilateral element as

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} \quad (5.7.17)$$

Putting eq. (5.7.17) in eq. (5.7.16) we get,

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \\ N_1 & N_2 & N_3 & N_4 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} \quad (5.7.18)$$

Thus, the strain displacement relationship matrix $[B]$ becomes

$$[B] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & 0 & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ 0 & 0 & 0 & 0 & \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \\ N_1 & N_2 & N_3 & N_4 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \quad (5.7.19)$$

For a four node quadrilateral element,

$$\begin{aligned} N_1 &= \frac{(1-\xi)(1-\eta)}{4} \Rightarrow \frac{\partial N_1}{\partial \xi} = -\frac{(1-\eta)}{4} \quad \text{and} \quad \frac{\partial N_1}{\partial \eta} = -\frac{(1-\xi)}{4} \\ N_2 &= \frac{(1+\xi)(1-\eta)}{4} \Rightarrow \frac{\partial N_2}{\partial \xi} = \frac{(1-\eta)}{4} \quad \text{and} \quad \frac{\partial N_2}{\partial \eta} = -\frac{(1+\xi)}{4} \\ N_3 &= \frac{(1+\xi)(1+\eta)}{4} \Rightarrow \frac{\partial N_3}{\partial \xi} = \frac{(1+\eta)}{4} \quad \text{and} \quad \frac{\partial N_3}{\partial \eta} = \frac{(1+\xi)}{4} \\ N_4 &= \frac{(1-\xi)(1+\eta)}{4} \Rightarrow \frac{\partial N_4}{\partial \xi} = -\frac{(1+\eta)}{4} \quad \text{and} \quad \frac{\partial N_4}{\partial \eta} = \frac{(1-\xi)}{4} \end{aligned} \quad (5.7.20)$$

Thus, the [B] matrix will become

$$[B] = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* & 0 \end{bmatrix} \times \begin{bmatrix} -\frac{(1-\eta)}{4} & \frac{(1-\eta)}{4} & \frac{(1+\eta)}{4} & -\frac{(1+\eta)}{4} & 0 & 0 & 0 & 0 \\ -\frac{(1-\xi)}{4} & -\frac{(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(1-\eta)}{4} & \frac{(1-\eta)}{4} & \frac{(1+\eta)}{4} & -\frac{(1+\eta)}{4} \\ 0 & 0 & 0 & 0 & -\frac{(1-\xi)}{4} & -\frac{(1+\xi)}{4} & \frac{(1+\xi)}{4} & \frac{(1-\xi)}{4} \\ \frac{(1-\xi)(1-\eta)}{4} & \frac{(1+\xi)(1-\eta)}{4} & \frac{(1+\xi)(1+\eta)}{4} & \frac{(1-\xi)(1+\eta)}{4} & \frac{(1-\xi)(1-\eta)}{4} & \frac{(1+\xi)(1-\eta)}{4} & \frac{(1+\xi)(1+\eta)}{4} & \frac{(1-\xi)(1+\eta)}{4} \end{bmatrix} \quad (5.7.21)$$

The stiffness matrix for the axisymmetric element finally can be found from the following expression after numerical integration.

$$[k] = \int_{\Omega} [B]^T [D] [B] d\Omega = \int_{-1}^{+1} \int_{-1}^{+1} [B]^T [D] [B] . 2d\xi d\eta \quad (5.7.22)$$

Lecture 8: Finite Element Formulation for 3 Dimensional Elements

5.8.1 Introduction

Solid elements can easily be formulated by the extension of the procedure followed for two dimensional solid elements. A domain in 3D can be discretized using tetrahedral or hexahedral elements. For example, the eight node solid brick element is analogous to the four node rectangular element. Regardless of the possible curvature of edges or number of nodes, the solid element can be mapped into the space of natural co-ordinates, i.e the $\xi = \pm 1, \eta = \pm 1, \zeta = \pm 1$ just like a plane element.

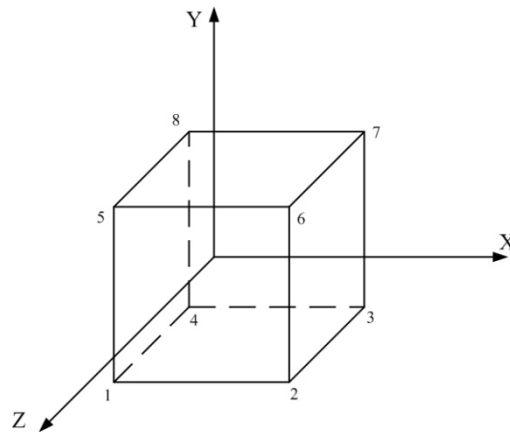


Fig. 5.8.1 Eight node brick element

For three dimensional cases, each node has three degrees of freedom having u , v , and w as displacement field in three perpendicular directions (X , Y and Z). In this case, one additional dimension increases the computational expense manifolds.

5.8.2 Strain Displacement Relation

The strain vector for three dimensional cases can be written in the following form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{Bmatrix} \quad (5.8.1)$$

The following relation exists between the derivative operators in the global co-ordinates and the natural co-ordinate system by the use of chain rule of partial differentiation.

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} \quad (5.8.2)$$

Where the Jacobian Matrix will be

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad (5.8.3)$$

For an isoparametric element the coordinates at a point inside the element can be expressed by its nodal coordinate.

$$x = \sum_{i=1}^n N_i x_i; \quad y = \sum_{i=1}^n N_i y_i \quad \text{and} \quad z = \sum_{i=1}^n N_i z_i \quad (5.8.4)$$

Substituting the above equations into the Jacobian matrix for an eight node brick element, we get

$$[J] = \sum_{i=1}^8 \begin{bmatrix} \frac{\partial N_i}{\partial \xi} x_i & \frac{\partial N_i}{\partial \xi} y_i & \frac{\partial N_i}{\partial \xi} z_i \\ \frac{\partial N_i}{\partial \eta} x_i & \frac{\partial N_i}{\partial \eta} y_i & \frac{\partial N_i}{\partial \eta} z_i \\ \frac{\partial N_i}{\partial \zeta} x_i & \frac{\partial N_i}{\partial \zeta} y_i & \frac{\partial N_i}{\partial \zeta} z_i \end{bmatrix} \quad (5.8.5)$$

The strain displacement relation is given by $\{\varepsilon\} = [B]\{d\}$, where, $\{d\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$.

The displacements in the x, y and z direction are u , v , and w respectively. Let consider the inverse of Jacobian matrix as

$$[J]^{-1} = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix} \quad (5.8.6)$$

Thus, the relation between two coordinate systems can be rewritten as

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* \\ J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} \quad (5.8.7)$$

Thus, one can write the following relations

$$\begin{aligned} \frac{\partial u}{\partial x} &= J_{11}^* \frac{\partial u}{\partial \xi} + J_{12}^* \frac{\partial u}{\partial \eta} + J_{13}^* \frac{\partial u}{\partial \zeta} = \sum_{i=1}^8 \left(J_{11}^* \frac{\partial N_i}{\partial \xi} + J_{12}^* \frac{\partial N_i}{\partial \eta} + J_{13}^* \frac{\partial N_i}{\partial \zeta} \right) u_i = \sum_{i=1}^8 a_i u_i \\ \frac{\partial u}{\partial y} &= J_{21}^* \frac{\partial u}{\partial \xi} + J_{22}^* \frac{\partial u}{\partial \eta} + J_{23}^* \frac{\partial u}{\partial \zeta} = \sum_{i=1}^8 \left(J_{21}^* \frac{\partial N_i}{\partial \xi} + J_{22}^* \frac{\partial N_i}{\partial \eta} + J_{23}^* \frac{\partial N_i}{\partial \zeta} \right) u_i = \sum_{i=1}^8 b_i u_i \\ \frac{\partial u}{\partial z} &= J_{31}^* \frac{\partial u}{\partial \xi} + J_{32}^* \frac{\partial u}{\partial \eta} + J_{33}^* \frac{\partial u}{\partial \zeta} = \sum_{i=1}^8 \left(J_{31}^* \frac{\partial N_i}{\partial \xi} + J_{32}^* \frac{\partial N_i}{\partial \eta} + J_{33}^* \frac{\partial N_i}{\partial \zeta} \right) u_i = \sum_{i=1}^8 c_i u_i \end{aligned}$$

Similarly,

$$\frac{\partial v}{\partial x} = \sum_{i=1}^8 a_i v_i; \quad \frac{\partial v}{\partial y} = \sum_{i=1}^8 b_i v_i; \quad \frac{\partial v}{\partial z} = \sum_{i=1}^8 c_i v_i; \quad \frac{\partial w}{\partial x} = \sum_{i=1}^8 a_i w_i; \quad \frac{\partial w}{\partial y} = \sum_{i=1}^8 b_i w_i \quad \text{and} \quad \frac{\partial w}{\partial z} = \sum_{i=1}^8 c_i w_i$$

Using above relations, the strain vector can be written as

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} = \sum_{i=1}^8 \begin{bmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & c_i \\ b_i & a_i & 0 \\ 0 & c_i & b_i \\ c_i & 0 & a_i \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \quad (5.8.8)$$

Now, the strain displacement relationship matrix $[B]$ can be identified from the above equation by comparing it to $\{\varepsilon\} = [B]\{d\}$.

5.8.3 Element Stiffness Matrix

The element stiffness matrix can be generated similar to two dimensional case using the following relations.

$$[k] = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} [B]^T [D] [B] d\xi d\eta d\zeta |J| \quad (5.8.9)$$

The size of the constitutive matrix $[D]$ for solid element will be 6×6 and is already discussed in module 1, lectures 3. For eight node brick element, the size of stiffness matrix will become 24×24 as number of nodes in one element is 8 and the degrees of freedom at each node is 3. It is well established that $2 \times 2 \times 2$ Gauss integration points are the optimal sampling points for eight node isoparametric brick elements.

5.8.4 Element Load Vector

The forces on an element can be generated due to its self weight or externally applied force which may be concentrated or distributed in nature. The distributed load may be uniform or non-uniform. All these types of loads are to redistributed to the nodes using finite element formulation.

5.8.4.1 Gravity load

The load vector due to body forces in general is given by

$$\{Q\} = \int_{\Omega} [N]^T \{X\} d\Omega \quad (5.8.10)$$

where $\{X\}$ is the body forces per unit volume. The nodal load vector at any node i may be expressed as

$$\{Q_i\} = \int_{\Omega} [N_i]^T \{X\} d\Omega \quad (5.8.11)$$

In case of gravity load, the force will act in the global negative Z direction. Therefore,

$$[N_i] = \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \quad \text{and} \quad \{X\} = \begin{Bmatrix} 0 \\ 0 \\ -\rho g \end{Bmatrix} \quad (5.8.12)$$

Here, the mass density of the material is ρ and the acceleration due to gravity is g . Thus, eq.(5.8.11) will become

$$\{Q_i\} = \int_{\Omega} \begin{Bmatrix} 0 \\ 0 \\ -N_i \rho g \end{Bmatrix} d\Omega \quad (5.8.13)$$

For isoparametric element the, the above expression will become

$$\{Q_i\} = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} \begin{Bmatrix} 0 \\ 0 \\ -N_i \rho \end{Bmatrix} |J| d\xi d\eta d\zeta \quad (5.8.14)$$

Using Gauss Quadrature integration rule, the above expression may be evaluated as

$$\{Q_i\} = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n w_i w_j w_k |J|(\xi_i, \eta_j, \zeta_k) \begin{Bmatrix} 0 \\ 0 \\ -N_i \rho g \end{Bmatrix}_{(\xi_i, \eta_j, \zeta_k)} \quad (5.8.15)$$

Where, n is the number of nodes in an element. For eight node linear brick element the value of n will be 8 and the integration order suggested is $2 \times 2 \times 2$. Similarly, for twenty node quadratic brick element, the value of n will be 20 and the integration order suggested is $3 \times 3 \times 3$.

5.8.4.2 Surface pressure

Let assume a uniform surface pressure of intensity q is acting normal to the element face. The load vector due to surface pressure is given by

$$\{Q\} = \int \int [N^s]^T \{p\} dA \quad (5.8.16)$$

The nodal load at any node i may be expressed as

$$\{Q_i\} = \int \int [N_i^s]^T \{p\} dA \quad (5.8.17)$$

In case of surface load, the value of $[N_i^s]$ in the above equation will become

$$[N_i^s] = \begin{bmatrix} N_i^s & 0 & 0 \\ 0 & N_i^s & 0 \\ 0 & 0 & N_i^s \end{bmatrix} \quad (5.8.18)$$

Here N_i^s is the interpolation function for the node i . For example, the value of N_i^s can be obtained by substituting $\xi=1$ in N_i^s for face 1. Thus, the surface pressure is expressed as,

$$\{p\} = \begin{Bmatrix} ql_3 \\ qm_3 \\ qn_3 \end{Bmatrix} \quad (5.8.19)$$

Where, l_3, m_3, n_3 are the direction cosines. Thus, eq.(5.8.17) can be expressed using eq.(5.8.19) in the following form.

$$\{Q_i\} = \int_{-1}^{+1} \int_{-1}^{+1} \begin{Bmatrix} N_i^s ql_3 \\ N_i^s qm_3 \\ N_i^s qn_3 \end{Bmatrix} dA \quad (5.8.20)$$

The value of dA is can be evaluated considering the cross product of vectors along the natural coordinates parallel to the loaded faces of the element. Thus,

$$dA = |e_2 \times e_3| d\eta d\zeta \quad (5.8.21)$$

5.8.5 Stress Computation

Using the relation of $\{F\} = [K]\{d\}$, the unknown nodal displacement vector $\{d\}$ are calculated in global coordinate system. Once the nodal displacements are obtained, the strain components as each node can be computed using strain-displacement relations for each element. Similarly element stress can be calculated using stress-strain relation. These stresses at Gauss points are extrapolated to the corresponding nodes of the element to find the nodal stresses. In general, for three dimensional state of stress there are at least three planes, called principal planes. The corresponding stress vector is perpendicular to the plane and where there are no normal shear stresses. These three stresses which are normal to these principal planes are called principal stresses. The principal stresses σ_1, σ_2 , and σ_3 are computed from the roots of the cubic equation represented by the determinant of the flowing.

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0 \quad (5.8.22)$$

The characteristic equation has three real roots σ_i , due to the symmetry of the stress tensor. The principal stresses are arranged so that $\sigma_1 > \sigma_2 > \sigma_3$. The maximum shear stress can be computed from the following relations.

$$\tau_{\max} = \text{largest of } \left(\frac{|\sigma_1 - \sigma_2|}{2}, \frac{|\sigma_2 - \sigma_3|}{2} \text{ and } \frac{|\sigma_3 - \sigma_1|}{2} \right) \quad (5.8.23)$$

These three shear stress components will occur on planes oriented at 45° from the principal planes. The distortion energy theory suggests that the total strain energy can be divided into two components. They are (i) volumetric strain energy and (ii) distortion or shear strain energy. It is anticipated that yield develops if the distortion component exceeds that at the yield point for a simple tensile test. From the concept of distortion energy theory, the equivalent stress which is historically known as Von Mises stress are defined as

$$\sigma_e = \sqrt{\left(\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right)} \quad (5.8.24)$$

The Von Mises stresses offer a measure of the shear or distortional stress in the material. In general, this type of stress tends to cause yielding in metals.

Worked out Examples

Example 5.1 Calculation of nodal loads on a triangular element

A CST element as shown in Fig. 5.I gets axial loading of (F_{x1}) 10 kN/m in X direction and (F_{y1}) 20 kN/m in Y direction. Compute the nodal loads in the element.

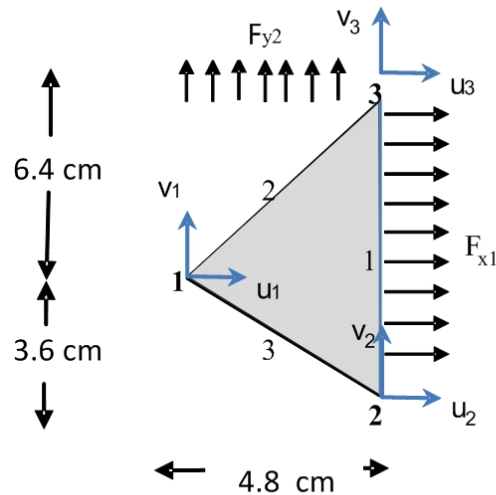


Fig. 5.I Distributed loading on a triangular element

From the above figure, the length of sides 1, 2 and 3 are calculated and will be 10, 8 and 6 cm respectively. First, let consider side 1:

$$\{F\}_1 = \int_0^{l_1} [N]^T \{F_r\} ds$$

Or,

$$\{F\}_1 = \int_0^{l_1} \begin{bmatrix} 0 & 0 \\ L_2 & 0 \\ L_3 & 0 \\ 0 & 0 \\ 0 & L_2 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} F_{x1} \\ F_{y1} \end{Bmatrix} ds = \int_0^{l_1} \begin{bmatrix} 0 & 0 \\ L_2 & 0 \\ L_3 & 0 \\ 0 & 0 \\ 0 & L_2 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} 10 \\ 0 \end{Bmatrix} ds = 10 \times \int_0^{l_1} \begin{Bmatrix} 0 \\ L_2 \\ L_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} ds$$

Putting, $\oint L_1^p L_2^q ds = \frac{p!q!}{(p+q+1)!} l$, we will get,

$$\{F\}_1 = 10 \times \frac{l_1}{2} \times \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 10 \times \frac{0.1}{2} \times \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Similarly for side 2,

$$\{F\}_2 = \int_0^{l_2} \begin{bmatrix} L_1 & 0 \\ 0 & 0 \\ L_3 & 0 \\ 0 & L_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} F_{x2} \\ F_{y2} \end{Bmatrix} ds = \int_0^{l_2} \begin{bmatrix} L_1 & 0 \\ 0 & 0 \\ L_3 & 0 \\ 0 & L_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} 0 \\ 20 \end{Bmatrix} ds = 20 \times \frac{l_2}{2} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.8 \\ 0 \\ 0 \\ 0.8 \end{Bmatrix} \text{ kN}$$

Since no force is acting on side 3,

$$\{F\}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Hence, the nodal load vector in all the nodes in x and y directions will become,

$$\{F\} = \begin{Bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0.8 \\ 0 \\ 0.8 \end{Bmatrix} \text{ kN.}$$