Lecture 1: Stiffness of Truss Members

4.1.1 Introduction

Analysis of frame structures can be carried out by the approach of stiffness method. However, such types of structures can also be analyzed by finite element method. A unified formulation will be demonstrated based on finite element concept in this module for the analysis of frame like structures. A truss structure is composed of slender members pin jointed together at their end points. Truss element can resist only axial forces (tension or compression) and can deform only in its axial direction. Therefore, in case of a planar truss, each node has components of displacements parallel to X and Y axis. Planar trusses lie in a single plane and are used to support roofs and bridges. Such members will not be able to carry transverse load or bending moment. The major benefits of use of truss structures are: lightweight, reconstructable, reconfigurable and mobile. Configuration of few standard truss structures are shown in Fig. 4.1.1.

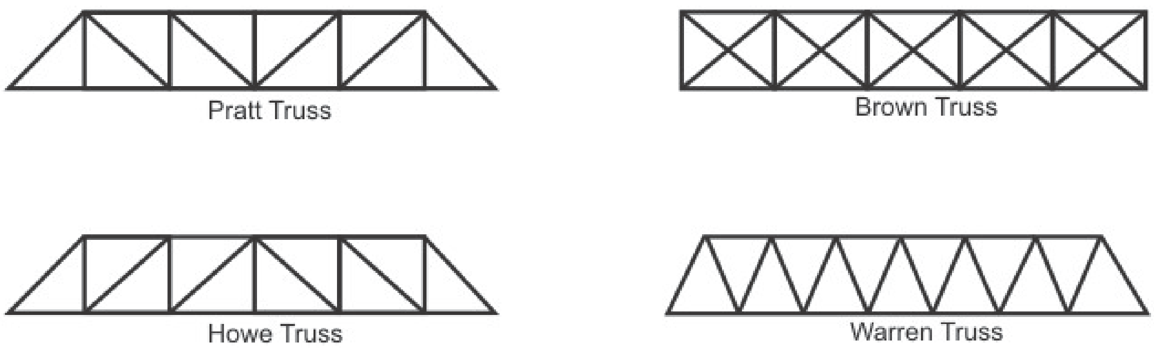

Fig. 4.1.1 Configuration of various truss structures

4.1.2 Element Stiffness of a Truss Member

Since, the truss is an axial force resisting member, the displacement along its axis only will be developed due to axial load. Therefore, using Pascal's triangle, the displacement function of truss member for development of shape function can be expressed as:

$$u(x) = \alpha_0 + \alpha_1 x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \end{Bmatrix} \tag{4.1.1}$$

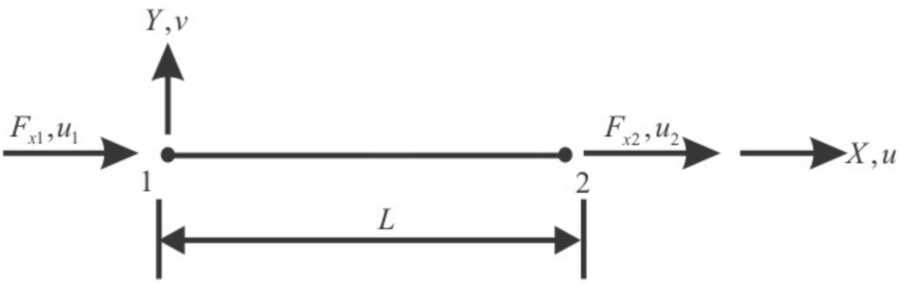

Fig. 4.1.2 Axial force on the member along X axis

Applying boundary conditions as shown in Fig. 4.1.2:

At $x = 0$, $u(0) = u_1$ and at $x = L$, $u(L) = u_2$

Thus, $\alpha_0 = u_1$ and $\alpha_1 = \dfrac{u_2 - u_1}{L}$. Therefore,

$$u(x) = \left(1 - \frac{x}{L}\right)u_1 + \frac{x}{L}u_2 = [N]\{u\} \tag{4.1.2}$$

Here, N is the shape function of the element and is expressed as:

$$[N] = \begin{bmatrix} 1 - \dfrac{x}{L} & \dfrac{x}{L} \end{bmatrix} \tag{4.1.3}$$

So we get the element stiffness matrix as

$$[k] = \iiint_{\Omega} [B]^T [D][B] d\Omega \tag{4.1.4}$$

Where, $[B] = \dfrac{d[N]}{dx} = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}$

So, the stiffness matrix will become:

$$= \int_0^L [B]^T E[B] A dx = AE \int_0^L \begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix} \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} dx = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Thus, the stiffness matrix of the truss member along its member axis will be:

$$[k] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.1.5}$$

4.1.3 Element Stiffness of Truss Member with Varying Cross Section

Now, let us find the stiffness matrix of a pin-jointed member of length L with respect to local axis, having cross sectional areas A_1 and A_2 at the two ends of the member as shown in the figure below.

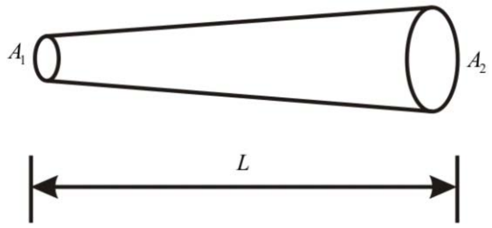

Fig. 4.1.3 Member with varying cross section

From the above figure, the cross sectional area at a distance of *x* from left end can be expressed as:

$$A_x = A_1 + \frac{A_2 - A_1}{L} x \qquad (4.1.6)$$

As it is a pin-jointed member, the displacement at any point may be expressed in terms of nodal displacement as $u = N_1 u_1 + N_2 u_2$.

Similarly the cross sectional area at any point may be represented in terms of the cross sectional area of the two ends. Thus $A_x = N_1 A_1 + N_2 A_2$

Where the shape functions are: $N_1 = 1 - \frac{x}{L}; N_2 = \frac{x}{L}$

Now, the strain may be written as:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 = -\frac{1}{L} u_1 + \frac{1}{L} u_2 = \frac{1}{L}[-1 \quad 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [B]\{u\} \qquad (4.1.7)$$

As the stress is proportional to strain according to Hook’s law, the stress-strain relationship will be as follows:

$$\sigma_x = E\varepsilon_x = \frac{E}{L}[-1 \quad 1]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = E[B]\{u\} \qquad (4.1.8)$$

Now the strain energy may be expressed as

$$U = \frac{1}{2}\int_V \varepsilon_x{}^T \sigma_x dv = \frac{1}{2}\int_0^L \varepsilon_x{}^T E\varepsilon_x A_x dx = \frac{1}{2}\int_0^L \{u\}^T [B]^T E[B]\{u\} A_x dx \qquad (4.1.9)$$

Applying Castigliano’s theorem, the force will become:

$$\{F\} = \frac{\partial U}{\partial \{u\}} = \int_0^L [B]^T E[B]\{u\} A_x dx = \frac{E}{L^2}\int_0^L [-1 \quad 1]^T [-1 \quad 1] A_x dx \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [k]\{d\}$$

(4.1.10)

Thus, the stiffness matrix will be:

$$[k] = \frac{E}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^L \left(A_1 + \frac{A_2 - A_1}{L} x \right) dx \quad = \frac{E}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[A_1 x + \frac{A_2 - A_1}{2L} x^2 \right]_0^L$$

$$= \frac{E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left[A_1 + \frac{A_2 - A_1}{2} \right] = \frac{E}{2L}(A_1 + A_2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (4.1.11)$$

4.1.4 Generalized Stiffness Matrix of a Plane Truss Member

Let us consider a member making an angle 'θ' with *X* axis as shown in the figure below. By resolving the forces along local *X* and *Y* direction, the following relations are obtained.

$$\left.\begin{aligned} \overline{F}_{x1} &= F_{x1}\cos\theta + F_{y1}\sin\theta \\ \overline{F}_{x2} &= F_{x2}\cos\theta + F_{y2}\sin\theta \\ \overline{F}_{y1} &= -F_{x1}\sin\theta + F_{y1}\cos\theta \\ \overline{F}_{y2} &= -F_{x2}\sin\theta + F_{y2}\cos\theta \end{aligned}\right\} \qquad (4.1.12)$$

Where, $\overline{F}_{x1}$ and $\overline{F}_{x2}$ are the axial forces along the member axis $\overline{X}$. Similarly, $\overline{F}_{y1}$ and $\overline{F}_{y2}$ are the forces perpendicular to the member axis $\overline{X}$.

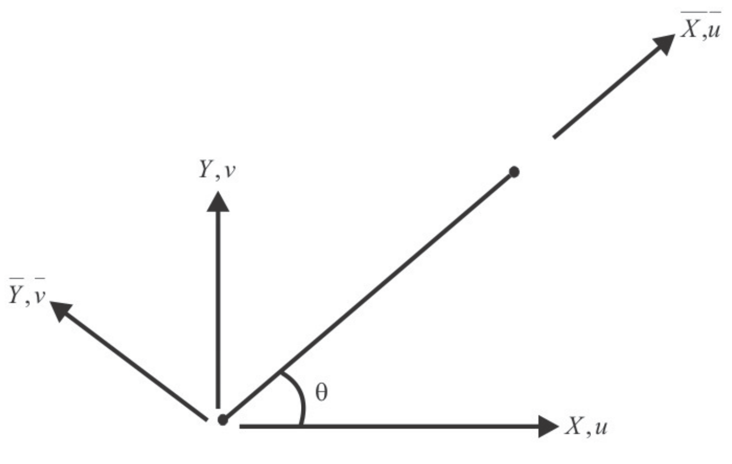

Fig. 4.1.4 Inclined truss member

The relationship expressed in eq. (4.1.12) can be rewritten in matrix form as follows:

$$\begin{Bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ \bar{F}_{x2} \\ \bar{F}_{y2} \end{Bmatrix} = \begin{bmatrix} \cos\theta & sin\theta & 0 & 0 \\ -sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & sin\theta \\ 0 & 0 & -sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} \tag{4.1.13}$$

Now, the above equation can be expressed in short as:

$$\{\bar{F}\} = [T]\{F\} \tag{4.1.14}$$

Here, $[T]$ is called transformation matrix. This relates between the global $(X, Y$ axis) and member axis $(\bar{X}, \bar{Y}$ axis). Similarly, the relations of nodal displacements between two coordinate systems may be written as:

$$\{\bar{d}\} = [T]\{d\} \tag{4.1.15}$$

Again, the equation stated in (4.1.5) can be generalized and expressed with respect to the member axis including force and displacement vector as:

$$\begin{Bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ \bar{F}_{x2} \\ \bar{F}_{y2} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{u}_2 \\ \bar{v}_2 \end{Bmatrix} \tag{4.1.16}$$

Where, the nodal forces in $\bar{Y}$ direction are zero. The above equation may also be expressed in short as:

$$\{\bar{F}\} = [\bar{k}]\{\bar{d}\} \tag{4.1.17}$$

Where, the matrices in the above equation are written with respect to the member axis. Now, eq. (4.1.17) can be rewritten with the use of eq. (4.1.14) and (4.1.15) as given below.

$$[T]\{F\} = [\bar{k}][T]\{d\} \tag{4.1.18}$$

Or,

$$\{F\} = [T]^{-1}[\bar{k}][T]\{d\} \tag{4.1.19}$$

Here, the transformation matrix [T] is orthogonal, i.e., $[T]^{-1}$ is equal to $[T]^{T}$. Therefore, from the above relationship, the generalized stiffness matrix can be expressed as:

$$[k] = [T]^T [\bar{k}][T] \tag{4.1.20}$$

Thus,

$$[k] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{4.1.21}$$

Or,

$$[k] = \frac{AE}{L} \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix} \tag{4.1.22}$$

The above stiffness matrix can be used for the analysis of two-dimensional truss problems.

Lecture 2: Analysis of Truss

4.2.1 Element Stiffness of a 3 Node Truss Member

Fig. 4.2.1 3-node truss member

Here, the displacement function using Pascal's triangle can be expressed as:

$$u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{Bmatrix} \tag{4.2.1}$$

Applying boundary conditions:
At x= 0, u(0)= u_1, x=L/2, u(L/2) = u_2 and at x=L, u(L) = u_3

And solving for α_0, α_1 and α_2

$$\alpha_0 = u_1 \ , \quad \alpha_1 = \frac{-3u_1 + 4u_2 - u_3}{L} \quad \text{and} \quad \alpha_2 = \frac{2u_1 - 4u_2 + 2u_3}{L^3}$$

Therefore,

$$u(x) = \left(1 - \frac{3x}{L} + \frac{2x^2}{L^2}\right) u_1 + \left(\frac{4x}{L} - \frac{4x^2}{L^2}\right) u_2 + \left(-\frac{x}{L} + \frac{2x^2}{L^2}\right) u_3 = [N]\{u\} \tag{4.2.2}$$

Here, *N* is the shape function of the element and is expressed as:

$$[N] = \left[\left(1 - \frac{3x}{L} + \frac{2x^2}{L^2}\right) \ \left(\frac{4x}{L} - \frac{4x^2}{L^2}\right) \ \left(-\frac{x}{L} + \frac{2x^2}{L^2}\right) \right] \tag{4.2.3}$$

Now, the element stiffness matrix can be written as

$$[k] = \iiint_{\Omega} [B]^T [D][B] \, d\Omega \tag{4.2.4}$$

Where, $[B] = \dfrac{d[N]}{d \ x} = \left[-\dfrac{3}{L} + \dfrac{4x}{L^2} \quad \dfrac{4}{L} - \dfrac{8x}{L^2} \quad -\dfrac{1}{L} + \dfrac{4x}{L^2} \right]$

So, the stiffness matrix will be:

$$[k] = \iiint_{\Omega} [B]^T [D][B] \, d\Omega = \int_0^L [B]^T E [B] A dx$$

$$= AE\int_0^L \begin{Bmatrix} -\frac{3}{L}+\frac{4x}{L^2} \\ \frac{4}{L}-\frac{8x}{L^2} \\ -\frac{1}{L}+\frac{4x}{L^2} \end{Bmatrix} \times \begin{bmatrix} -\frac{3}{L}+\frac{4x}{L^2} & \frac{4}{L}-\frac{8x}{L^2} & -\frac{1}{L}+\frac{4x}{L^2} \end{bmatrix} dx$$

$$= \frac{AE}{L^2}\int_0^L \begin{bmatrix} 9+\frac{16x^2}{L^2}-\frac{24x}{L} & -12+\frac{40x}{L}-\frac{32x^2}{L^2} & 3-\frac{16x}{L}+\frac{16x^2}{L^2} \\ -12+\frac{40x}{L}-\frac{32x^2}{L^2} & 16-\frac{64x}{L}+\frac{64x^2}{L^2} & -4+\frac{24x}{L}-\frac{32x^2}{L^2} \\ 3-\frac{16x}{L}+\frac{16x^2}{L^2} & -4+\frac{24x}{L}-\frac{32x^2}{L^2} & 1-\frac{8x}{L}-\frac{16x^2}{L^2} \end{bmatrix} dx \tag{4.2.5}$$

After integrating the above equation, the stiffness matrix of the 3-node truss member will become:

$$[k] = \frac{\mathrm{AE}}{3\mathrm{L}}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{4.2.6}$$

4.2.2 Worked Out Example

Analyze the truss shown below by finite element method. Assume the cross sectional area of the inclined member as 1.5 times the area (A) of the horizontal and vertical members. Assume modulus of elasticity is constant for all the members and is E.

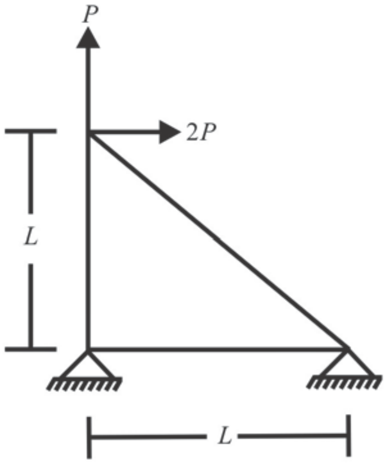

Fig. 4.2.2 Plane truss

Solution

The analysis of truss starts with the numbering of members and joints as shown below:

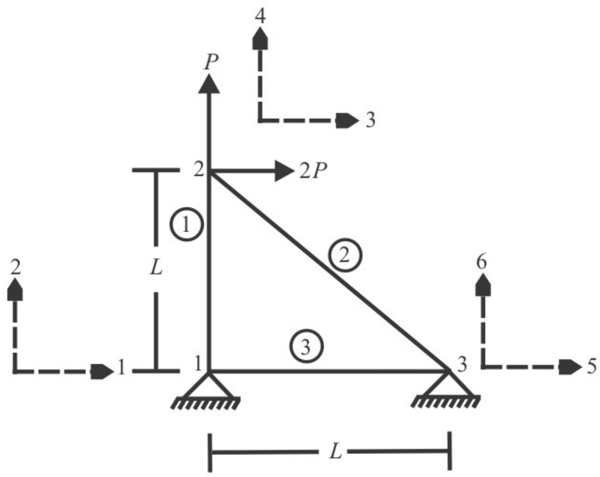

Fig. 4.2.3 Numbering of members and nodes

The member information for the truss is shown in Table 4.2.1. The member and node numbers, modulus of elasticity, cross sectional areas are the necessary input data. From the coordinate of the nodes of the respective members, the length of each member is computed. Here, the angle θ has been calculated considering anticlockwise direction. The signs of the direction cosines depend on the choice of numbering the nodal connectivity.

Table 4.2.1 Member Information for Truss

Member No.	Starting Node	Ending Node	Value of θ	Area	Modulus of Elasticity
1	1	2	90°	*A*	*E*
2	2	3	315°	1.5*A*	*E*
3	3	1	180°	*A*	*E*

Now, let assume the coordinate of node 1 as (0, 0). The coordinate and restraint joint information are given in Table 4.2.2. The integer 1 in the restraint list indicates the restraint exists and 0 indicates the restraint at that particular direction does not exist. Thus, in node no. 2, the integer 0 in *x* and *y* indicates that the joint is free in *x* and *y* directions.

Table 4.2.2 Nodal Information for Plane Truss

Node No.	Coordinates		Restraint List	
	x	*y*	*x*	*y*
1	0	0	1	1
2	0	*L*	0	0
3	*L*	0	1	1

The stiffness matrices of each individual member can be found out from the stiffness matrix equation as shown below.

$$[k] = \frac{AE}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Thus the local stiffness matrices of each member are calculated based on their individual member properties and orientations and written below.

$$[k]_1 = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad [k]_2 = \frac{3AE}{4\sqrt{2}L}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

and

$$[k]_3 = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} 5 & \mathbf{6} & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 5 \\ 6 \\ 1 \\ 2 \end{array}$$

Global stiffness matrix can be formed by assembling the local stiffness matrices into globally. Thus the global stiffness matrix are calculated from the above relations and obtained as follows:

$$\begin{array}{cccccc|c} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \\ & & & & & & \mathbf{1} \\ & & & & & & \mathbf{2} \\ & & & & & & \mathbf{3} \\ & & & & & & \mathbf{4} \\ & & & & & & \mathbf{5} \\ & & & & & & \mathbf{6} \end{array}$$

$$[K]=\frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ 0 & -1 & -\frac{3}{4\sqrt{2}} & 1+\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} \\ -1 & 0 & -\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}}+1 & -\frac{3}{4\sqrt{2}} \\ 0 & 0 & \frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \end{bmatrix}$$

The equivalent load vector for the given truss can be written as: $\{F\}=\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 2P \\ P \\ 0 \\ 0 \end{Bmatrix}$

Let us assume that u and v are the horizontal and vertical displacements respectively at joints. Thus the displacement vector will be expressed as follows:

$$\{d\}=\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the relationship between the force and the displacement will be:

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ 2P \\ P \\ F_{x3} \\ F_{y3} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ 0 & -1 & -\frac{3}{4\sqrt{2}} & 1+\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} \\ -1 & 0 & -\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}}+1 & -\frac{3}{4\sqrt{2}} \\ 0 & 0 & \frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & -\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix}$$

From the above relation, the unknown displacements u_2 and v_2 can be found out through computer programming. However, as numbers of unknown displacements in this case are only two, the solution can be obtained by manual calculations. The above equation may be rearranged with respect to unknown and known displacements in the following form:

$$\begin{Bmatrix} F_\alpha \\ F_\beta \end{Bmatrix} = \begin{bmatrix} k_{\alpha\alpha} & k_{\alpha\beta} \\ k_{\beta\alpha} & k_{\beta\beta} \end{bmatrix} \begin{Bmatrix} d_\alpha \\ d_\beta \end{Bmatrix}$$

Thus the developed matrices for the truss problem can be rearranged as:

$$\left\{ \begin{array}{c} 2P \\ P \\ \hline F_{x1} \\ F_{y1} \\ F_{x3} \\ F_{y3} \end{array} \right\} = \frac{AE}{L} \left[\begin{array}{cc|cccc} \frac{3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} & 0 & 0 & \frac{-3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ \frac{-3}{4\sqrt{2}} & 1+\frac{3}{4\sqrt{2}} & 0 & -1 & \frac{3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} \\ \hline 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ \frac{-3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} & -1 & 0 & \frac{3}{4\sqrt{2}}+1 & -\frac{3}{4\sqrt{2}} \\ \frac{3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} & 0 & 0 & -\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \end{array} \right] \left\{ \begin{array}{c} u_2 \\ v_2 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array} \right\}$$

.

The above relation may be condensed into following

$$\begin{Bmatrix} 2P \\ P \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} \frac{3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} \\ \frac{-3}{4\sqrt{2}} & 1+\frac{3}{4\sqrt{2}} \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$$

The unknown displacements can be derived from the relationships expressed in the above equation.

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} \frac{3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} \\ \frac{-3}{4\sqrt{2}} & 1+\frac{3}{4\sqrt{2}} \end{bmatrix}^{-1}\begin{Bmatrix} 2P \\ P \end{Bmatrix} = \frac{4\sqrt{2}L}{3AE}\begin{bmatrix} 1+\frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ \frac{3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \end{bmatrix}\begin{Bmatrix} 2P \\ P \end{Bmatrix}$$

Thus the unknown displacement at node 2 of the truss structure will become:

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{PL}{AE}\begin{bmatrix} 3+\frac{8\sqrt{2}}{3} \\ 3 \end{bmatrix}$$

<u>Support Reactions:</u>

The support reactions $\{P_s\}$ can be determined from the following relation:

$$\{P_s\} = -\{P_{cs}\} + \left[K_{\beta\alpha}\right]\{d_\alpha\}$$

Where, $\{P_{cs}\}$ correspond to equivalent loadings at supports. Thus, the support reaction of the present truss structure will be:

$$\{P_s\} = -\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \frac{AE}{L}\begin{bmatrix} 0 & 0 \\ 0 & -1 \\ \frac{-3}{4\sqrt{2}} & \frac{3}{4\sqrt{2}} \\ \frac{3}{4\sqrt{2}} & \frac{-3}{4\sqrt{2}} \end{bmatrix}\frac{PL}{AE}\begin{bmatrix} 3+\frac{8\sqrt{2}}{3} \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3P \\ -2P \\ 2P \end{bmatrix}$$

<u>Member End Actions:</u>

Now, the member end actions can be obtained from the corresponding member stiffness and the nodal displacements. The member end forces are derived as shown below.

<u>Member –1</u>

$$\begin{Bmatrix} F_{mx1} \\ F_{my1} \\ F_{mx2} \\ F_{my2} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 3+\frac{8\sqrt{2}}{3} \\ 3 \end{Bmatrix}\frac{PL}{AE} = \begin{Bmatrix} 0 \\ -3P \\ 0 \\ 3P \end{Bmatrix}$$

Member – 2

$$\begin{Bmatrix} F_{mx2} \\ F_{my2} \\ F_{mx3} \\ F_{my3} \end{Bmatrix} = \frac{3AE}{4\sqrt{2}L} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 3+\frac{8\sqrt{2}}{3} \\ 3 \\ 0 \\ 0 \end{Bmatrix} \frac{PL}{AE} = \begin{Bmatrix} 2P \\ -2P \\ -2P \\ 2P \end{Bmatrix}$$

Member –3

$$\begin{Bmatrix} F_{mx3} \\ F_{my3} \\ F_{mx1} \\ F_{my1} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \frac{PL}{AE} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Thus the member forces in all members of the truss will be:

$$\{F_m\} = \begin{Bmatrix} 3P \\ -\sqrt{(2P)^2 + (2P)^2} \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3P \\ -2\sqrt{2}P \\ 0 \end{Bmatrix}$$

The reaction forces at the supports of the truss structure will be:

$$\{F_R\} = \begin{Bmatrix} 0 \\ -3P \\ -2P \\ 2P \end{Bmatrix}$$

Thus the member force diagram will be as shown in Fig. 4.2.4.

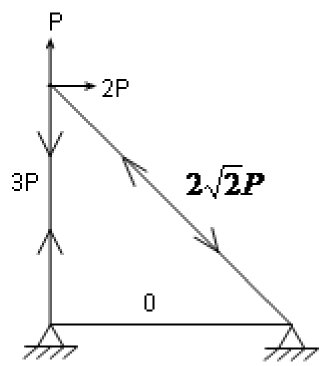

Fig. 4.2.4 Member Force Diagram

Lecture 3: Stiffness of Beam Members

4.3.1 Introduction

A beam is a structural member which is capable of withstanding load primarily by resisting bending. The primary tool for analysis of beam is the Euler–Bernoulli beam equation. Other methods for determining the deflection of beams include "slope deflection method" and "method of virtual work". For calculation of internal forces of beam include "moment distribution method", force or flexibility method and stiffness method. However, all these methods have limitations if either of geometry, loading, material properties or boundary conditions becomes arbitrary in nature. Finite element techniques can well handle such cases and relieve the analyzer of making simplifications to arrive approximate solutions.

4.3.2 Derivation of Shape Function

The degrees of freedom at each node for a beam member will be (i) vertical deflection and (ii) rotation. For a beam member, the slope of the elastic curve θ is given by: $\theta = \dfrac{dv}{dx}$, where the variable *v* is the displacement function of the beam. As the beam has two degrees of freedom at each node, the variation of *v* will be cubic and can be expressed using Pascal's triangle as:

$$v(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{4.3.1}$$

and

$$\theta = \frac{dv}{dx} = \begin{bmatrix} 0 & 1 & 2x & 3x^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{4.3.2}$$

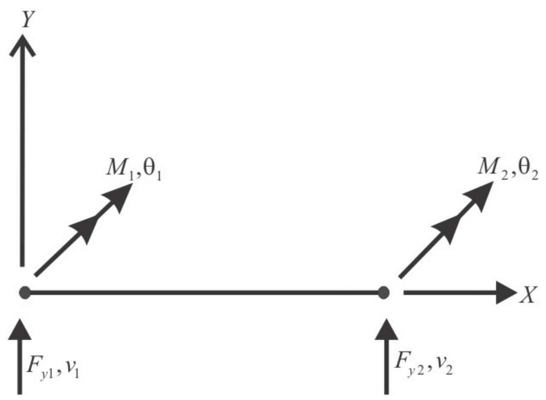

Fig. 4.3.1 Beam element

Now, applying boundary conditions, the following expressions from the above relations can be obtained:

At x=0:

$$V_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}; \; \theta_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix};$$

At x=L:

$$V_2 = \begin{bmatrix} 1 & L & L^2 & L^3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}; \; \theta_2 = \begin{bmatrix} 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

Thus combining the above expressions one can write:

$$\begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = [A]\{\alpha\} \qquad (4.3.3)$$

So,

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}^{-1} \begin{Bmatrix} V_1 \\ \theta_2 \\ V_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix} \begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix} \qquad (4.3.4)$$

Therefore,

$$v(x) = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{l^2} & -\frac{2}{l} & \frac{3}{l^2} & -\frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & -\frac{2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix} \qquad (4.3.5)$$

Where,

$$N_1 = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3; \; N_2 = x - \frac{2}{L}x^2 + \frac{x^3}{L^2}; \; N_3 = \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \text{ and } N_4 = -\frac{x^2}{L} + \frac{x^3}{L^2} \qquad (4.3.6)$$

N is called shape function which interpolates the beam displacement in terms of its nodal displacements.

4.3.3 Derivation of Element Stiffness Matrix

Now, the strain displacement relationship matrix [B] can be expressed from the following expressions with the help of eq. (4.3.1):

$$\chi = \frac{d^2 v}{dx^2} = [0 \quad 0 \quad 2 \quad 6x]\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_4 \end{Bmatrix} = [B]\{\alpha\} = [B][A]^{-1}\{d\} \tag{4.3.7}$$

Where, $[B] = [0 \quad 0 \quad 2 \quad 6x]$; $[A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}$; $\{d\} = \begin{Bmatrix} V_1 \\ \theta_2 \\ V_2 \\ \theta_2 \end{Bmatrix}$

From the moment curvature relationship, we can write:

$$M = EI\chi = EI\frac{d^2 v}{dx^2} = EI[B][A]^{-1}\{d\} \tag{4.3.8}$$

Strain energy,

$$U = \int_0^L \frac{1}{2}[\chi]^T [M]dx = \frac{EI}{2}\int_0^L \{d\}^T \left[A^{-1}\right]^T [B]^T [B]\left[A^{-1}\right]\{d\}dx \tag{4.3.9}$$

Thus,

$$\{F\} = \frac{\partial U}{\partial\{d\}} = EI\int_0^L \left[A^{-1}\right]^T [B]^T [B]\left[A^{-1}\right]\{d\}dx \tag{4.3.10}$$

So, the stiffness matrix will be:

$$[k] = EI\int_0^L \left[A^{-1}\right]^T [B]^T [B]\left[A^{-1}\right]dx = EI\left[A^{-1}\right]^T \int_0^L [B]^T [B]dx[A]^{-1} \tag{4.3.11}$$

Now, $\int_0^L [B^T][B]dx = \int_0^L \begin{Bmatrix} 0 \\ 0 \\ 2 \\ 6x \end{Bmatrix}[0 \quad 0 \quad 2 \quad 6x]dx = \int_0^L \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 12x \\ 0 & 0 & 12x & 36x^2 \end{bmatrix}dx = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4L & 6L^2 \\ 0 & 0 & 6L^2 & 12L^3 \end{bmatrix}$

(4.3.12)

So,

$$[k] = EI\left[A^{-1}\right]^{T}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4L & 6L^2 \\ 0 & 0 & 6L^2 & 12L^3 \end{bmatrix}[A]^{-1}$$

$$[k] = EI\begin{bmatrix} 1 & 0 & -\frac{3}{L^2} & \frac{2}{L^3} \\ 0 & 1 & -\frac{2}{L} & \frac{1}{L^2} \\ 0 & 0 & \frac{3}{L^2} & -\frac{2}{L^3} \\ 0 & 0 & -\frac{1}{L} & \frac{1}{L^2} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4L & 6L^2 \\ 0 & 0 & 6L^2 & 12L^3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix}$$

$$= EI\begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 2 & 6l \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & \frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix} = EI\begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & \frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & -\frac{6}{L^2} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix}$$

Thus, the element stiffness of a beam member is:

$$[k] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad (4.3.13)$$

4.3.4 Generalized Stiffness Matrix of a Beam Member

Consider a beam member making an angle ‘θ’ with *X* axis as shown in Fig 4.3.2 below. By resolving the forces along local *X* and *Y* direction, the following relations are obtained.

$$\left.\begin{aligned}
\bar{F}_{x1} &= F_{x1}\cos\theta + F_{y1}\sin\theta \\
\bar{F}_{x2} &= F_{x2}\cos\theta + F_{y2}\sin\theta \\
\bar{F}_{y1} &= -F_{x1}\sin\theta + F_{y1}\cos\theta \\
\bar{F}_{y2} &= -F_{x2}\sin\theta + F_{y2}\cos\theta \\
\bar{M}_1 &= M_1 \\
\bar{M}_2 &= M_2
\end{aligned}\right\} \tag{4.3.14}$$

Where, $\bar{F}_{x1}$ and $\bar{F}_{x2}$ are the axial forces along the member axis $\bar{X}$. Similarly, $\bar{F}_{y1}$ and $\bar{F}_{y2}$ are the forces perpendicular to the member axis $\bar{X}$. $\bar{M}_1$ and $\bar{M}_2$ are the moment about its axis at node 1 and 2 respectively.

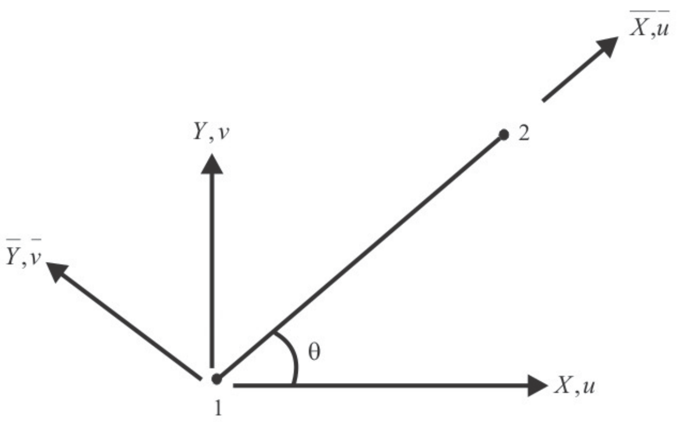

Fig. 4.3.2 Inclined beam member

The relationship expressed in eq. (4.3.14) can be rewritten in matrix form as follows:

$$\begin{Bmatrix} \bar{F}_{x1} \\ \bar{F}_{y1} \\ \bar{M}_1 \\ \bar{F}_{x2} \\ \bar{F}_{y2} \\ \bar{M}_2 \end{Bmatrix} = \begin{bmatrix} \cos\theta & sin\theta & 0 & 0 & 0 & 0 \\ -sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & sin\theta & 0 \\ 0 & 0 & 0 & -sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ F_{x2} \\ F_{y2} \\ M_2 \end{Bmatrix} \tag{4.3.15}$$

Now, the above equation can be expressed in short as:

$$\{\bar{F}\} = [T]\{F\} \tag{4.3.16}$$

Similarly, the displacement vector in local coordinate system $(\bar{X}, \bar{Y})$ may be transformed to global (X, Y) coordinate system by the following relation.

$$\{\bar{d}\} = [T]\{d\} \tag{4.3.17}$$

The force-displacement relation in local coordinate system may be expressed as:

$$\begin{Bmatrix} \overline{Fx_1} \\ \overline{Fy_1} \\ \overline{M_1} \\ \overline{Fx_2} \\ \overline{Fy_2} \\ \overline{M_2} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} \overline{u_1} \\ \overline{v_1} \\ \overline{\theta_1} \\ \overline{u_2} \\ \overline{v_2} \\ \overline{\theta_2} \end{Bmatrix} \tag{4.3.18}$$

The matrices in the above equation are written with respect to the member axis. Now, the eq. (4.3.18) can be rewritten as follows with the use of eqs. (4.3.16) and (4.3.17).

$$[T]\{F\} = [\bar{k}][T]\{d\} \tag{4.3.19}$$

Or,

$$\{F\} = [T]^{-1}[\bar{k}][T]\{d\} \tag{4.3.20}$$

Here, the transformation matrix $[T]$ is orthogonal. Thus, from the above relationship, the generalized stiffness matrix can be expressed as:

$$[k] = [[T]^T[\bar{k}]][T] \tag{4.3.21}$$

Considering $\lambda = \cos\theta$ and $\mu = \sin\theta$ the above expression can be written as follows:

$$[k]=EI\begin{bmatrix}\lambda & -\mu & 0 & 0 & 0 & 0\\ \mu & \lambda & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda & -\mu & 0\\ 0 & 0 & 0 & \mu & \lambda & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{12}{L^3} & \frac{6}{L^2} & 0 & -\frac{12}{L^3} & \frac{6}{L^2}\\ 0 & \frac{6}{L^2} & \frac{4}{L} & 0 & -\frac{6}{L^2} & \frac{2}{L}\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{12}{L^3} & -\frac{6}{L^2} & 0 & \frac{12}{L^3} & -\frac{6}{L^2}\\ 0 & \frac{6}{L^2} & \frac{2}{L} & 0 & -\frac{6}{L^2} & \frac{4}{L}\end{bmatrix}\begin{bmatrix}\lambda & \mu & 0 & 0 & 0 & 0\\ -\mu & \lambda & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda & \mu & 0\\ 0 & 0 & 0 & -\mu & \lambda & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \tag{4.3.22}$$

Thus, the generalized stiffness matrix of a beam member is derived as:

$$[k]=EI\begin{bmatrix}\frac{12\mu^2}{L^3} & -\frac{12\mu\lambda}{L^3} & -\frac{6\mu}{L^2} & -\frac{12\mu^2}{L^3} & \frac{12\mu\lambda}{L^3} & -\frac{6\mu}{L^2}\\ -\frac{12\mu\lambda}{L^3} & \frac{12\lambda^2}{L^3} & \frac{6\lambda}{L^2} & \frac{12\mu\lambda}{L^3} & -\frac{12\lambda^2}{L^3} & \frac{6\lambda}{L^2}\\ -\frac{6\mu}{L^2} & \frac{6\lambda}{L^2} & \frac{4}{L} & \frac{6\mu}{L^2} & -\frac{6\lambda}{L^2} & \frac{2}{L}\\ -\frac{12\mu^2}{L^3} & \frac{12\mu\lambda}{L^3} & \frac{6\mu}{L^2} & \frac{12\mu^2}{L^3} & -\frac{12\mu\lambda}{L^3} & \frac{6\mu}{L^2}\\ \frac{12\mu\lambda}{L^3} & -\frac{12\lambda^2}{L^3} & -\frac{6\lambda}{L^2} & -\frac{12\mu\lambda}{L^3} & \frac{12\lambda^2}{L^3} & -\frac{6\lambda}{L^2}\\ -\frac{6\mu}{L^2} & \frac{6\lambda}{L^2} & \frac{2}{L} & \frac{6\mu}{L^2} & -\frac{6\lambda}{L^2} & \frac{4}{L}\end{bmatrix} \tag{4.3.23}$$

Lecture 4: Analysis of Continuous Beam

4.4.1 Equivalent Loading on Beam Member

In finite element analysis, the external loads are necessary to be acting at the joints, which does not happen always; as some forces may act on the member. The forces acting on the member should be replaced by equivalent forces acting at the joints. These joint forces obtained from the forces on the members are called equivalent joint loads. These joint loads are combined with the actual joint loads to provide the combined joint loads, which are then utilized in the analysis.

4.4.1.1 Varying Load

Let a beam is loaded with a linearly varying load as shown in the figure below. The equivalent forces at nodes can be expressed using finite element technique. If w(x) is the function of load, then the nodal load can be expressed as follows.

$$\{Q\} = \int [N]^T w(x)\, dx \tag{4.4.1}$$

The loading function for the present case can be written as:

$$w(x) = w_1 + \frac{w_2 - w_1}{L} x \tag{4.4.2}$$

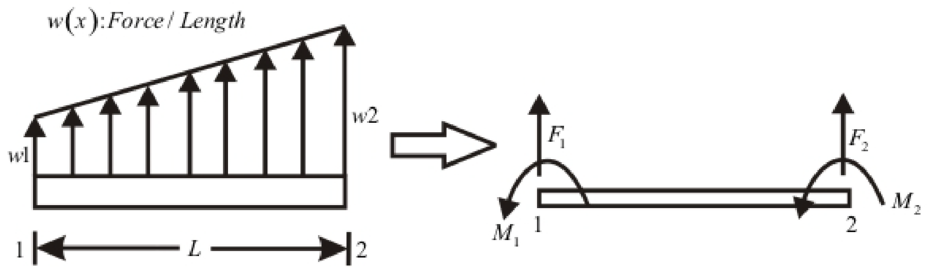

Fig. 4.4.1 Varying load on beam

From eqs. (4.4.1) and (4.4.2), the equvalent nodal load will become

$$\{Q\} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} \int_0^L \left(\frac{2x^3}{L^3} - \frac{3x^2}{L^2} + 1 \right) w(x) dx \\ \int_0^L \left(\frac{x^3}{L^2} - \frac{2x^2}{L} + x \right) w(x) dx \\ \int_0^L \left(-\frac{2x^3}{L^3} + \frac{3x^2}{L^2} \right) w(x) dx \\ \int_0^L \left(\frac{x^3}{L^2} - \frac{x^2}{L} \right) w(x) dx \end{Bmatrix} = \begin{Bmatrix} \left(\frac{7w_1}{20} + \frac{3w_2}{20} \right) L \\ \left(\frac{w_1}{20} + \frac{w_2}{30} \right) L^2 \\ \left(\frac{3w_1}{20} + \frac{7w_2}{20} \right) L \\ \left(-\frac{w_1}{30} - \frac{w_2}{20} \right) L^2 \end{Bmatrix} \tag{4.4.3}$$

Now, if $w_1 = w_2 = w$, then the equivalent nodal force will be:

$$\{Q\} = \begin{Bmatrix} \dfrac{wL}{2} \\ \dfrac{wL^2}{12} \\ \dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \end{Bmatrix} \tag{4.4.4}$$

4.4.1.2 Concentrated Load

Consider a force F is applied at a point is regarded as a limiting case of intense pressure over infinitesimal length, so that p(x)dx approaches F. Therefore,

$$\{Q\} = \int [N]^T p(x)\,dx = \left[N^*\right]^T F \tag{4.4.5}$$

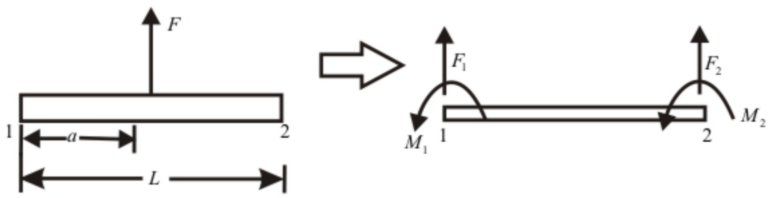

Fig. 4.4.2 Concentrated load on beam

Here, [N*] is obtained by evaluating [N] at point where the concentrated load F is applied. Thus,

$$[N^*] = \begin{Bmatrix} \dfrac{2x^3}{L^3} - \dfrac{3x^2}{L^2} + 1 \\ \dfrac{x^3}{L^2} - \dfrac{2x^2}{L} + x \\ -\dfrac{2x^3}{L^3} + \dfrac{3x^2}{L^2} \\ \dfrac{x^3}{L^2} - \dfrac{x^2}{L} \end{Bmatrix} at\ distance\ a = \begin{Bmatrix} \dfrac{2a^3}{L^3} - \dfrac{3a^2}{L^2} + 1 \\ \dfrac{a^3}{L^2} - \dfrac{2a^2}{L} + a \\ -\dfrac{2a^3}{L^3} + \dfrac{3a^2}{L^2} \\ \dfrac{a^3}{L^2} - \dfrac{a^2}{L} \end{Bmatrix} \tag{4.4.6}$$

Therefore, $$\{Q\}=\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}=\begin{Bmatrix} \left(\left(\dfrac{2a^3}{L^3}-\dfrac{3a^2}{L^2}+1\right)\right) \\ \left(\dfrac{a^3}{L^2}-\dfrac{2a^2}{L}+a\right) \\ \left(-\dfrac{2a^3}{L^3}+\dfrac{3a^2}{L^2}\right) \\ \left(\dfrac{a^3}{L^2}-\dfrac{a^2}{L}\right) \end{Bmatrix} \quad (4.4.7)$$

Now, if load F is acting at midspan (i.e., a=L/2), then equivalent nodal load will be

$$\{Q\}=\begin{Bmatrix} \dfrac{F}{2} \\ \dfrac{FL}{8} \\ \dfrac{F}{2} \\ -\dfrac{FL}{8} \end{Bmatrix} \quad (4.4.8)$$

With the above approach, the equivalent nodal load can be found for various loading function acting on beam members.

4.4.2 Worked Out Example

Analyze the beam shown below by the stiffness method. Assume the moment of inertia of member 2 as twice that of member 1. Find the bending moment and reactions at supports of the beam assuming the length of span, *L* as 4 m, concentrated load (*P)* as 15 kN and udl, *w* as 4 kN/m.

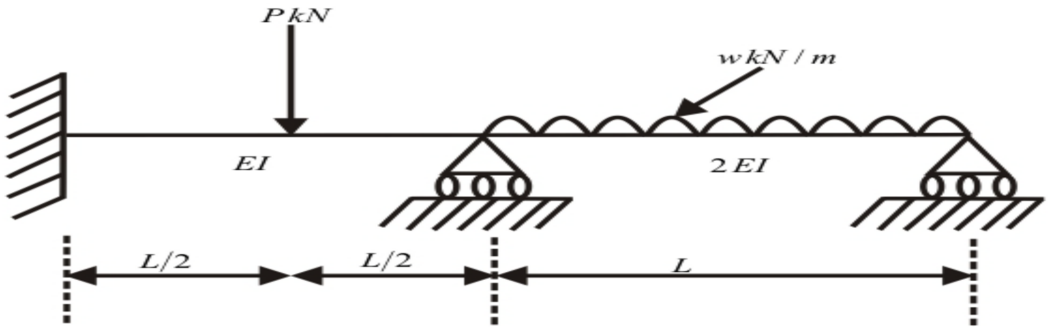

Fig. 4.4.3 Example of a continuous beam

Solution

Step 1: Numbering of Nodes and Members

The analysis of beam starts with the numbering of members and joints as shown below:

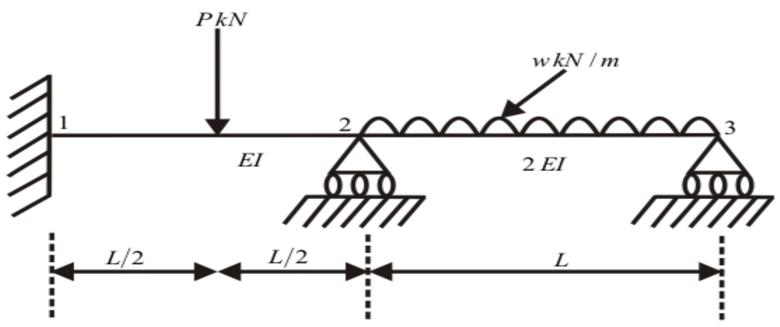

Fig. 4.4.4 Numbering of nodes and members

The member **AB** and **BC** are designated as (**1**) and (**2**). The points **A,B,C** are designated by nodes 1, 2 and 4. The member information for beam is shown in tabulated form as shown in Table 4.4.1. The coordinate of node 1 is assumed as (0, 0). The coordinate and restraint joint information are shown in Table 4.4.2. The integer 1 in the restraint list indicates the restraint exists and 0 indicates the restraint at that particular direction does not exist. Thus, in node no. 2, the integer 0 in rotation indicates that the joint is free rotation.

Table 4.4.1Member Information for Beam

Member number	**Starting node**	**Ending node**	**Rigidity modulus**
1	1	2	EI
2	2	3	2EI

Table 4.4.2 Nodal Information for Beam

Node No.	Coordinates		Restraint List	
	x	*y*	*Vertical*	*Rotation*
1	*0*	0	1	1
2	*L*	0	1	0
3	*2L*	0	1	0

Step 2: Formation of member stiffness matrix:

The local stiffness matrices of each member are given below based on their individual member properties and orientations. Thus the local stiffness matrix of member (1) is:

$$[k]_1 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Similarly, the local stiffness matrix of member (2) is:

$$[k]_2 = \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{24EI}{L^3} & \frac{12EI}{L^2} & -\frac{24EI}{L^3} & \frac{12EI}{L^2} \\ \frac{12EI}{L^2} & \frac{8EI}{L} & -\frac{12EI}{L^2} & \frac{4EI}{L} \\ -\frac{24EI}{L^3} & -\frac{12EI}{L^2} & \frac{24EI}{L^3} & -\frac{12EI}{L^2} \\ \frac{12EI}{L^2} & \frac{4EI}{L} & -\frac{12EI}{L^2} & \frac{8EI}{L} \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Step 3: Formation of global stiffness matrix:

The global stiffness matrix is obtained by assembling the local stiffness matrix of members (1) and (2) as follows:

$$[K] = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & 0 \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} & 0 & 0 \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{36EI}{L} & \frac{6EI}{L^2} & -\frac{24EI}{L^3} & \frac{12EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{12EI}{L} & -\frac{12EI}{L^2} & \frac{4EI}{L} \\ 0 & 0 & -\frac{24EI}{L^3} & -\frac{12EI}{L^2} & \frac{24EI}{L^3} & -\frac{12EI}{L^2} \\ 0 & 0 & \frac{12EI}{L^2} & \frac{4EI}{L} & -\frac{12EI}{L^2} & \frac{8EI}{L} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Step 4: Boundary condition:

The boundary conditions according to the support of the beam can be expressed in terms of the displacement vector. The displacement vector will be as follows

$$\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix}$$

Step 5: Load vector:

The concentrated load on member (1) and the distributed load on member (2) are replaced by equivalent joint load. The equivalent joint load vector can be written as

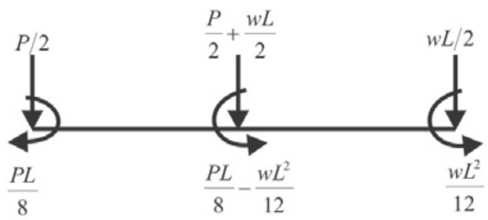

Fig. 4.4.5 Equivalent Load

$$\{F\} = \begin{Bmatrix} -\dfrac{P}{2} \\ -\dfrac{PL}{8} \\ -\left(\dfrac{P}{2} + \dfrac{wL}{2}\right) \\ \left(\dfrac{PL}{8} - \dfrac{wL^2}{12}\right) \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{Bmatrix}$$

Step 6 : Determination of unknown displacements:.

The unknown displacement can be obtained from the relationship as given below:

$$\{F\} = [K]\{d\}$$
$$\{d\} = [K]^{-1}\{F\}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \\ 0 \\ \theta_3 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & 0 \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} & 0 & 0 \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{36EI}{L} & \frac{6EI}{L^2} & -\frac{24EI}{L^3} & \frac{12EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{6EI}{L^2} & \frac{12EI}{L} & -\frac{12EI}{L^2} & \frac{4EI}{L} \\ 0 & 0 & -\frac{24EI}{L^3} & -\frac{12EI}{L^2} & \frac{24EI}{L^3} & -\frac{12EI}{L^2} \\ 0 & 0 & \frac{12EI}{L^2} & \frac{4EI}{L} & -\frac{12EI}{L^2} & \frac{8EI}{L} \end{bmatrix}^{-1} \times \begin{Bmatrix} -\frac{P}{2} \\ -\frac{PL}{8} \\ -\left(\frac{P}{2} + \frac{wL}{2}\right) \\ \left(\frac{PL}{8} - \frac{wL^2}{12}\right) \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

The above relation may be condensed into following

$$\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L} & \frac{4EI}{L} \\ \frac{4EI}{L} & \frac{8EI}{L} \end{bmatrix}^{-1} \times \begin{Bmatrix} \frac{PL}{8} - \frac{wL^2}{12} \\ \frac{wL^2}{12} \end{Bmatrix} = \frac{L}{20EI} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} \frac{PL}{8} - \frac{wL^2}{12} \\ \frac{wL^2}{12} \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{L}{20EI} \begin{bmatrix} \frac{PL}{4} - \frac{wL^2}{4} \\ -\frac{PL}{8} + \frac{wL^2}{3} \end{bmatrix}$$

$$\theta_2 = \frac{PL^2}{80EI} - \frac{wL^3}{80EI}$$

$$\theta_3 = -\frac{PL}{160EI} + \frac{wL^3}{60EI}$$

Step 7: Determination of member end actions:

The member end actions can be obtained from the corresponding member stiffness and the nodal displacements. The member end actions for each member are derived as shown below.

Member-(1)

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{L}{20EI}\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{PL}{4} - \frac{wL^2}{4} \end{Bmatrix} = \begin{Bmatrix} \frac{3P}{40} - \frac{3wL}{40} \\ \frac{PL}{40} - \frac{wL^2}{40} \\ -\frac{3P}{40} + \frac{3wL}{40} \\ \frac{PL}{20} - \frac{wL^2}{20} \end{Bmatrix}$$

Member-(2)

$$\begin{Bmatrix} F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \frac{EI}{L}\begin{bmatrix} \frac{24}{L^2} & \frac{12}{L} & -\frac{24}{L^2} & \frac{12}{L} \\ \frac{12}{L} & 8 & -\frac{12}{L} & 4 \\ -\frac{24}{L^2} & -\frac{12}{L} & \frac{24}{L^2} & -\frac{12}{L} \\ \frac{12}{L} & 4 & -\frac{12}{L} & 8 \end{bmatrix} \times \begin{Bmatrix} 0 \\ \frac{PL}{4} - \frac{wL^2}{4} \\ 0 \\ -\frac{PL}{8} + \frac{wL^2}{3} \end{Bmatrix} = \begin{bmatrix} \frac{wL}{20} + \frac{3P}{40} \\ \frac{3PL}{40} - \frac{wL^2}{30} \\ -\frac{wL}{20} - \frac{3P}{40} \\ \frac{wL^2}{12} \end{bmatrix}$$

Actual member end actions:

Member (1)

$$\begin{Bmatrix} \overline{F_1} \\ \overline{M_1} \\ \overline{F_2} \\ \overline{M_2} \end{Bmatrix} = \begin{Bmatrix} \frac{3P}{40} - \frac{3wL}{40} \\ \frac{PL}{40} - \frac{wL^2}{40} \\ -\frac{3P}{40} + \frac{3wL}{40} \\ \frac{PL}{20} - \frac{wL^2}{20} \end{Bmatrix} + \begin{Bmatrix} \frac{P}{2} \\ \frac{PL}{8} \\ \frac{P}{2} \\ -\frac{PL}{8} \end{Bmatrix} = \begin{Bmatrix} \frac{23P}{40} - \frac{3wL}{40} \\ \frac{6PL}{40} - \frac{wL^2}{40} \\ \frac{17P}{40} + \frac{3wL}{40} \\ -\frac{3PL}{40} - \frac{wL^2}{20} \end{Bmatrix}$$

Member (2)

$$\begin{Bmatrix} \overline{F_2} \\ \overline{M_2} \\ \overline{F_3} \\ \overline{M_3} \end{Bmatrix} = \begin{bmatrix} \frac{wL}{20} + \frac{3P}{40} \\ \frac{3PL}{40} - \frac{wL^2}{30} \\ -\frac{wL}{20} - \frac{3P}{40} \\ \frac{wL^2}{12} \end{bmatrix} + \begin{bmatrix} \frac{wL}{2} \\ \frac{wL^2}{12} \\ \frac{wL}{2} \\ -\frac{wL^2}{12} \end{bmatrix} = \begin{Bmatrix} \frac{11wL}{20} + \frac{3P}{40} \\ \frac{3PL}{40} + \frac{wL^2}{20} \\ \frac{9wL}{20} - \frac{3P}{40} \\ 0 \end{Bmatrix}$$

The support reactions at the supports A, B and C are $\{F_R\} = \begin{Bmatrix} R_A \\ R_B \\ R_C \end{Bmatrix} = \begin{Bmatrix} \frac{23P}{40} - \frac{3wL}{40} \\ \frac{25wL}{40} + \frac{P}{2} \\ \frac{9wL}{20} + \frac{3P}{4} \end{Bmatrix}$

Putting the numerical values of *L*, *P* and *w* (P=15, L=4, w=4) the member actions and support reactions will be as follows:

Member end actions:

$$\begin{Bmatrix} F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 9.925 \\ 7.7 \\ 6.075 \\ 0 \end{Bmatrix} \quad , \quad \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{Bmatrix} 7.425 \\ 7.4 \\ 7.575 \\ -7.7 \end{Bmatrix}$$

Support reactions:

$$\{F_R\} = \begin{Bmatrix} R_A \\ R_B \\ R_C \end{Bmatrix} = \begin{Bmatrix} 7.425 \\ 17.5 \\ 6.075 \end{Bmatrix}$$

Lecture 5: Plane Frame Analysis

4.5.1 Introduction

The plane frame is a combination of plane truss and two dimensional beam. All the members lie in the same plane and are interconnected by rigid joints in case of plane frame. The internal stress resultants at a cross-section of a plane frame member consist of axial force, bending moment and shear force.

4.5.2 Member Stiffness Matrix

In case of plane frame, the degrees of freedom at each node will be (i) axial deformation, (ii) vertical deformation and (iii) rotation. Thus the frame members have three degrees of freedom at each node as shown in Fig. 4.5.1 below.

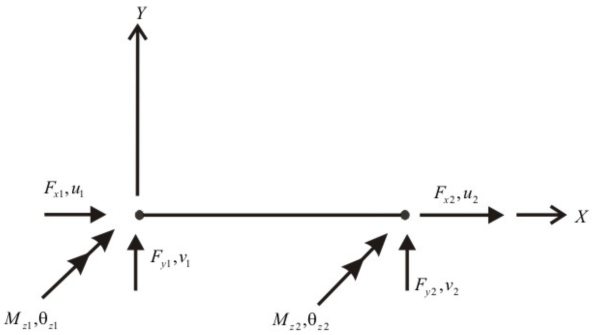

Fig. 4.5.1 Plane frame element

Therefore, the stiffness matrix of the frame in its local coordinate system will be the combination of 2-d truss and 2-d beam matrices:

$$\overline{[k]}= \begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} \\ \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \quad (4.5.1)$$

4.5.3 Generalized Stiffness Matrix

In plane frame the members are oriented in different directions and hence it is necessary to transform stiffness matrix of individual members from local to global co-ordinate system before formulating the global stiffness matrix by assembly. The generalized stiffness matrix of a frame member can be obtained by transferring the matrix of local coordinate system into its global coordinate system. The transformation matrix can be expressed as:

$$[\mathrm{T}]=\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4.5.2)$$

Now, the generalized stiffness matrix of the member can be obtained from the relation of $[K]=[T]^T[\bar{K}][T]$. Thus considering $\lambda=\cos\theta$ and $\mu=\sin\theta$ the stiffness matrix in global coordinate system can be written as follows:

$$[\mathrm{K}] = \mathrm{EI}\begin{bmatrix} \lambda & -\mu & 0 & 0 & 0 & 0 \\ \mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\mu & 0 \\ 0 & 0 & 0 & \mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{\mathrm{AE}}{\mathrm{L}} & 0 & 0 & -\frac{\mathrm{AE}}{\mathrm{L}} & 0 & 0 \\ 0 & \frac{12\mathrm{EI}}{\mathrm{L}^3} & \frac{6\mathrm{EI}}{\mathrm{L}^2} & 0 & -\frac{12\mathrm{EI}}{\mathrm{L}^3} & \frac{6\mathrm{EI}}{\mathrm{L}^2} \\ 0 & \frac{6\mathrm{EI}}{\mathrm{L}^2} & \frac{4\mathrm{EI}}{\mathrm{L}} & 0 & -\frac{6\mathrm{EI}}{\mathrm{L}^2} & \frac{2\mathrm{EI}}{\mathrm{L}} \\ -\frac{\mathrm{AE}}{\mathrm{L}} & 0 & 0 & \frac{\mathrm{AE}}{\mathrm{L}} & 0 & 0 \\ 0 & -\frac{12\mathrm{EI}}{\mathrm{L}^3} & -\frac{6\mathrm{EI}}{\mathrm{L}^2} & 0 & \frac{12\mathrm{EI}}{\mathrm{L}^3} & -\frac{6\mathrm{EI}}{\mathrm{L}^2} \\ 0 & \frac{6\mathrm{EI}}{\mathrm{L}^2} & \frac{2\mathrm{EI}}{\mathrm{L}} & 0 & -\frac{6\mathrm{EI}}{\mathrm{L}^2} & \frac{4\mathrm{EI}}{\mathrm{L}} \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{\mathrm{EA}}{\mathrm{L}}\lambda^2+\frac{12\mathrm{EI}}{\mathrm{L}^3}\mu^2\right) & \left(\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu-\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & -\frac{6\mathrm{EI}}{\mathrm{L}^2}\mu & \left(-\frac{\mathrm{EA}}{\mathrm{L}}\lambda^2-\frac{12\mathrm{EI}}{\mathrm{L}^3}\mu^2\right) & \left(-\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu+\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & -\frac{6\mathrm{EI}}{\mathrm{L}^2}\mu \\ \left(\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu-\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & \left(\frac{\mathrm{EA}}{\mathrm{L}}\mu^2+\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda^2\right) & \frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda & \left(-\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu+\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & \left(-\frac{\mathrm{EA}}{\mathrm{L}}\mu^2-\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda^2\right) & \frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda \\ -\frac{6\mathrm{EI}}{\mathrm{L}^2}\mu & \frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda & \frac{4\mathrm{EI}}{\mathrm{L}} & \frac{6\mathrm{EI}}{\mathrm{L}^2}\mu & -\frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda & \frac{2\mathrm{EI}}{\mathrm{L}} \\ \left(-\frac{\mathrm{EA}}{\mathrm{L}}\lambda^2-\frac{12\mathrm{EI}}{\mathrm{L}^3}\mu^2\right) & \left(-\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu+\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & \frac{6\mathrm{EI}}{\mathrm{L}^2}\mu & \left(\frac{\mathrm{EA}}{\mathrm{L}}\lambda^2+\frac{12\mathrm{EI}}{\mathrm{L}^3}\mu^2\right) & \left(\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu-\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & \frac{6\mathrm{EI}}{\mathrm{L}^2}\mu \\ \left(-\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu+\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & \left(-\frac{\mathrm{EA}}{\mathrm{L}}\mu^2-\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda^2\right) & -\frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda & \left(\frac{\mathrm{EA}}{\mathrm{L}}\lambda\mu-\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda\mu\right) & \left(\frac{\mathrm{EA}}{\mathrm{L}}\mu^2+\frac{12\mathrm{EI}}{\mathrm{L}^3}\lambda^2\right) & -\frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda \\ -\frac{6\mathrm{EI}}{\mathrm{L}^2}\mu & \frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda & \frac{2\mathrm{EI}}{\mathrm{L}} & \frac{6\mathrm{EI}}{\mathrm{L}^2}\mu & -\frac{6\mathrm{EI}}{\mathrm{L}^2}\lambda & \frac{4\mathrm{EI}}{\mathrm{L}} \end{bmatrix}$$

(4.5.3)

4.5.4 Worked Out Example

Analyse the plane frame shown below by the stiffness method. Assume the modulus of elasticity of the horizontal member is 1.5 times that of the vertical member and length of the vertical member is 1.5 times that of horizontal member. Find the bending moment and reactions at support assuming the length, cross section area and modulus of elasticity of vertical member as 3.0 m, 0.4 x 0.4 m^2 and 2 x 10^{11} N/mm^2, respectively.

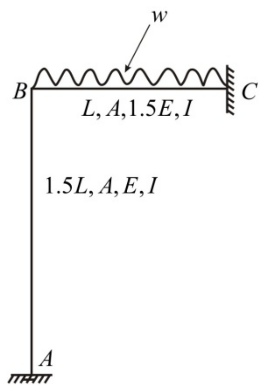

Fig. 4.5.2 Plane frame

Solution

Step 1: Numbering of Nodes and Members

The numbering of members and joints of the plane frame are as shown below:

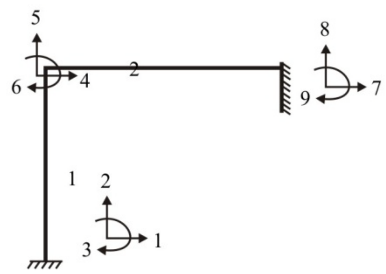

Fig. 4.5.3 Numbering of Nodes and Members

The members **AB** and **BC** are designated as (**1**) and (**2**). The points **A, B** and **C** are designated by nodes 1, 2 and 3. The member information for the frame is shown in tabulated form as shown in Table 1(a). The coordinate of node 1 is assumed as (0,0). The coordinate and restraint joint information are shown in Table 1(b). The integer 1 in the restraint list indicates the restraint exists and 0 indicates the restraint at that particular direction does not exist. Thus, in node no. 2, the integer 0 all the restraint type indicates that the joint is in free all the three directions.

Table 4.5.1 Member Information for Beam

Member number	Starting node	Ending node	Rigidity modulus
1	1	2	EI
2	2	3	1.5EI

Table 4.5.2 Nodal Information for Beam

Node no.	Coordinates		Restraint list		
	X	Y	Axial	Vertical	Rotation
1	0	0	1	1	1
2	0	1.5L	0	0	0
3	L	1.5L	1	1	1

Step 2: Formation of member stiffness matrix:

The individual member stiffness matrices can be found out directly from eqn. shown above. Thus the stiffness matrices of each member in global coordinate system are given below based on their individual member properties and orientations. Thus the stiffness matrix of member (1) is:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
[k]_1= & \begin{bmatrix}
\frac{12EI}{(1.5L)^3} & 0 & -\frac{6EI}{(1.5L)^2} & -\frac{12EI}{(1.5L)^3} & 0 & -\frac{6EI}{(1.5L)^2} \\
0 & \frac{AE}{(1.5L)} & 0 & 0 & -\frac{AE}{(1.5L)} & 0 \\
-\frac{6EI}{(1.5L)^2} & 0 & \frac{4EI}{(1.5L)} & \frac{6EI}{(1.5L)^2} & 0 & \frac{2EI}{1.5L} \\
-\frac{12EI}{(1.5L)^3} & 0 & \frac{6EI}{(1.5L)^2} & \frac{12EI}{(1.5L)^3} & 0 & \frac{6EI}{(1.5L)^2} \\
0 & -\frac{AE}{(1.5L)} & 0 & 0 & \frac{AE}{(1.5L)} & 0 \\
-\frac{6EI}{(1.5L)^2} & 0 & \frac{2EI}{(1.5L)} & \frac{6EI}{(1.5L)^2} & 0 & \frac{4EI}{(1.5L)}
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

Similarly, the stiffness matrix of member (2) is :

4 5 6 7 8 9

$$[k]_2 = \begin{array}{c} \begin{bmatrix} \frac{A(1.5\,E)}{L} & 0 & 0 & -\frac{A(1.5\,E)}{L} & 0 & 0 \\ 0 & \frac{12(1.5\,E)I}{L^3} & \frac{6(1.5\,E)I}{L^2} & 0 & -\frac{12(1.5\,E)I}{L^3} & \frac{6(1.5\,E)I}{L^2} \\ 0 & \frac{6(1.5\,E)I}{L^2} & \frac{4(1.5\,E)I}{L} & 0 & -\frac{6(1.5\,E)I}{L^2} & \frac{2(1.5\,E)I}{L} \\ -\frac{A(1.5\,E)}{L} & 0 & 0 & \frac{A(1.5\,E)}{L} & 0 & 0 \\ 0 & -\frac{12(1.5\,E)I}{L^3} & -\frac{6(1.5\,E)I}{L^2} & 0 & \frac{12(1.5\,E)I}{L^3} & -\frac{6(1.5\,E)I}{L^2} \\ 0 & \frac{6(1.5\,E)I}{L^2} & \frac{2(1.5\,E)I}{L} & 0 & -\frac{6(1.5\,E)I}{L^2} & \frac{4(1.5\,E)I}{L} \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

Step 3 : Formulation of global stiffness matrix:

The global stiffness matrix is obtained by assembling by assembling the local stiffness matrix of member (1) and (2) as follows:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

$$[K] = \begin{bmatrix} \frac{32EI}{9L^3} & 0 & -\frac{8EI}{3L^2} & -\frac{32EI}{9L^3} & 0 & -\frac{8EI}{3L^2} & 0 & 0 & 0 \\ 0 & \frac{2AE}{3L} & 0 & 0 & -\frac{2AE}{3L} & 0 & 0 & 0 & 0 \\ \frac{8EI}{3L^2} & 0 & \frac{8EI}{3L} & \frac{8EI}{3L^2} & 0 & \frac{4EI}{3L} & 0 & 0 & 0 \\ -\frac{32EI}{9L^3} & 0 & \frac{8EI}{3L^2} & (\frac{32EI}{9L^3} + \frac{1.5EA}{L}) & 0 & \frac{8EI}{3L^2} & -\frac{1.5EA}{L} & 0 & 0 \\ 0 & -\frac{2AE}{3L} & 0 & 0 & (\frac{2AE}{3L} + \frac{18EI}{L^3}) & \frac{9EI}{L^2} & 0 & -\frac{18EI}{L^3} & \frac{9EI}{L^2} \\ -\frac{8EI}{3L^2} & 0 & \frac{4EI}{3L} & \frac{8EI}{3L^2} & \frac{9EI}{L^2} & (\frac{8EI}{3L^2} + \frac{6EI}{L}) & 0 & -\frac{9EI}{L^2} & \frac{3EI}{L} \\ 0 & 0 & 0 & -\frac{1.5AE}{L} & 0 & 0 & \frac{1.5AE}{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{18EI}{L^3} & -\frac{9EI}{L^2} & 0 & \frac{18EI}{L^3} & -\frac{9EI}{L^2} \\ 0 & 0 & 0 & 0 & \frac{9EI}{L^2} & \frac{3EI}{L} & 0 & -\frac{9EI}{L^2} & \frac{6EI}{L} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

Step 4: Boundary conditions:

The boundary conditions according to the support of the frame can be expressed in terms of the displacement vector. The displacement vector will be as follows:

$$\{d\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \delta x_B \\ \delta y_B \\ \theta_B \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, δx_B , δy_B and θ_B indicate the displacement in X-direction, displacement in Y-direction and rotation at point B.

Step 5: Load vector:

The distributed load on member (2) can be replaced by its equivalent joint load as shown in the figure below.

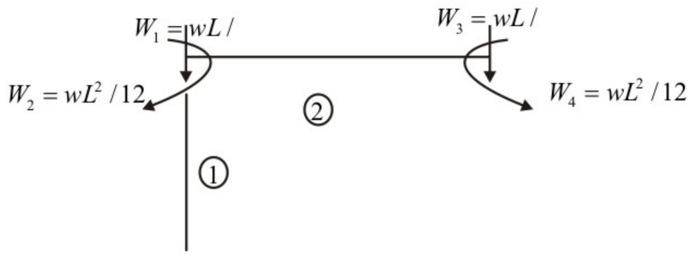

Fig. 4.5.4 Equivalent Joint Loads

Thus, the equivalent joint load vector can be written as

$$\{F\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \\ 0 \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \end{bmatrix}$$

Step 6: **Determination of unknown displacements:**

The unknown displacements can be obtained from the relationship of {F} = [K]{d} or {d} =$[k]^{-1}$ {F}. Now eliminating the rows and columns in the stiffness matrix and force matrix, corresponding to zero elements in displacement matrix, the reduced matrix will be as follows.

$$\begin{bmatrix} \delta x_B \\ \delta y_B \\ \theta_B \end{bmatrix} = \begin{bmatrix} \left(\dfrac{32EI}{9L^3} + \dfrac{1.5EA}{L}\right) & 0 & \dfrac{8EI}{3L^2} \\ 0 & \left(\dfrac{2AE}{3L} + \dfrac{18EI}{L^3}\right) & \dfrac{9EI}{L^2} \\ \dfrac{8EI}{3L^2} & \dfrac{9EI}{L^2} & \left(\dfrac{8EI}{3L} + \dfrac{6EI}{L}\right) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \end{bmatrix}$$

Thus, the unknown displacements will be:

$$\begin{bmatrix} \delta x_B \\ \delta y_B \\ \theta_B \end{bmatrix} = \frac{1}{10^{10}} \begin{bmatrix} 0.04327w \\ -1.7127w \\ -5.4978w \end{bmatrix}$$

Step 7: Determination of member end actions:

The member end actions can be obtained from the corresponding member stiffness and the nodal displacements. The member end actions for each member are derived as shown below.

<u>Member – (1)</u>

In case of member (1), the member forces will be: $\{F_m\}_1 = [K]_{(1)}\{d\}_{(1)}$

$$\begin{bmatrix} F_{x_1} \\ F_{y_1} \\ M_1 \\ F_{x_2} \\ F_{y_2} \\ M_2 \end{bmatrix} = 10^6 \begin{bmatrix} 56.17 & 0 & -126.4 & -56.17 & 0 & -126.4 \\ 0 & 7110 & 0 & 0 & -7110 & 0 \\ -126.4 & 0 & 379.2 & 126.4 & 0 & 189.6 \\ -56.17 & 0 & 379.2 & 56.17 & 0 & 126.4 \\ 0 & -7110 & 0 & 0 & 7110 & 0 \\ -126.4 & 0 & 189.6 & 126.4 & 0 & 379.2 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4.327\ X\ 10^{-12}w \\ -1.7127\ X\ 10^{-10}w \\ -5.4978\ X\ 10^{-10}w \end{bmatrix}$$

$$= \begin{bmatrix} 0.0697w \\ 1.2177w \\ -0.10479w \\ -0.06925w \\ -1.21661w \\ -0.20793w \end{bmatrix}$$

It is to be noted that $\{F_m\}$ are the end actions due to joint loads. Hence it must be added to the corresponding end actions in the restrained structure in order to obtain the end actions due to the loads. Therefore, $\{F_m\}_{actual}$ are the true member end actions due to actual loading system can be expressed as

$$\{F_m\}_{actual} = \{F_m\} + \{F_{fm}\}$$

Where, $\{F_{fm}\}$ are the end actions in the restrained structure. Since there is no load acting on member (1), the actual end actions will be:

$$\{F_m\}_{actual} = \begin{bmatrix} 0.0697w \\ 1.2177w \\ -0.10479w \\ -0.06925w \\ -1.21661w \\ -0.20793w \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0697w \\ 1.2177w \\ -0.10479w \\ -0.06925w \\ -1.21661w \\ -0.20793w \end{bmatrix}$$

<u>Member (2)</u>

In similar way, the member forces in member (2) will be $\{F_m\}_{(2)} = [K]_{(2)}\{d\}_{(2)}$

$$\begin{bmatrix} F_{x_2} \\ F_{y_2} \\ M_2 \\ F_{x_3} \\ F_{y_3} \\ M_3 \end{bmatrix} = 10^9 \begin{bmatrix} 16 & 0 & 0 & -16 & 0 & 0 \\ 0 & 0.284 & 0.426 & 0 & -0.284 & 0.426 \\ 0 & 0.426 & 0.853 & 0 & -0.426 & 0.426 \\ -16 & 0 & 0 & 16 & 0 & 0 \\ 0 & -0.284 & -0.426 & 0 & 0.284 & -0.426 \\ 0 & 0.426 & 0.426 & 0 & -0.426 & 0.853 \end{bmatrix} \times \begin{bmatrix} 4.327\ X\ 10^{-12}w \\ -1.7127\ X\ 10^{-10}w \\ -5.4978\ X\ 10^{-10}w \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.069232w \\ -0.28325w \\ -0.54215w \\ -0.06923w \\ 0.283245w \\ -0.3076w \end{bmatrix}$$

The actual member forces in the member (2) will be:

$$\{F_m\}_{actual} = \begin{bmatrix} 0.069232w \\ -0.28325w \\ -0.54215w \\ -0.06923w \\ 0.283245w \\ -0.3076w \end{bmatrix} + \begin{bmatrix} 0 \\ 1.5w \\ 0.75w \\ 0 \\ 1.5w \\ -0.75w \end{bmatrix} = \begin{bmatrix} 0.0692w \\ 1.2167w \\ 0.2078w \\ -0.0692w \\ 1.7832w \\ -1.0576w \end{bmatrix}$$

<u>Lecture 6 Analysis of Grid and Space Frame</u>

4.6.1 Introduction

The property of a grid member is basically a combination of 2-d beam with torsional effect. The plane frame is assumed to be loaded in its own plane where as loading in the grid is normal to its plane. As a result torsional effects are included in the grid analysis. Thus the grid member can withstand bending moment, shear force as well as torsional moment.

4.6.2 Element Stiffness Matrix for Grid Members

The degrees of freedom at each node of the grid member will be (i) vertical deformation and (ii) rotation in two different directions.

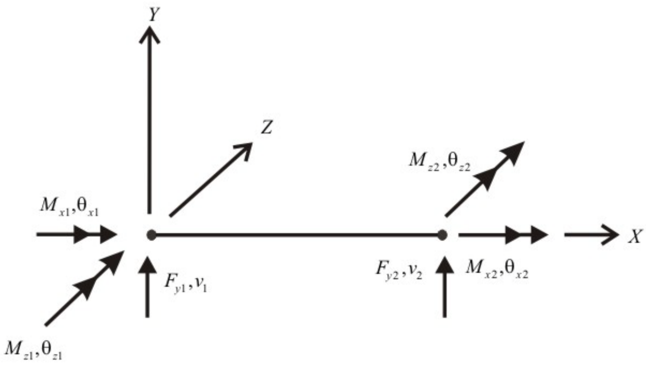

Fig. 4.6.1 Degrees of freedom of grid element

Therefore, the stiffness matrix of the grid in its local coordinate system will be:

$$[k]=\begin{array}{c} \begin{array}{cccccc} \theta_{x1} & \theta_{z1} & v_1 & \theta_{x2} & \theta_{y2} & v_2 \end{array} \\ \begin{bmatrix} \frac{GI_x}{L} & 0 & 0 & -\frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{4EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & -\frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} \\ -\frac{GI_x}{L} & 0 & 0 & \frac{GI_x}{L} & 0 & 0 \\ 0 & \frac{2EI_y}{L} & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} \\ 0 & \frac{6EI_y}{L^2} & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} \end{bmatrix} \end{array} \begin{array}{c} \\ M_{x1} \\ M_{z1} \\ F_{y1} \\ M_{x2} \\ M_{z2} \\ F_{y2} \end{array} \qquad (4.6.1)$$

Here, the G is the modulus of torsional rigidity.

4.6.3 Generalized Stiffness Matrix

The generalized stiffness matrix of a grid member can be obtained by transferring the matrix of local coordinate system into its global coordinate system. The transformation matrix can be expressed as:

$$[T]=\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, the generalized stiffness matrix of the member can be obtained from the relation of $[k]=[T]^T[\bar{k}][T]$. Thus considering $\lambda=\cos\theta$ and $\mu=\sin\theta$ the stiffness matrix in global coordinate system can be written as follows:

$$[k]=\begin{bmatrix} \lambda & -\mu & 0 & 0 & 0 & 0\\ \mu & \lambda & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda & -\mu & 0\\ 0 & 0 & 0 & \mu & \lambda & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \frac{\mathrm{GI_x}}{\mathrm{L}} & 0 & 0 & -\frac{\mathrm{GI_x}}{\mathrm{L}} & 0 & 0\\ 0 & \frac{\mathrm{4EI_y}}{\mathrm{L}} & -\frac{\mathrm{6EI_y}}{\mathrm{L^2}} & 0 & \frac{\mathrm{2EI_y}}{\mathrm{L}} & \frac{\mathrm{6EI_y}}{\mathrm{L^2}}\\ 0 & -\frac{\mathrm{6EI_y}}{\mathrm{L^2}} & \frac{\mathrm{12EI_y}}{\mathrm{L^3}} & 0 & -\frac{\mathrm{6EI_y}}{\mathrm{L^2}} & -\frac{\mathrm{12EI_y}}{\mathrm{L^3}}\\ -\frac{\mathrm{GI_x}}{\mathrm{L}} & 0 & 0 & \frac{\mathrm{GI_x}}{\mathrm{L}} & 0 & 0\\ 0 & \frac{\mathrm{2EI_y}}{\mathrm{L}} & -\frac{\mathrm{6EI_y}}{\mathrm{L^2}} & 0 & \frac{\mathrm{4EI_y}}{\mathrm{L}} & -\frac{\mathrm{6EI_y}}{\mathrm{L^2}}\\ 0 & \frac{\mathrm{6EI_y}}{\mathrm{L^2}} & -\frac{\mathrm{12EI_y}}{\mathrm{L^3}} & 0 & \frac{\mathrm{6EI_y}}{\mathrm{L^2}} & \frac{\mathrm{12EI_y}}{\mathrm{L^3}} \end{bmatrix}$$

$$\times \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0\\ -\mu & \lambda & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda & \mu & 0\\ 0 & 0 & 0 & -\mu & \lambda & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{GI_x}{L}\lambda^2+\frac{4EI_y}{L}\mu^2 & \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)\lambda\mu & \frac{6EI_y}{L^2}\mu & -\frac{GI_x}{L}\lambda^2+\frac{4EI_y}{L}\mu^2 & -\left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)\lambda\mu & -\frac{6EI_y}{L^2}\mu\\ \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)\lambda\mu & \frac{GI_x}{L}\mu^2+\frac{4EI_y}{L}\lambda^2 & -\frac{6EI_y}{L^2}\lambda & -\left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)\lambda\mu & -\frac{GI_x}{L}\mu^2+\frac{2EI_y}{L}\lambda^2 & \frac{6EI_y}{L^2}\lambda\\ \frac{6EI_y}{L^2}\mu & -\frac{6EI_y}{L^2}\lambda & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2}\mu & -\frac{6EI_y}{L^2}\lambda & -\frac{12EI_y}{L^3}\\ -\frac{GI_x}{L}\lambda^2+\frac{4EI_y}{L}\mu^2 & -\left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)\lambda\mu & \frac{6EI_y}{L^2}\mu & \frac{GI_x}{L}\lambda^2+\frac{4EI_y}{L}\mu^2 & \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)\lambda\mu & -\frac{6EI_y}{L^2}\mu\\ \left(\frac{GI_x}{L}+\frac{2EI_y}{L}\right)\lambda\mu & -\frac{GI_x}{L}\mu^2+\frac{2EI_y}{L}\lambda^2 & -\frac{6EI_y}{L^2}\lambda & \left(\frac{GI_x}{L}-\frac{4EI_y}{L}\right)\lambda\mu & \frac{GI_x}{L}\mu^2+\frac{4EI_y}{L}\lambda^2 & \frac{6EI_y}{L^2}\lambda\\ -\frac{6EI_y}{L^2}\mu & \frac{6EI_y}{L^2}\lambda & -\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2}\mu & \frac{6EI_y}{L^2}\lambda & \frac{12EI_y}{L^3} \end{bmatrix} \quad (4.6.2)$$

4.6.4 Worked Out Example

Analyze the grid shown below by the stiffness method. Draw the shear force and bending moment diagram assuming the cross sectional area and modulus of elasticity of each member as 0.3×0.3 m^2 and 2×10^{11} N/m^2 respectively. Assume *EI* = 3*GJ*. The length of member AB and BC is 4 m and 5 m respectively.

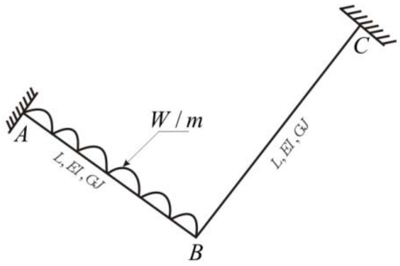

Fig. 4.6.2 Grid structure

<u>Solution</u>

Step 1: Numbering of Nodes and Members

The numbering of members and joints of the plane frame are as shown in the figure below:

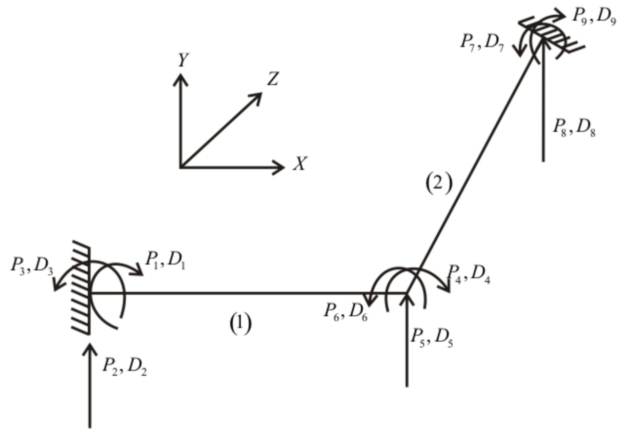

Fig. 4.6.3 Numbering of nodes and members

The member **AB** and **BC** are designated as (**1**) and (**2**). The points **A, B** and **C** are designated by nodes 1, 2 and 3. The member information for the grid is shown in tabulated form as shown in Table 4.6.1. The coordinate of node 1 is assumed as (0, 0). The coordinate and restraint joint information are shown in Table 4.6.2. The integer 1 in the restraint list indicates the restraint exists and 0 indicates the restraint at that particular direction does not exist. Thus, in node no. 2, the integer 0 all the restraint type indicates that the joint is free in all the three directions.

Table 4.6.1 Member Information

Member number	Starting node	Ending node
1	1	2
2	2	3

Table 4.6.2 Member Coordinates

ode No	coordinates		Restraint list		
	x	z	Vertical	Rotation	Rotation
1	0	0	1	1	1
2	4	0	0	0	0
3	4	5	1	1	1

Step 2: Formation of member stiffness matrix:

The individual member stiffness matrices can be found out directly. Thus the stiffness matrices of each member in global coordinate system are given below based on their individual member properties and orientations. As the member **AB** is horizontal, *i.e.*, θ **=** 0, the values of Cos θ = 1 and Sin θ = 0. Thus the stiffness matrix of member (1) is:

$$
[k]_{AB} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

Assuming *EI=3GJ=3K*, the above equation can be written as

$$
[k]_{AB} = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{K}{L} & 0 & 0 & -\frac{K}{L} & 0 & 0 \\ 0 & \frac{36K}{L^3} & \frac{18K}{L^2} & 0 & -\frac{36K}{L^3} & \frac{18K}{L^2} \\ 0 & \frac{18K}{L^2} & \frac{12K}{L} & 0 & -\frac{18K}{L^2} & \frac{6K}{L} \\ -\frac{K}{L} & 0 & 0 & \frac{K}{L} & 0 & 0 \\ 0 & -\frac{36K}{L^3} & -\frac{18K}{L^2} & 0 & \frac{36K}{L^3} & -\frac{18K}{L^2} \\ 0 & \frac{18K}{L^2} & \frac{6K}{L} & 0 & -\frac{18K}{L^2} & \frac{12K}{L} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

As the member *BC* member is also horizontal, the value of Cos θ = 1 and Sin θ = 0 and thus, the stiffness matrix will be:

$$[k]_{BC} = \begin{array}{c c} \begin{array}{cccccc} 6 & 5 & 4 & 7 & 8 & 9 \end{array} & \\ \begin{bmatrix} \frac{K}{L} & 0 & 0 & -\frac{K}{L} & 0 & 0 \\ 0 & \frac{36K}{L^3} & \frac{18K}{L^2} & 0 & -\frac{36K}{L^3} & \frac{18K}{L^2} \\ 0 & \frac{18K}{L^2} & \frac{12K}{L} & 0 & -\frac{18K}{L^2} & \frac{6K}{L} \\ -\frac{K}{L} & 0 & 0 & \frac{K}{L} & 0 & 0 \\ 0 & -\frac{36K}{L^3} & -\frac{18K}{L^2} & 0 & \frac{36K}{L^3} & -\frac{18K}{L^2} \\ 0 & \frac{18K}{L^2} & \frac{6K}{L} & 0 & -\frac{18K}{L^2} & \frac{12K}{L} \end{bmatrix} & \begin{array}{c} 6 \\ 5 \\ 4 \\ 7 \\ 8 \\ 9 \end{array} \end{array}$$

…ss matrix of members (AB) and (BC). Now looking at the grid structure, the displacements at the fixed supports, are known and all are equal to zero. Only the displacement at co-ordinates 4, 5, 6 are unknown. So the global system stiffness matrix, corresponding to the displacement at co-ordinate 4, 5, 6 will be:

$$[K] = \begin{bmatrix} \frac{K}{L_{AB}} + \frac{12K}{L_{BC}} & \frac{18K}{L_{BC}{}^2} & 0 \\ \frac{18K}{L_{BC}{}^2} & \frac{36K}{L_{AB}{}^3} + \frac{36K}{L_{BC}{}^3} & -\frac{18K}{L_{AB}{}^2} \\ 0 & -\frac{18K}{L_{AB}{}^2} & \frac{K}{L_{BC}} + \frac{12K}{L_{AB}} \end{bmatrix}$$

$$= \text{K} \begin{bmatrix} 2.65 & 0.72 & 0 \\ 0.72 & 0.8505 & -1.125 \\ 0 & -1.125 & 3.2 \end{bmatrix}$$

Step 4: Boundary condition:

The boundary conditions according to the support of the grid structure can be expressed in terms of the displacement vector. The displacement vector will be as follows

$$\{d\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ d_4 \\ d_5 \\ d_6 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Here, d_4, d_5, and d_6 indicate the displacement vectors at point B.

Step 5: Load vector:
The distributed load on member (1) can be replaced by its equivalent joint load as shown in the figure below.

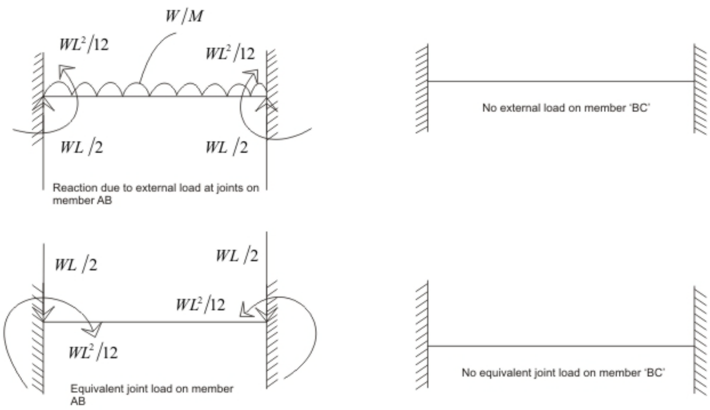

Fig. 4.6.4 Equivalent load

Thus the equivalent load vector will be:

$$\{P\} = \begin{Bmatrix} 0 \\ -\dfrac{wL}{2} \\ -\dfrac{wL^2}{12} \\ 0 \\ -\dfrac{wL}{2} \\ \dfrac{wL^2}{12} \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Step 6: Determination of unknown displacements:.

The unknown displacements can be obtained from the relationship of $\{F\}=[K]\{d\}$ or $\{d\}=[K]^{-1}\{F\}$. Now, eliminating the rows in the force matrix, corresponding to zero element in displacement matrix, the reduced matrix will be as follows.

$$\begin{bmatrix} d_4 \\ d_5 \\ d_6 \end{bmatrix} = \mathrm{k}^{-1} \begin{bmatrix} 2.65 & 0.72 & 0 \\ 0.72 & 0.8505 & -1.125 \\ 0 & -1.125 & 3.2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\dfrac{4w}{2} \\ \dfrac{16w}{12} \end{bmatrix}$$

$$= \frac{1}{k} \begin{bmatrix} 0.662 & -1.047 & -0.368 \\ -1.047 & 3.856 & 1.3556 \\ -0.368 & 1.355 & 0.789 \end{bmatrix} \begin{bmatrix} 0 \\ -2w \\ \dfrac{4}{3}w \end{bmatrix}$$

Thus, the unknown displacements will be:

$$\begin{bmatrix} d_4 \\ d_5 \\ d_6 \end{bmatrix} = \frac{1}{K} \begin{bmatrix} 1.603w \\ -5.905w \\ -\!-1.658w \end{bmatrix}$$

Step 7: Determination of member end actions:

The member end actions can be obtained from the corresponding member stiffness and the nodal displacements. The member end actions for each member are derived as shown below.

<u>Member - AB</u>

In case of member (AB), the member forces will be: $\{F_m\}_{(AB)} = [K]_{(AB)}\{d\}_{(AB)}$

$$\begin{bmatrix} T_1 \\ F_1 \\ M_1 \\ T_2 \\ F_2 \\ M_2 \end{bmatrix} = K \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{-1}{4} & 0 & 0 \\ 0 & \frac{36}{4^3} & \frac{18}{4^2} & 0 & -\frac{36}{4^3} & \frac{18}{4^2} \\ 0 & \frac{18}{4^2} & \frac{12}{4} & 0 & -\frac{18}{4^2} & \frac{6}{4} \\ -\frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{36}{4^3} & -\frac{18}{4^2} & 0 & \frac{36}{4^3} & -\frac{18}{4^2} \\ 0 & \frac{18}{4^2} & \frac{6}{4} & 0 & -\frac{18}{4^2} & \frac{12}{4} \end{bmatrix} \times \frac{1}{K} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.603w \\ -5.905w \\ -1.658w \end{bmatrix}$$

Thus,

$$\begin{bmatrix} T_1 \\ F_1 \\ M_1 \\ T_2 \\ F_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} -0.4w \\ 1.456w \\ 4.156w \\ 0.4w \\ -1.456w \\ 1.70w \end{bmatrix}$$

It is to be noted that $\{F_m\}$ are the end actions due to joint loads. Hence it must be added to the corresponding end actions in the restrained structure in order to obtain the end actions due to the loads. Therefore, $\{F_m\}_{Actual}$ are the true member end actions due to actual loading system and can be expressed as

$$\{F_m\}_{Actual} = \{F_m\} + \{F_{fm}\}$$

Where, $\{F_{fm}\}$are the end actions in the restrained structure. Since there is no load acting on member (1), the actual end action will be:

$$\{F_m\}_{Actual} = \begin{bmatrix} -0.4w \\ 1.456w \\ 4.156w \\ 0.4w \\ -1.456w \\ 1.70w \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{4w}{2} \\ \frac{4^2 w}{12} \\ 0 \\ \frac{4w}{2} \\ -\frac{4^2 w}{12} \end{bmatrix} = \begin{bmatrix} -0.4w \\ 3.46w \\ 5.49w \\ 0.40w \\ 0.54w \\ 0.34w \end{bmatrix}$$

Member - BC

In similar way, the member forces in member (BC) will be: $\{F_m\}_{(BC)} = [K]_{(BC)}\{d\}_{(BC)}$

$$\begin{bmatrix} T_2 \\ F_2 \\ M_2 \\ T_3 \\ F_3 \\ M_3 \end{bmatrix} = K \begin{bmatrix} \frac{1}{5} & 0 & 0 & \frac{-1}{4} & 0 & 0 \\ 0 & \frac{36}{5^3} & \frac{18}{5^2} & 0 & -\frac{36}{5^3} & \frac{18}{5^2} \\ 0 & \frac{18}{5^2} & \frac{12}{5} & 0 & -\frac{18}{5^2} & \frac{6}{5} \\ -\frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & -\frac{36}{5^3} & -\frac{18}{5^2} & 0 & \frac{36}{5^3} & -\frac{18}{5^2} \\ 0 & \frac{18}{5^2} & \frac{6}{5} & 0 & -\frac{18}{5^2} & \frac{12}{5} \end{bmatrix} \times \frac{1}{K} \begin{bmatrix} -1.658w \\ -5.905w \\ 1.603 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} T_2 \\ F_2 \\ M_2 \\ T_3 \\ F_3 \\ M_3 \end{bmatrix} = \begin{bmatrix} -0.33w \\ -0.55w \\ -0.40w \\ 0.33w \\ 0.55w \\ -2.33w \end{bmatrix}$$

Since there is no load acting on member (BC), the actual end action will be:

$$\{F_m\}_{Actual} = \begin{bmatrix} -0.33w \\ -0.55w \\ -0.40w \\ 0.33w \\ 0.55w \\ -2.33w \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.33w \\ -0.55w \\ -0.40w \\ 0.33w \\ 0.55w \\ -2.33w \end{bmatrix}$$

Thus, the reaction forces at the support and load at the joints will be:

$$\begin{bmatrix} T_1 \\ F_1 \\ M_1 \\ T_2 \\ F_2 \\ M_2 \\ T_3 \\ F_3 \\ M_3 \end{bmatrix} = \begin{bmatrix} -0.4w \\ 3.46w \\ 5.49w \\ 0 \\ 0 \\ 0 \\ 0.33w \\ 0.55w \\ -2.33w \end{bmatrix}$$

4.6.5 Analysis of Space Frame

Space frames are an increasingly common architectural technique especially for large roof spans in commercial and industrial buildings. The rigid jointed frames such as building frames are usually three dimensional space structures. Thus in case of certain structures like multi-storeyed buildings, it is necessary consider 3-dimensional effects for analysis. The space frame constitutes the final step of increasing complexity. It consists of plane frame and grid actions. The displacement and rotation vector associated with each joint have three components in case of space frame structures. There are six equilibrium equations associated with each joint. The degrees of freedom at each node of the space frame member will be (i) displacement in three perpendicular directions and (ii) rotations in three different directions. Therefore, the degrees of freedom in each node of the member will be six as shown in the figure below. The stiffness matrix in local coordinate system considering all possible degrees of freedom will be as given in Table 4.6.1.

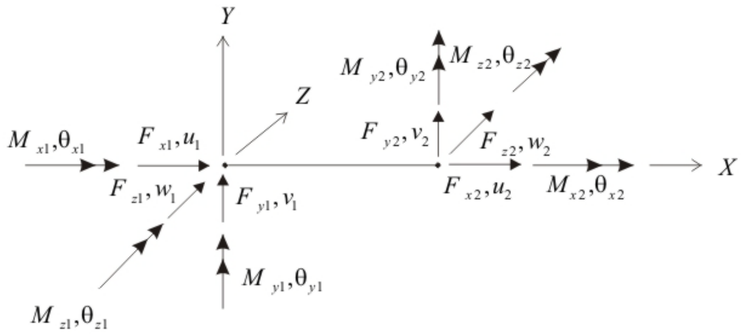

Fig. 4.6.5 Degrees of freedom for space frame member

Table 4.6.1 Stiffness matrix of space frame member

$$
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\left[\begin{array}{cccccc|cccccc}
\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & \frac{6EI_Z}{L^2} & 0 & -\frac{12EI_Z}{L^3} & 0 & 0 & 0 & \frac{6EI_Z}{L^2} \\
0 & 0 & \frac{12EI_Y}{L^3} & 0 & -\frac{6EI_Y}{L^2} & 0 & 0 & 0 & -\frac{12EI_Y}{L^3} & 0 & -\frac{6EI_Y}{L^2} & 0 \\
0 & 0 & 0 & \frac{GI_X}{L} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_X}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0 \\
0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{4EI_Z}{L} & 0 & -\frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L} \\
\hline
-\frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 & \frac{EA_X}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2} & 0 & \frac{12EI_Z}{L^3} & 0 & 0 & 0 & -\frac{6EI_Z}{L^2} \\
0 & 0 & -\frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 & 0 & 0 & \frac{12EI_Y}{L^3} & 0 & \frac{6EI_Y}{L^2} & 0 \\
0 & 0 & 0 & -\frac{GI_X}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI_X}{L} & 0 & 0 \\
0 & 0 & -\frac{6EI_Y}{L^2} & 0 & \frac{2EI_Y}{L} & 0 & 0 & 0 & \frac{6EI_Y}{L^2} & 0 & \frac{4EI_Y}{L} & 0 \\
0 & \frac{6EI_Z}{L^2} & 0 & 0 & 0 & \frac{2EI_Z}{L} & 0 & -\frac{6EI_Z}{L} & 0 & 0 & 0 & \frac{4EI_Z}{L}
\end{array}\right]
\end{array}
$$

The generalized stiffness matrix of a rigid jointed space frame member can be obtained by transferring the matrix of local coordinate system into its global coordinate system. The transformation matrix will become a square matrix of size 12×12 in this case as the degrees of freedom for each node/joint is six.