Lecture 1: Natural Coordinates

Natural coordinate system is basically a local coordinate system which allows the specification of a point within the element by a set of dimensionless numbers whose magnitude never exceeds unity. This coordinate system is found to be very effective in formulating the element properties in finite element formulation. This system is defined in such that the magnitude at nodal points will have unity or zero or a convenient set of fractions. It also facilitates the integration to calculate element stiffness.

3.1.1 One Dimensional Line Elements

The line elements are used to represent spring, truss, beam like members for the finite element analysis purpose. Such elements are quite useful in analyzing truss, cable and frame structures. Such structures tend to be well defined in terms of the number and type of elements used. For example, to represent a truss member, a two node linear element is sufficient to get accurate results. However, three node line elements will be more suitable in case of analysis of cable structure to capture the nonlinear effects. The natural coordinate system for one dimensional line element with two nodes is shown in Fig. 3.1.1. Here, the natural coordinates of any point P can be defined as follows.

$$N_1 = 1 - \frac{x}{l} \quad and \quad N_2 = \frac{x}{l} \tag{3.1.1}$$

Where, x is represented in Cartesian coordinate system. Similarly, x/l can be represented as ξ in natural coordinate system. Thus the above expression can be rewritten in the form of natural coordinate system as given below.

$$N_1 = 1 - \xi \quad and \quad N_2 = \xi \tag{3.1.2}$$

Now, the relationship between natural and Cartesian coordinates can be expressed from eq. (3.1.1) as

$$\begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & l \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \tag{3.1.3}$$

Here, N_1 and N_2 is termed as shape function as well. The variation of the magnitude of two linear shape functions (N_1 and N_2) over the length of bar element are shown in Fig. 3.1.2. This example displays the simplest form of interpolation function. The linear interpolation used for field variable ϕ can be written as

$$\phi(\xi) = \phi_1 N_1 + \phi_2 N_2 \tag{3.1.4}$$

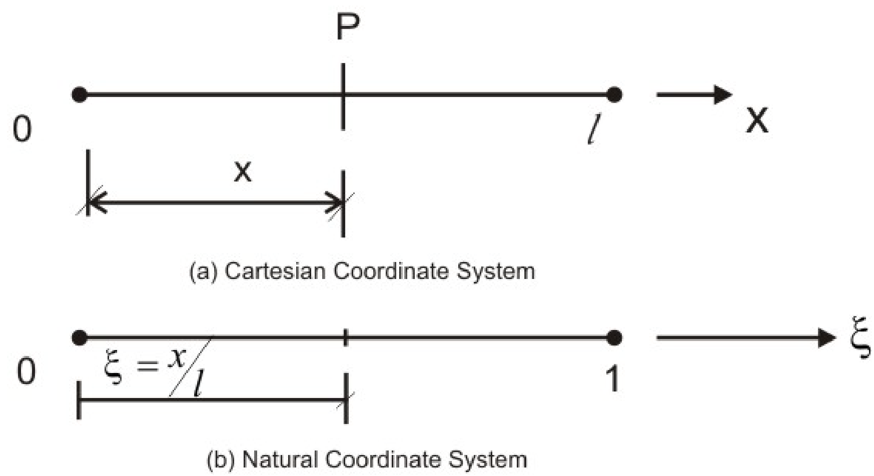


Fig. 3.1.1 Two node line element

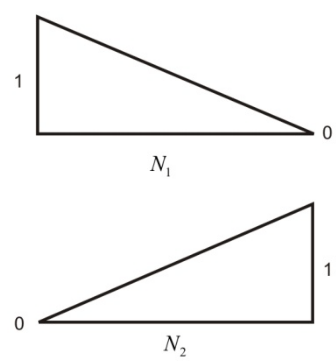


Fig. 3.1.2 Linear interpolation function for two node line element

Similarly, for three node line element, the shape function can be derived with the help of natural coordinate system which may be expressed as follows:

$$\{N\}=\begin{Bmatrix}N_1\\N_2\\N_3\end{Bmatrix}=\begin{Bmatrix}1-\dfrac{3x}{l}+\dfrac{2x^2}{l^2}\\ \dfrac{4x}{l}-\dfrac{4x^2}{l^2}\\ -\dfrac{x}{l}+\dfrac{2x^2}{l^2}\end{Bmatrix}=\begin{Bmatrix}1-3\xi+2\xi^2\\4\xi-4\xi^2\\-\xi+2\xi^2\end{Bmatrix} \quad (3.1.5)$$

The detailed derivation of the interpolation function will be discussed in subsequent lecture. The variation of the shape functions over the length of the three node element are shown in Fig. 3.1.3

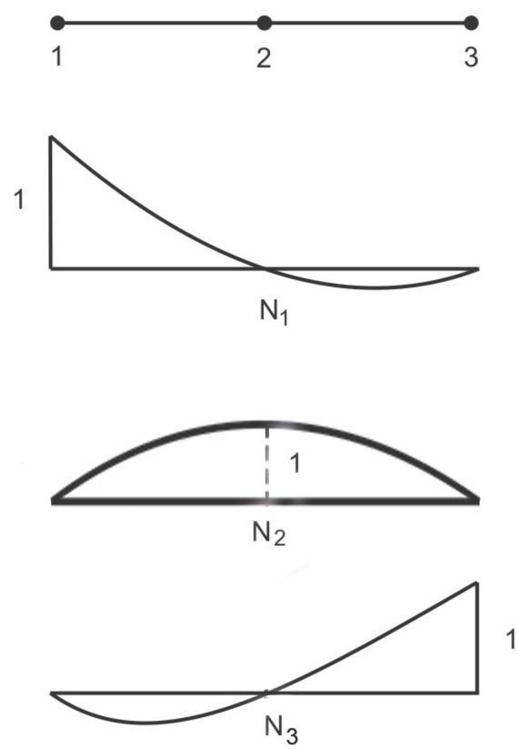


Fig. 3.1.3 Variation of interpolation function for three node line element

Now, if ϕ is considered to be a function of L_1 and L_2, the differentiation of ϕ with respect to x for two node line element can be expressed by the chain rule formula as

$$\frac{\mathrm{d}\phi}{\mathrm{dx}}=\frac{\partial\phi}{\partial \mathrm{L}_1}.\frac{\partial \mathrm{L}_1}{\partial \mathrm{x}}+\frac{\partial\phi}{\partial \mathrm{L}_2}.\frac{\partial \mathrm{L}_2}{\partial \mathrm{x}} \quad (3.1.6)$$

Thus, eq.(3.1.4) can be written as

$$\frac{\partial L_1}{\partial x}=-\frac{1}{l} \;\; and \;\; \frac{\partial L_2}{\partial x}=\frac{1}{l} \quad (3.1.7)$$

Therefore,

$$\frac{d}{dx}=\frac{1}{l}\left(\frac{\partial}{\partial L_2}-\frac{\partial}{\partial L_1}\right) \quad (3.1.7)$$

The integration over the length *l* in natural coordinate system can be expressed by

$$\int_l L_1^{\,p} L_2^{\,q}\, dl=\frac{p!q!}{(p+q+1)!}l \quad (3.1.9)$$

Here, p! is the factorial product p(p-1)(p-2)….(1) and 0! is defined as equal to unity.

3.1.2 Two Dimensional Triangular Elements

The natural coordinate system for a triangular element is generally called as triangular coordinate system. The coordinate of any point *P* inside the triangle is *x,y* in Cartesian coordinate system. Here, three coordinates, L_1, L_2 and L_3 can be used to define the location of the point in terms of natural coordinate system. The point *P* can be defined by the following set of area coordinates:

$$L_1=\frac{A_1}{A} \;\; ; \;\; L_2=\frac{A_2}{A} \;\; ; \;\; L_3=\frac{A_3}{A} \quad (3.1.10)$$

Where,

A_1= Area of the triangle P23

A_2= Area of the triangle P13

A_3= Area of the triangle P12

A=Area of the triangle 123

Thus,

$$A=A_1+A_2+A_3$$

and

$$L_1+L_2+L_3=1 \quad (3.1.11)$$

Therefore, the natural coordinate of three nodes will be: node 1 (1,0,0); node 2 (0,1,0); and node 3 (0,0,1).

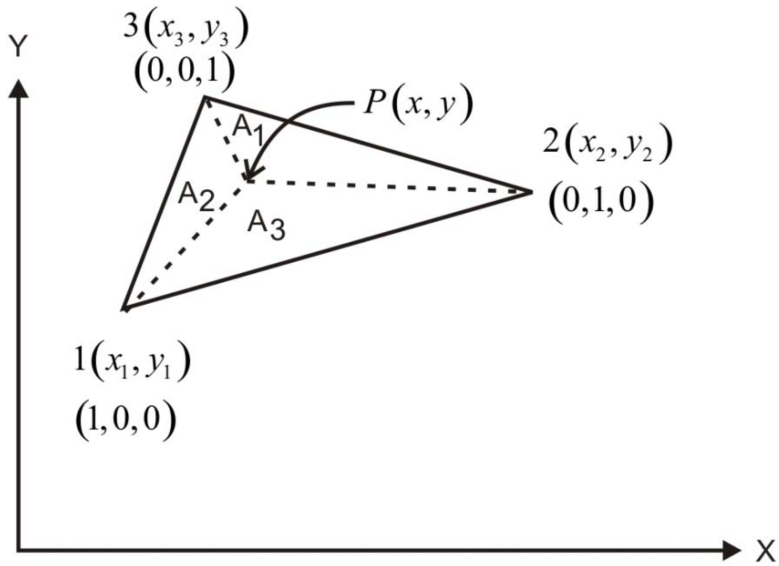


Fig. 3.1.4 Triangular coordinate system

The area of the triangles can be written using Cartesian coordinates considering *x*, *y* as coordinates of an arbitrary point *P* inside or on the boundaries of the element:

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$A_1 = \frac{1}{2}\begin{bmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$A_2 = \frac{1}{2}\begin{bmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{bmatrix}$$

$$A_3 = \frac{1}{2}\begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}$$

The relation between two coordinate systems to define point P can be established by their nodal coordinates as

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \qquad (3.1.12)$$

Where,

$$x = L_1 x_1 + L_2 x_2 + L_3 x_3$$

$$y = L_1 y_1 + L_2 y_2 + L_3 y_3$$

The inverse between natural and Cartesian coordinates from eq.(3.1.12) may be expressed as

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \tag{3.1.13}$$

The derivatives with respect to global coordinates are necessary to determine the properties of an element. The relationship between two coordinate systems may be computed by using the chain rule of partial differentiation as

$$\begin{aligned} \frac{\partial}{\partial \mathrm{x}} &= \frac{\partial}{\partial \mathrm{L}_1}.\frac{\partial \mathrm{L}_1}{\partial \mathrm{x}} + \frac{\partial}{\partial \mathrm{L}_2}.\frac{\partial \mathrm{L}_2}{\partial \mathrm{x}} + \frac{\partial}{\partial \mathrm{L}_3}.\frac{\partial \mathrm{L}_3}{\partial \mathrm{x}} \\ &= \frac{\mathrm{b}_1}{2\mathrm{A}}.\frac{\partial}{\partial \mathrm{L}_1} + \frac{\mathrm{b}_2}{2\mathrm{A}}.\frac{\partial}{\partial \mathrm{L}_2} + \frac{\mathrm{b}_3}{2\mathrm{A}}.\frac{\partial}{\partial \mathrm{L}_3} \\ &= \sum_{\mathrm{i}=1}^{3} \frac{\mathrm{b}_\mathrm{i}}{2\mathrm{A}}.\frac{\partial}{\partial \mathrm{L}_\mathrm{i}} \end{aligned} \tag{3.1.14}$$

Where, $b_1 = y_2 - y_3$; $b_2 = y_3 - y_1$ and $b_3 = y_1 - y_2$. Similarly, following relation can be obtained.

$$\frac{\partial}{\partial \mathrm{y}} = \sum_{\mathrm{i}=1}^{3} \frac{\mathrm{c}_\mathrm{i}}{2\mathrm{A}}.\frac{\partial}{\partial \mathrm{L}_\mathrm{i}} \tag{3.1.15}$$

Where, $c_1 = x_3 - x_2$; $c_2 = x_1 - x_3$ and $c_3 = x_2 - x_1$. The above expressions are looked cumbersome. However, the main advantage is the ease with which polynomial terms can be integrated using following area integral expression.

$$\int_{\mathrm{A}} \mathrm{L}_1^{\mathrm{p}} \mathrm{L}_2^{\mathrm{q}} \mathrm{L}_3^{\mathrm{r}} \mathrm{dA} = \frac{\mathrm{p}!\mathrm{q}!\mathrm{r}!}{(\mathrm{p}+\mathrm{q}+\mathrm{r}+2)!} 2\mathrm{A} \tag{3.1.16}$$

Where 0! is defined as unity.

3.1.3 Shape Function using Area Coordinates

The interpolation functions for the triangular element are algebraically complex if expressed in Cartesian coordinates. Moreover, the integration required to obtain the element stiffness matrix is quite cumbersome. This will be discussed in details in next lecture. The interpolation function and subsequently the required integration can be obtained in a simplified manner by the concept of area coordinates. Considering a linear displacement variation of a triangular element as shown in Fig. 3.1.5, the displacement at any point can be written in terms of its area coordinates.

$$u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3$$

Or, $$u = \{\phi\}^T \{\alpha\} \tag{3.1.17}$$

where, $\{\phi\}^T = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}$ and $\{\alpha\}^T = \{\alpha_1 \quad \alpha_2 \quad \alpha_3\}$

And $$L_1 = \frac{A_1}{A} \;\; ; \;\; L_2 = \frac{A_2}{A} \;\; ; \;\; L_3 = \frac{A_3}{A} \tag{3.1.18}$$

Here, A is the total area of the triangle. It is important to note that the area coordinates are dependent as $L_1 + L_2 + L_3 = 1$. It may be seen from figure that at node 1, $L_1 = 1$ while $L_2 = L_3 = 0$. Similarly for other two nodes: at node 2, $L_2 = 1$ while $L_1 = L_3 = 0$, and $L_3 = 1$ while $L_2 = L_1 = 0$. Now, substituting the area coordinates for node 1, 2 and 3, the displacement components at nodes can be written as

$$\{u_i\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{\alpha\} \tag{3. 1.19}$$

Thus, from the above expression, one can obtain the unknown coefficient α :

$$\{\alpha\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{3. 1.20}$$

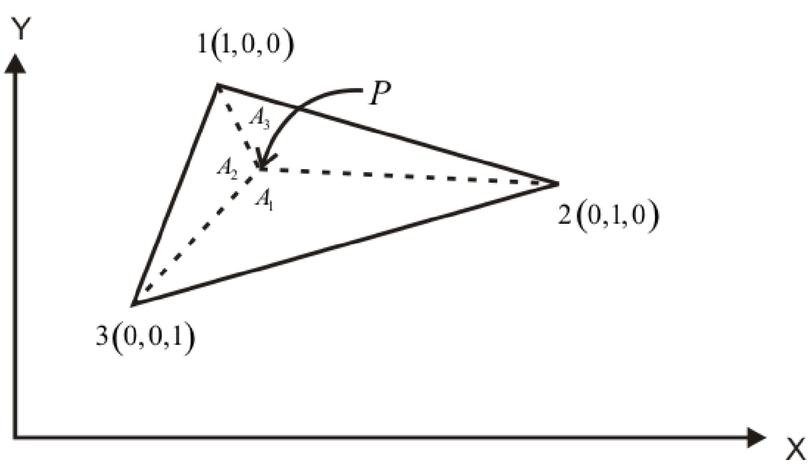


Fig. 3.1.5 Area coordinates for triangular element

Now, eq.(3.1.17) can be written as:

$$\{u\} = \{\phi\}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \{\phi\}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \{u_i\} \tag{3. 1.21}$$

The above expression can be written in terms of interpolation function as $u = \{N\}^T \{u_i\}$

Where,

$$\{N\}^T = [L_1 \quad L_2 \quad L_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [L_1 \quad L_2 \quad L_3] \tag{3. 1.22}$$

Similarly, the displacement variation *v* in Y direction can be expressed as follows.

$$v = \{N\}^T \{v_i\} \tag{3.1.23}$$

Thus, for two displacement components *u* and *v* of any point inside the element can be written as:

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \{N\}^T & \{0\}^T \\ \{0\}^T & \{N\}^T \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \tag{3.1.24}$$

Thus, the shape function of the element will become

$$[N] = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix} \tag{3.1.25}$$

It is important to note that the shape function N_i become unity at node *i* and zero at other nodes of the element. The displacement at any point of the element can be expressed in terms of its nodal displacement and the interpolation function as given below.

$$\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 \\ v &= N_1 v_1 + N_2 v_2 + N_3 v_3 \end{aligned} \tag{3.1.26}$$

Lecture 2: Triangular Elements

The triangular element can be used to represent the arbitrary geometry much easily. On the other hand, rectangular elements, in general, are of limited use as they are not well suited for representing curved boundaries. However, an assemblage of rectangular and triangular element with triangular elements near the boundary can be very effective (Fig. 3.2.1). Triangular elements may also be used in 3-dimensional axi-symmetric problems, plates and shell structures. The shape function for triangular elements (linear, quadratic and cubic) with various nodes (Fig. 3.2.2) can be formulated. An internal node will exist for cubic element as seen in Fig. 3.2.2(c).

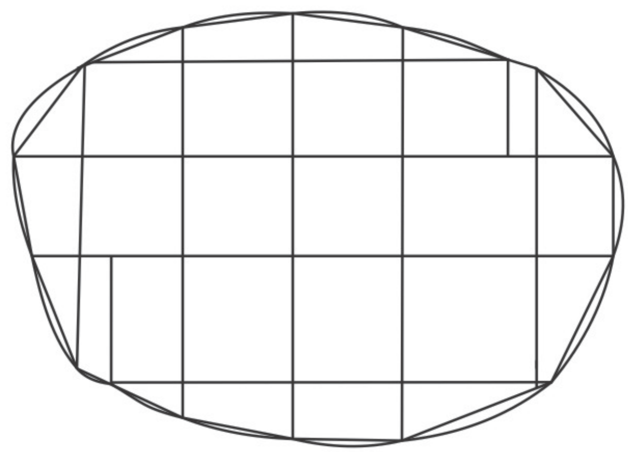


Fig. 3.2.1 Finite element mesh consisting of triangular and rectangular element

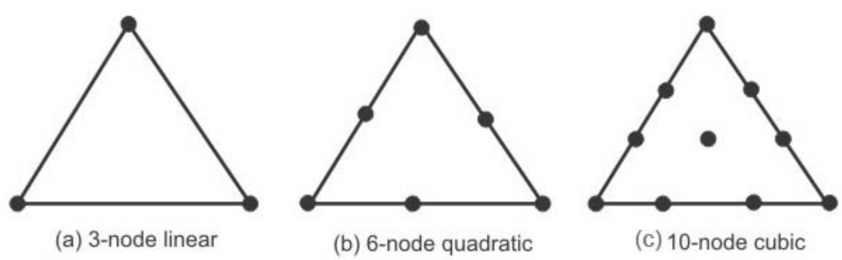


Fig. 3.2.2 Triangular elements

In displacement formulation, it is very important to approximate the variation of displacement in the element by suitable function. The interpolation function can be derived either using the Cartesian coordinate system or by the area coordinates.

3.2.1 Shape function using Cartesian coordinates

Polynomials are easiest way of mathematical operation for expressing variation of displacement. For example, the displacement variation within the element can be represented by the following function in case of two dimensional plane stress/strain problems.

$u = \alpha\alpha_0 + \alpha\alpha_1 x + \alpha\alpha_2 y$

(3.2.1)

$v = \alpha\alpha_3 + \alpha\alpha_4 x + \alpha\alpha_5 y$

(3.2.2)

where α_0, α_1, α_2 ….. are unknown coefficients. Thus the displacement vectors at any point *P*, in the element (Fig.3.2.3) can be expressed with the following relation.

$$\{d\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \quad (3.2.3)$$

Or, $\{d\}=[\ \phi]\{\ \alpha\ \}$ (3.2.4)

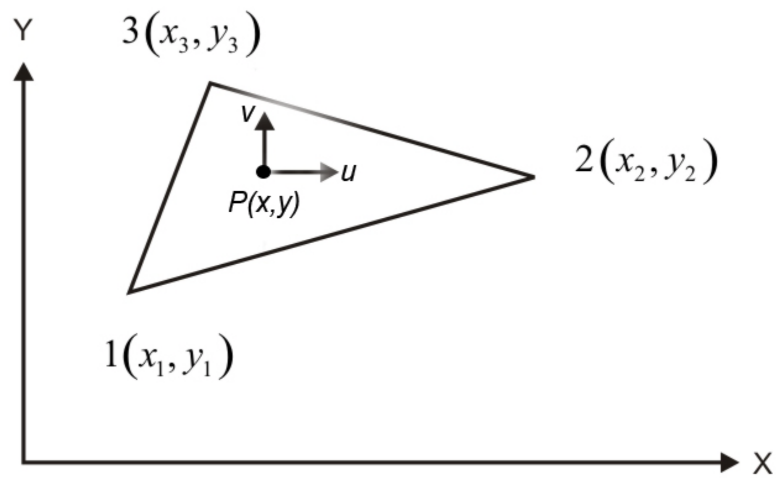


Fig. 3.2.3 Triangular element in Cartesian Coordinates

Similarly, for "*m*" node element having three degrees of freedom at each node, the displacement function can be expressed as

$$u = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + \ldots\ldots + \alpha_{m-1} y^n$$
$$v = \alpha_m + \alpha_{m+1} x + \alpha_{m+2} y + \alpha_{m+3} x^2 + \alpha_{m+4} xy + \ldots\ldots + \alpha_{2m-1} y^n \quad (3.2.5)$$
$$w = \alpha_{2m} + \alpha_{2m+1} x + \alpha_{2m+2} y + \alpha_{2m+3} x^2 + \alpha_{2m+4} xy + \ldots\ldots + \alpha_{3m-1} y^n$$

Hence, in such case,

$$\{d\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} \{\phi\}^T & 0 & 0 \\ 0 & \{\phi\}^T & 0 \\ 0 & 0 & \{\phi\}^T \end{bmatrix} \{\alpha\} \qquad (3.2.6)$$

Where, $\{\alpha\}^T = [\alpha_0\ \alpha_1 \ldots \alpha_{3m-1}]$ and, $[\phi]^T = [1 \quad x \quad y \quad x^2 \quad xy \ldots. \quad y^n]$

Now, for a linear triangular element with 2 degrees of freedom, eq. (3.2.3) can be written in terms of the nodal displacements as follows.

$$\{d\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} \qquad (3.2.7)$$

Where, $\{d\}$ is the nodal displacements. To simplify the above expression for finding out the shape function, the displacements in X direction can be separated out which will be as follows:

$$\{u_i\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_{\propto_0} \\ \alpha_{\propto_1} \\ \alpha_{\propto_2} \end{Bmatrix} \qquad (3.2.8)$$

To obtain the polynomial coefficients, $\{\alpha\}$ the matrix of the above equation are to be inverted. Thus,

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad (3.2.9)$$

Where, A is the area of the triangle and can be obtained as follows.

$$A = \frac{1}{2} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \qquad (3.2.10)$$

Now, eq. (3.2.1) can be written from the above polynomial coefficients.

$$
\begin{aligned}
u &= \frac{1}{2A}\left[\left(x_2y_3 - x_3y_2\right) + \left(y_2 - y_3\right)x + \left(x_3 - x_2\right)y\right]u_1 \\
&\quad + \frac{1}{2A}\left[\left(x_3y_1 - x_1y_3\right) + \left(y_3 - y_1\right)x + \left(x_1 - x_3\right)y\right]u_2 \\
&\quad + \frac{1}{2A}\left[\left(x_1y_2 - x_2y_1\right) + \left(y_1 - y_2\right)x + \left(x_2 - x_1\right)y\right]u_2
\end{aligned}
\tag{3.2.11}
$$

Thus, the interpolation function can be obtained from the above as:

$$
\{N\} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2A}\left[\left(x_2y_3 - x_3y_2\right) + \left(y_2 - y_3\right)x + \left(x_3 - x_2\right)y\right] \\ \frac{1}{2A}\left[\left(x_3y_1 - x_1y_3\right) + \left(y_3 - y_1\right)x + \left(x_1 - x_3\right)y\right] \\ \frac{1}{2A}\left[\left(x_1y_2 - x_2y_1\right) + \left(y_1 - y_2\right)x + \left(x_2 - x_1\right)y\right] \end{Bmatrix}
\tag{3.2.12}
$$

Such three node triangular element is commonly known as constant strain triangle (CST) as its strain is assumed to be constant inside the element. This property may be derived from eq. (3.2.1) and eq.(3.2.2). For example, in case of 2-D plane stress/strain problem, one can express the strain inside the triangle with the help of eq.(3.2.1) and eq.(3.2.2):

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial(\alpha_0 + \alpha_1 x + \alpha_2 y)}{\partial x} = \alpha_1 \\
\varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial(\alpha_3 + \alpha_4 x + \alpha_5 y)}{\partial y} = \alpha_5 \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \alpha_2 + \alpha_2
\end{aligned}
\tag{3.2.13}
$$

CST is the simplest element to develop mathematically. As there is no variation of strain inside the element, the mesh size of the triangular element should be small enough to get correct results. This element produces constant temperature gradients ensuring constant heat flow within the element for heat transfer problems.

3.2.2 Higher Order Triangular Elements

Higher order elements are useful if the boundary of the geometry is curve in nature. For curved case, higher order triangular element can be suited effectively while generating the finite element mesh. Moreover, in case of flexural action in the member, higher order elements can produce more accurate results compare to those using linear elements. Various types of higher order triangular elements are used in practice. However, most commonly used triangular element is the six node element for which development of shape functions are explained below.

3.2.2.1 Shape function for six node element

Fig. 3.2.4 shows a triangular element with six nodes. The additional three nodes (4, 5, and 6) are situated at the midpoints of the sides of the element. A complete polynomial representation of the field variable can be expressed with the help of Pascal triangle:

$$\phi(x,y)=\alpha_0+\alpha_1 x+\alpha_2 y+\alpha_3 x^2+\alpha_4 xy+\alpha_5 y^2 \tag{3.2.14}$$

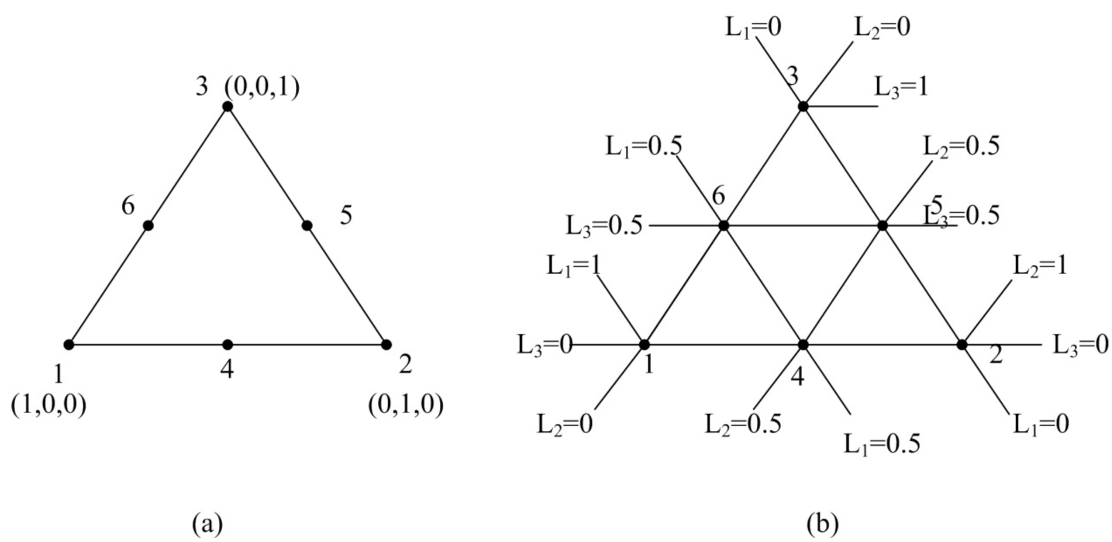


Fig. 3.2.4 (a) Six node triangular element (b) Lines of constant values of the area coordinates

Using the above field variable function, one can reach the following expression using interpolation function and the nodal values.

$$\phi(x,y)=\sum_{i=1}^{6} N_i(x,y)\phi_i \tag{3.2.15}$$

Here, the every shape function must be such that its value will be unity if evaluated at its related node and zero if evaluated at any of the other five nodes. Moreover, as the field variable representation is quadratic, each interpolation function will also become quadratic. Fig. 3.2.4(a) shows the six node element with node numbering convention along with the area coordinates at three corners. The six node element with lines of constant values of the area coordinates passing through the nodes is shown in Fig. 3.2.4(b). Now the interpolation functions can be constructed with the help of area coordinates from the above diagram. For example, the interpolation function N_1 should be unity at node 1 and zero at all other five nodes. According to the above diagram, the value of L_1 is 1 at node 1 and ½ at node 4 and 6. Again, L_1 will be 0 at nodes 2, 3 and 5. To satisfy all these conditions, one can propose following expression:

$$N_1(x,y) = N_1(L_1,L_2,L_3) = L_1\left(L_1 - \frac{1}{2}\right) \tag{3.2.16}$$

Evaluating the above expression, the value of N_1 is becoming ½ at node 1 though it must become unity. Therefore, the above expression is slightly modified satisfying all the conditions and will be as follows:

$$N_1 = 2L_1\left(L_1 - \frac{1}{2}\right) = L_1(2L_1 - 1) \tag{3.2.17}$$

Eq. (3.2.17) assures the required conditions at all the six nodes and is a quadratic function, as L_1 is a linear function of x and y. The remaining five interpolation functions can also be obtained in similar fashion applying the required nodal conditions. Thus, the shape function for the six node triangle element can be written as given below.

$$\begin{aligned}
N_1 &= L_1(2L_1 - 1)\\
N_2 &= L_2(2L_2 - 1)\\
N_3 &= L_3(2L_3 - 1)\\
N_4 &= 4L_1L_2\\
N_5 &= 4L_2L_3\\
N_6 &= 4L_3L_1
\end{aligned} \tag{3.2.18}$$

Such six node triangular element is commonly known as linear strain triangle (LST) as its strain is assumed to vary linearly inside the element. In case of 2-D plane stress/strain problem, the element displacement field for such quadratic triangle may be expressed as

$$\begin{aligned}
u(x,y) &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2\\
v(x,y) &= \alpha_6 + \alpha_7 x + \alpha_8 y + \alpha_9 x^2 + \alpha_{10} xy + \alpha_{11} y^2
\end{aligned} \tag{3.2.19}$$

So the element strain can be derived from the above displacement field as follows.

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = \alpha_1 + 2\alpha_3 x + \alpha_4 y\\
\varepsilon_y &= \frac{\partial v}{\partial y} = \alpha_8 + \alpha_{10} x + 2\alpha_{11} y\\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \alpha_2 + \alpha_4 x + 2\alpha_5 y + \alpha_7 + 2\alpha_9 x + \alpha_{10} y
\end{aligned} \tag{3.2.20}$$

The above expression shows that the strain components are linearly varying inside the element. Therefore, this six node element is called linear strain triangle. The main advantage of this element is that it can capture the variation of strains and therefore stresses of the element.

3.2.3 Construction of Shape Function by Degrading Technique

Sometimes, the geometry of the structure or its loading and boundary conditions are such that the stresses developed in few locations are quite high. On the other hand, variations of stresses are less in some areas and as a result, refinement of finite element mesh is not necessary. It would be economical in terms of computation if higher order elements are chosen where stress concentration is high and lower order elements at area away from the critical area. Fig. 3.2.5 shows graphical representations where various order of triangular elements are used for generating a finite element mesh.

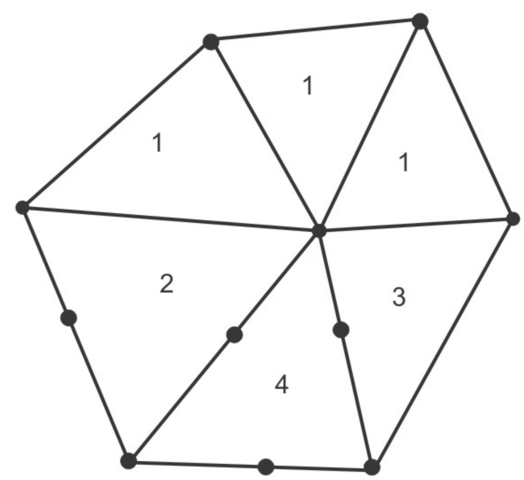


Fig. 3.2.5 Triangular elements with different number of nodes

Fig. 3.2.5 contains four types of element. Type 1 has only three nodes, type 2 element has five nodes, type 3 has four nodes and type 4 has six nodes. The shape function for 3-node and 6-node triangular elements has already been derived. The shape functions of 6-node element can suitably be degraded to derive shape functions of other two types of triangular elements.

3.2.3.1 Five node triangular element

Let consider a six node triangular element as shown in Fig. 3.2.6(a) whose shape functions and nodal displacements are (N_1, N_2, N_3, N_4, N_5, N_6) and (u_1, u_2, u_3, u_4, u_5, u_6) respectively. Similarly, for a five node triangular element as shown in Fig. 3.2.6(b), the shape functions and nodal displacements are considered as (N'_1, N'_2, N'_3, N'_4, N'_5) and (u'_1, u'_2, u'_3, u'_4, u'_5) respectively. Thus, the displacement at any point in a six node triangular element will become

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6u_6 \tag{3.2.21}$$

Where, N_1, N_2, …, N_6 are the shape functions and is given in eq.(3.2.18). If there is no node between 2 and 3, the displacement along line 2-3 is considered to vary linearly. Thus the displacement at an assumed node 5´ may be written as

$$u'_5 = \frac{u_2 + u_3}{2} \tag{3.2.22}$$

Substituting, the value of u'_5 for u_5 in eq.(3.2.21) the following expression will be obtained.

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5\frac{u_2 + u_3}{2} + N_6u_5 \tag{3.2.23}$$

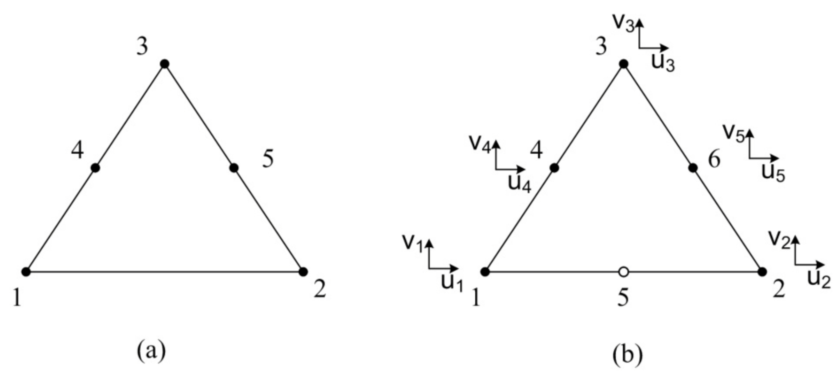


Fig. 3.2.6 Degrading for five node element

Thus, the displacement function can be expressed by five nodal displacements as:

$$u = N_1u_1 + \left(N_2 + \frac{N_5}{2}\right)u_2 + \left(N_3 + \frac{N_5}{2}\right)u_3 + N_4u_4 + N_6u_5 \tag{3.2.24}$$

However, the displacement function for the five node triangular element can be expressed as

$$u = N_1'u_1 + N_2'u_2 + N_3'u_3 + N_4'u_4 + N_5'u_5 \tag{3.2.25}$$

Comparing eq.(3.2.24) and eq.(3.2.25) and observing node 6 of six node triangle corresponds to node 5 of five node triangle, we can write

$$N'_1 = N_{1,}\ \ N_2' = N_2 + \frac{N_5}{2},\ \ N_3' = N_3 + \frac{N_5}{2},\ \ N_4' = N_4 \text{ and } N_5' = N_5 \tag{3.2.26}$$

Hence, the shape function of a five node triangular element will be

$$\begin{aligned}
N_1' &= L_1\left(2L_1 - 1\right) \\
N_2' &= L_2\left(2L_2 - 1\right) + \frac{4L_2L_3}{2} = L_2\left(1 - 2L_1\right) \\
N_3' &= N_3 + \frac{N_5}{2} = L_3\left(2L_3 - 1\right) + \frac{4L_2L_3}{2} = L_3\left(1 - 2L_1\right) \\
N_4' &= 4L_1L_2 \\
N_5' &= 4L_3L_1
\end{aligned} \tag{3.2.27}$$

Thus, for a five node triangular element, the above shape function can be used for finite element analysis.

Lecture 3: Rectangular Elements

Rectangular elements are suitable for modelling regular geometries. Sometimes, it is used along with triangular elements to represent an arbitrary geometry. The simplest element in the rectangular family is the four node rectangle with sides parallel to x and y axis. Fig. 3.3.1 shows rectangular elements with varying nodes representing linear, quadratic and cubic variation of function.

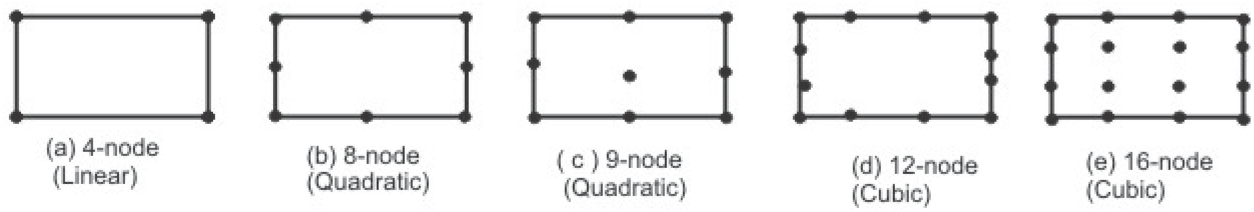


Fig. 3.3.1 Rectangular elements

3.3.1 Shape Function for Four Node Element

Shape functions of a rectangular element can be derived using both Cartesian and natural coordinate systems. A four term polynomial expression for the field variable will be required for a rectangular element with four nodes having four degrees of freedom. Since there is no complete four term polynomial in two dimensions, the incomplete, symmetric expression from the Pascal's triangle may be chosen to ensure geometric isotropy.

3.3.1.1 Shape function using Cartesian coordinates

For the derivation of interpolation function, the sides of the rectangular element (Fig. 3.3.2) are assumed to be parallel to the global Cartesian axes. From the Pascal's triangle, a linear variation may be assumed to define filed variable to ensure inter-element continuity.

$$\phi(x, y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy \qquad (3.3.1)$$

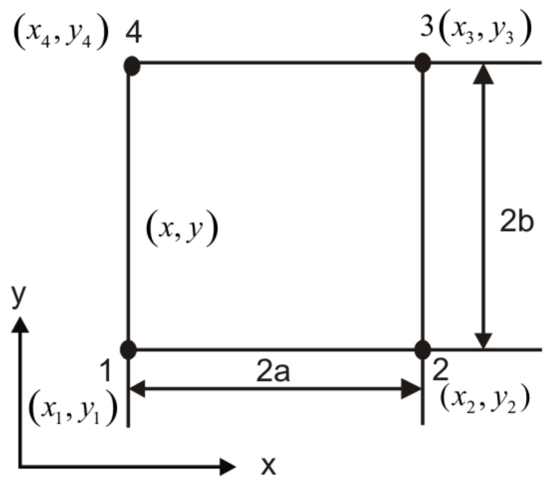


Fig. 3.3.2 Rectangular element in Cartesian coordinate

Applying nodal conditions, the above expression may be written in matrix form as

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{3.3.2}$$

The unknown polynomial coefficients may be obtained from the above equation with the use of nodal field variables.

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \tag{3.3.3}$$

Thus, the field variable at any point inside the element can be described in terms of nodal values as

$$\begin{aligned} \phi(x,y) &= \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}^{-1} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \\ &= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \end{aligned} \tag{3.3.4}$$

From the above expression, the shape function N_i can be derived and will be as follows.

$$N_1 = \left(\frac{x-x_2}{x_1-x_2}\right)\left(\frac{y-y_4}{y_1-y_4}\right)$$
$$N_2 = \left(\frac{x-x_1}{x_2-x_1}\right)\left(\frac{y-y_3}{y_2-y_3}\right)$$
$$N_3 = \left(\frac{x-x_4}{x_3-x_4}\right)\left(\frac{y-y_2}{y_3-y_2}\right) \qquad (3.3.5)$$
$$N_4 = \left(\frac{x-x_3}{x_4-x_3}\right)\left(\frac{y-y_1}{y_4-y_1}\right)$$

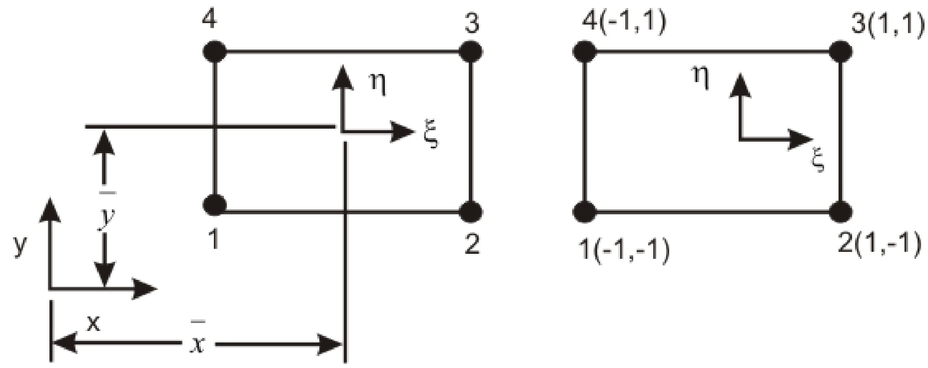
Now, substituting the nodal coordinates in terms of (x_1, y_1) as (–a, –b) at node 1; (x_2, y_2) as (a, –b) at node 2; (x_3, y_3) as (a, b) at node 3 and (x_4, y_4) as (–a, b) at node 4 the above expression can be re-written as:

$$N_1 = \frac{1}{4ab}(x-a)(y-b)$$
$$N_2 = \frac{1}{4ab}(x+a)(y-b)$$
$$N_3 = \frac{1}{4ab}(x+a)(y+b) \qquad (3.3.6)$$
$$N_4 = \frac{1}{4ab}(x-a)(y+b)$$

Thus, the shape function N can be found from the above expression in Cartesian coordinate system.

3.3.1.2 Shape function using natural coordinates

The derivation of interpolation function in terms of Cartesian coordinate system is algebraically complex as seen from earlier section. However, the complexity can be reduced by the use of natural coordinate system, where the natural coordinates will vary from -1 to +1 in place of –a to +a or –b to +b. The transformation of Cartesian coordinates to Natural coordinates are shown in Fig. 3.3.3.



(a) Transformation of Cartesian to natural coordinate (b) Natural coordinates at nodes

Fig. 3.3.3 Four node rectangular element

From the figure, the relation between two coordinate systems can be expressed as

$$\xi = \frac{x - \overline{x}}{a} \qquad and \quad \eta = \frac{y - \overline{y}}{b} \tag{3.3.7}$$

Here, 2*a* and 2*b* are the width and height of the rectangle. The coordinate of the center of the rectangle can be written as follows:

$$\overline{x} = \frac{x_1 + x_2}{2} \qquad and \quad \overline{y} = \frac{y_1 + y_4}{2} \tag{3.3.8}$$

Thus, from eq. (3.3.7) and eq.(3.3.8), the nodal values in natural coordinate systems can be derived which is shown in Fig. 3.3.4(b). With the above relations variations of ξ & η will be from -1 to +1. Now the interpolation function can be derived in a similar fashion as done in section 3.3.1.1. The filed variable can be written in natural coordinate system ensuring inter-element continuity as:

$$\phi(\xi, \eta) = \alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\xi\eta \tag{3.3.9}$$

The coordinates of four nodes of the element in two different systems are shown in Table 3.3.1 for ready reference for the derivation purpose. Applying the nodal values in the above expression one can get

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \tag{3.3.10}$$

Table 3.3.1 Cartesian and natural coordinates for four node element

Node	Cartesian Coordinate		Natural Coordinate	
	x	y	ξ	η
1	x_1	y_1	-1	-1
2	x_2	y_2	1	-1
3	x_3	y_3	1	1
4	x_4	y_4	-1	1

Thus, the unknown polynomial coefficients can be found as

$$\begin{Bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}=\begin{bmatrix}1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1\\ 1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}^{-1}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix}=\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix} \tag{3.3.11}$$

The field variable can be written as follows using eq.(3.3.9) and eq.(3.3.11).

$$\begin{aligned}\phi(\xi,\eta)&=\begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\begin{Bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix}=\begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix}\\ &=\begin{bmatrix}N_1 & N_2 & N_3 & N_4\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{Bmatrix}\end{aligned} \tag{3.3.12}$$

Where, N_i are the interpolation function of the element in natural coordinate system and can be found as:

$$N_i=\begin{Bmatrix}N_1\\ N_2\\ N_3\\ N_4\end{Bmatrix}=\begin{Bmatrix}\frac{(1-\xi)(1-\eta)}{4}\\ \frac{(1+\xi)(1-\eta)}{4}\\ \frac{(1+\xi)(1+\eta)}{4}\\ \frac{(1-\xi)(1+\eta)}{4}\end{Bmatrix} \tag{3.3.13}$$

3.3.2 Shape Function for Eight Node Element

The shape function of eight node rectangular element can be derived in similar fashion as done in case of four node element. The only difference will be on choosing of polynomial as this element is of quadratic in nature. The derivation will be algebraically complex in case of using Cartesian coordinate system. However, use of the natural coordinate system will make the process simpler as the natural coordinates vary from -1 to +1 in the element. The variation of filed variable ϕ can be expressed in natural coordinate system by the following polynomial.

$$\phi(\xi,\eta)=\alpha_0+\alpha_1\xi+\alpha_2\eta+\alpha_3\xi^2+\alpha_4\xi\eta+\alpha_5\eta^2+\alpha_6\xi^2\eta+\alpha_7\xi\eta^2 \qquad (3.3.14)$$

It may be noted that the cubic terms ξ^3 and η^3 are omitted and geometric invariance is ensured by choosing the above expression. Fig. 3.3.4 shows the natural nodal coordinates of the eight node rectangle element in natural coordinate system.

The nodal field variables can be obtained from the above expression after putting the coordinates at nodes.

$$\{\phi_i\}=\begin{Bmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\\\phi_5\\\phi_6\\\phi_7\\\phi_8\end{Bmatrix}=\begin{bmatrix}1&-1&-1&1&1&1&-1&-1\\1&1&-1&1&-1&1&-1&1\\1&1&1&1&1&1&1&1\\1&-1&1&1&-1&1&1&-1\\1&0&-1&0&0&1&0&0\\1&1&0&1&0&0&0&0\\1&0&1&0&0&1&0&0\\1&-1&0&1&0&0&0&0\end{bmatrix}\begin{Bmatrix}\alpha_0\\\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\\\alpha_5\\\alpha_6\\\alpha_7\end{Bmatrix}=[A]\{\alpha_i\} \qquad (3.3.15)$$

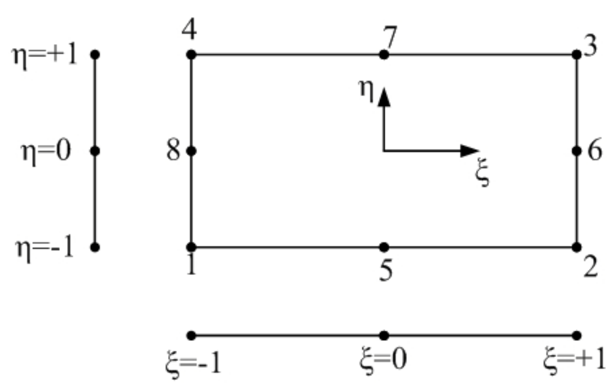


Fig. 3.3.4 Natural coordinates of eight node rectangular element

Replacing the unknown coefficient α_i in eq.(3.3.14) from eq.(3.3.15), the following relations will be obtained.

$$\phi(\xi,\eta)=\left[1\ \xi\ \eta\ \xi^2\ \ \xi\eta\ \ \eta^2\ \ \xi^2\eta\ \ \xi\eta^2\right][A]^{-1}\{\phi_i\}$$

$$=\left[1\ \xi\ \eta\ \xi^2\ \ \xi\eta\ \ \eta^2\ \ \xi^2\eta\ \ \xi\eta^2\right]\frac{1}{4}\begin{bmatrix} -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & -2 & 0 & -2 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & -2 & 0 & -2 \\ -1 & -1 & 1 & 1 & 2 & 0 & -2 & 0 \\ -1 & 1 & 1 & -1 & 0 & -2 & 0 & 2 \end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\end{Bmatrix}$$

$$=\left[N_1\quad N_2\quad N_3\quad N_4\quad N_5\quad N_6\quad N_7\quad N_8\right]\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\end{Bmatrix} \tag{3.3.16}$$

Thus, the interpolation function will become

$$\begin{aligned} &N_1=\frac{(1-\xi)(1-\eta)(-\xi-\eta-1)}{4};\ \ N_2\frac{(1+\xi)(1-\eta)(\xi-\eta-1)}{4};\\ &N_3=\frac{(1+\xi)(1+\eta)(\xi+\eta-1)}{4};N_4=\frac{(1-\xi)(1+\eta)(-\xi+\eta-1)}{4};\\ &N_5=\frac{(1+\xi)(1-\xi)(1-\eta)}{2};\ N_6=\frac{(1+\xi)(1+\eta)(1-\eta)}{2};\\ &N_7=\frac{(1+\xi)(1-\xi)(1+\eta)}{2};\ N_8=\frac{(1-\xi)(1+\eta)(1-\eta)}{2} \end{aligned} \tag{3.3.17}$$

The shape functions of rectangular elements with higher nodes can be derived in similar manner using appropriate polynomial satisfying all necessary criteria. However, difficulty arises due to the inversion of large size of the matrix because of higher degree of polynomial chosen. In next lecture, the shape functions of rectangular element with higher nodes will be derived in a much simpler way.

Lecture 4: Lagrange and Serendipity Elements

In last lecture note, the interpolation functions are derived on the basis of assumed polynomial from Pascal's triangle for the filed variable. As seen, the inverse of the large matrix is quite cumbersome if the element is of higher order.

3.4.1 Lagrange Interpolation Function

An alternate and simpler way to derive shape functions is to use Lagrange interpolation polynomials. This method is suitable to derive shape function for elements having higher order of nodes. The Lagrange interpolation function at node *i* is defined by

$$f_i(\xi)=\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{(\xi-\xi_j)}{(\xi_i-\xi_j)}=\frac{(\xi-\xi_1)(\xi-\xi_2)....(\xi-\xi_{i-1})(\xi-\xi_{i+1})....(\xi-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)....(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})....(\xi_i-\xi_n)} \tag{3.4.1}$$

The function f_i (ξ) produces the Lagrange interpolation function for i^{th} node, and ξ_j denotes ξ coordinate of j^{th} node in the element. In the above equation if we put $\xi = \xi_j$, and $j \neq i$, the value of the function $f_i(\xi)$ will be equal to zero. Similarly, putting $\xi = \xi_i$, the numerator will be equal to denominator and hence $f_i(\xi)$ will have a value of unity. Since, Lagrange interpolation function for i^{th} node includes product of all terms except j^{th} term; for an element with n nodes, $f_i(\xi)$ will have n-1 degrees of freedom. Thus, for one-dimensional elements with n-nodes we can define shape function as $N_i(\xi)=f_i(\xi)$.

3.4.1.1 Shape function for two node bar element

Consider the two node bar element discussed as in section 3.1.1. Let us consider the natural coordinate of the center of the element as 0, and the natural coordinate of the nodes 1 and 2 are -1 and +1 respectively. Therefore, the natural coordinate ξ at any point x can be represented by,

$$\xi=\frac{2(x-x_1)}{l}-1 \tag{3.4.2}$$

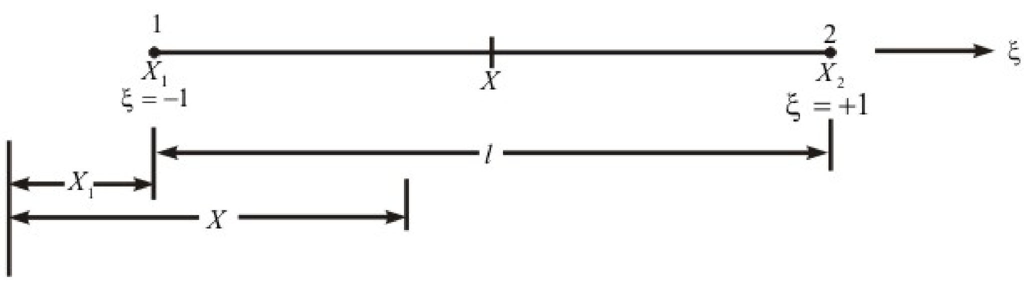


Fig. 3.4.1 Natural coordinates of bar element

The shape function for two node bar element as shown in Fig. 3.4.1 can be derived from eq.(3.4.1) as follows:

$$N_1 = f_1(\xi) = \frac{(\xi - \xi_2)}{(\xi_1 - \xi_2)} = \frac{(\xi - 1)}{-1 - (1)} = \frac{1}{2}(1 - \xi)$$
$$N_2 = f_1(\xi) = \frac{(\xi - \xi_1)}{(\xi_2 - \xi_1)} = \frac{(\xi + 1)}{1 - (1)} = \frac{1}{2}(1 + \xi) \qquad (3.4.3)$$

Graphically, these shape functions are represented in Fig.3.4.2.

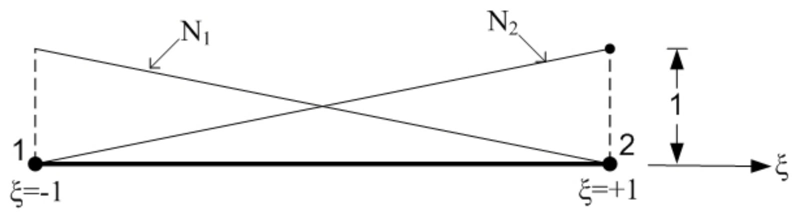


Fig. 3.4.2 Shape functions for two node bar element

3.4.1.2 Shape function for three node bar element

For a three node bar element as shown in Fig. 3.4.3, the shape function will be quadratic in nature. These can be derived in the similar fashion using eq.(3.4.1) which will be as follows:

$$N_1(\xi) = f_1(\xi) = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi)(\xi - 1)}{(-1)(-2)} = \frac{1}{2}\xi(\xi - 1)$$

$$N_2(\xi) = f_2(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = \frac{(\xi + 1)(\xi - 1)}{(1)(-1)} = (1 - \xi^2) \qquad (3.4.4)$$

$$N_3(\xi) = f_3(\xi) = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = \frac{(\xi + 1)(\xi)}{(2)(1)} = \frac{1}{2}\xi(\xi + 1)$$

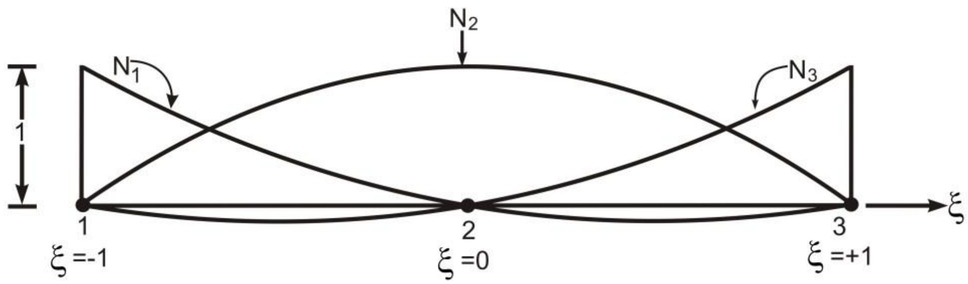


Fig. 3.4.3 Quadratic shape functions for three node bar element

3.4.1.3 Shape function for two dimensional elements

We can derive the Lagrange interpolation function for two or three dimensional elements from one dimensional element as discussed above. Those elements whose shape functions are derived from the products of one dimensional Lagrange interpolation functions are called Lagrange elements. The Lagrange interpolation function for a rectangular element can be obtained from the product of appropriate interpolation functions in the ξ direction [$f_i(\xi)$] and η direction [$f_i(\eta)$]. Thus,

$$N_i(\xi,\eta) = f_i(\xi) f_i(\eta) \text{ Where, } i = 1,2,3, \ldots, \text{n-node} \tag{3.4.5}$$

The procedure is described in details in following examples.

Four node rectangular element

The shape functions for the four node rectangular element as shown in the Fig. 3.4.4 can be derived by applying eq.(3.4.3) eq.(3.4.5) which will be as follows.

`

$$\begin{aligned} N_1(\xi,\eta) &= f_1(\xi) f_1(\eta) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)} \frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} \\ &= \frac{(\xi-1)}{-1-(1)} \times \frac{(\eta-1)}{-1-(1)} = \frac{1}{4}(1-\xi)(1-\eta) \end{aligned} \tag{3.4.6}$$

Similarly, other interpolation functions can be derived which are given below.

`

$$\begin{aligned} N_2(\xi,\eta) &= f_2(\xi) f_1(\eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) &= f_2(\xi) f_2(\eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) &= f_1(\xi) f_2(\eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{aligned} \tag{3.4.7}$$

These shape functions are exactly same as eq.(3.3.13) which was derived earlier by choosing polynomials.

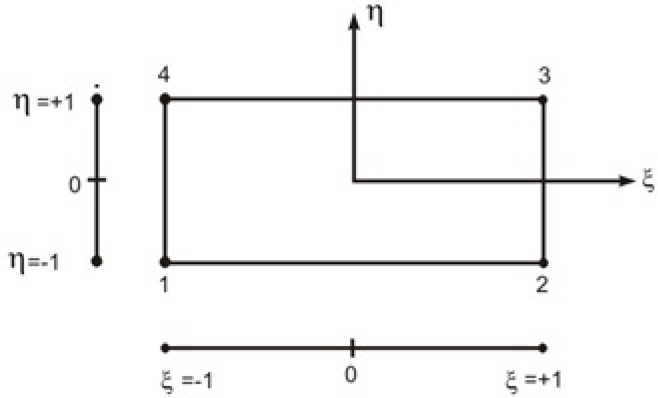


Fig. 3.4.4 Four node rectangular element

Nine node rectangular element

In a similar way, to the derivation of four node rectangular element, we can derive the shape functions for a nine node rectangular element. In this case, the shape functions can be derived using eq.(3.4.4) and eq.(3.4.5).

$$N_1(\xi,\eta) = f_1(\xi) f_1(\eta) = \frac{1}{2}\xi(\xi-1)\times\frac{1}{2}\eta(\eta-1) = \frac{1}{4}\xi\eta(\xi-1)(\eta-1) \tag{3.4.8}$$

In a similar way, all the other shape functions of the element can be derived. The shape functions of nine node rectangular element will be:

$$\begin{aligned}
&N_1 = \frac{1}{4}\xi\eta(\xi-1)(\eta-1), \qquad N_2 = \frac{1}{4}\xi\eta(\xi+1)(\eta-1) \\
&N_3 = \frac{1}{4}\xi\eta(\xi+1)(\eta+1), \qquad N_4 = \frac{1}{4}\xi\eta(\xi-1)(\eta+1) \\
&N_5 = \frac{1}{2}\eta\left(1-\xi^2\right)(\eta-1), \qquad N_6 = \frac{1}{2}\xi(\xi+1)\left(1-\eta^2\right) \\
&N_7 = \frac{1}{2}\eta\left(1-\xi^2\right)(\eta+1), \qquad N_8 = \frac{1}{2}\xi(\xi-1)\left(1-\eta^2\right) \\
&\qquad\qquad N_9 = \left(1-\xi^2\right)\left(1-\eta^2\right)
\end{aligned} \tag{3.4.9}$$

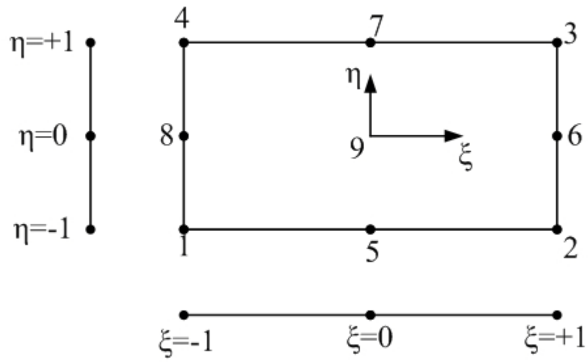


Fig. 3.4.5 Nine node rectangular element

Thus, it is observed that the two dimensional Lagrange element contains internal nodes (Fig. 3.4.6) which are not connected to other nodes.

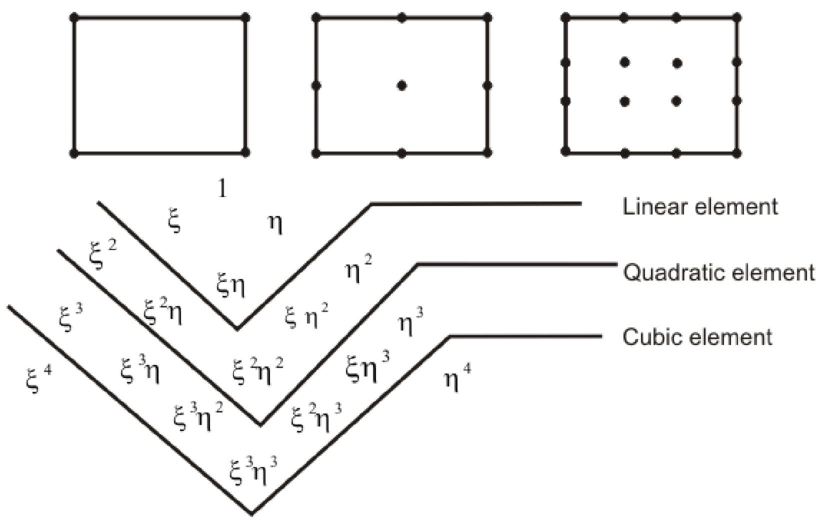


Fig. 3.4.6 Two dimensional Lagrange elements and Pascal triangle

3.4.2 Serendipity Elements

Higher order Lagrange elements contains internal nodes, which do not contribute to the inter-element connectivity. However, these can be eliminated by condensation procedure which needs extra computation. The elimination of these internal nodes results in reduction in size of the element matrices. Alternatively, one can develop shape functions of two dimensional elements which contain nodes only on the boundaries. These elements are called serendipity elements (Fig. 3.4.7) and their interpolation functions can be derived by inspection or the procedure described in previous lecture

(Module 3, lecture 3). The interpolation function can be derived by inspection in terms of natural coordinate system as follows:

(a) Linear element

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{3.4.10}$$

(b) Quadratic element

(i) For nodes at $\xi=\pm1,\ \eta=\pm1$

$$N_i(\xi,\eta)=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) \tag{3.4.11a}$$

(ii) For nodes at $\xi=\pm1,\ \eta=0$

$$N_i(\xi,\eta)=\frac{1}{2}(1+\xi\xi_i)(1+\eta^2) \tag{3.4.11b}$$

(iii) For nodes at $\xi=0,\ \eta=\pm1$

$$N_i(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta\eta_i) \tag{3.4.11c}$$

(c) Cubic element

(i) For nodes at $\xi=\pm1,\ \eta=\pm1$

$$N_i(\xi,\eta)=\frac{1}{32}(1+\xi\xi_i)(1+\eta\eta_i)\left[9\left(\xi^2+\eta^2\right)-10\right] \tag{3.4.12a}$$

(ii) For nodes at $\xi=\pm1,\ \eta=\pm\frac{1}{3}$

$$N_i(\xi,\eta)=\frac{9}{32}(1+\xi\xi_i)(1-\eta^2)(1+9\eta\eta_i) \tag{3.4.12b}$$

And so on for other nodes at the boundaries.

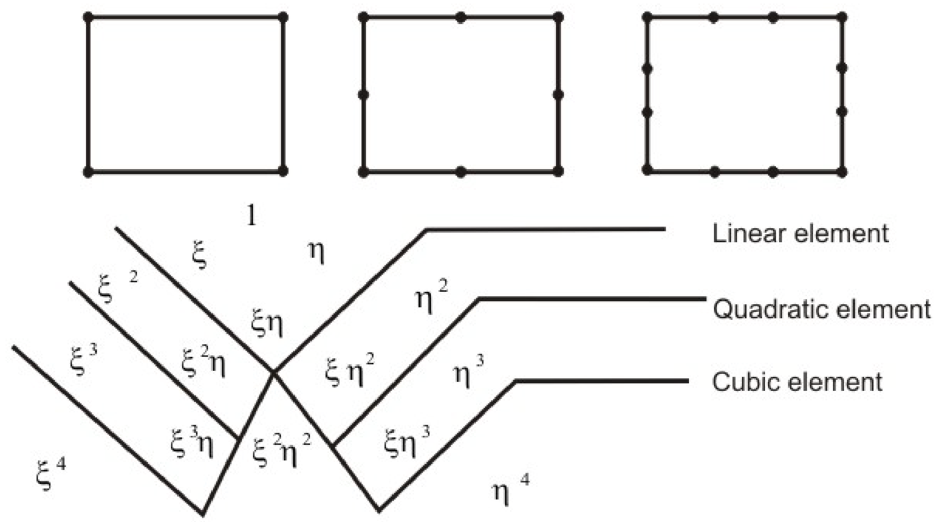


Fig. 3.4.7 Two dimensional serendipity elements and Pascal triangle

Thus, the nodal conditions must be satisfied by each interpolation function to obtain the functions serendipitously. For example, let us consider an eight node element as shown in Fig. 3.4.8 to derive its shape function. The interpolation function N_1 must become zero at all nodes except node 1, where its value must be unity. Similarly, at nodes 2, 3, and 6, $\xi = 1$, so including the term $\xi - 1$ satisfies the zero condition at those nodes. Similarly, at nodes 3, 4 and 7, $\eta = 1$ so the term $\eta - 1$ ensures the zero condition at these nodes.

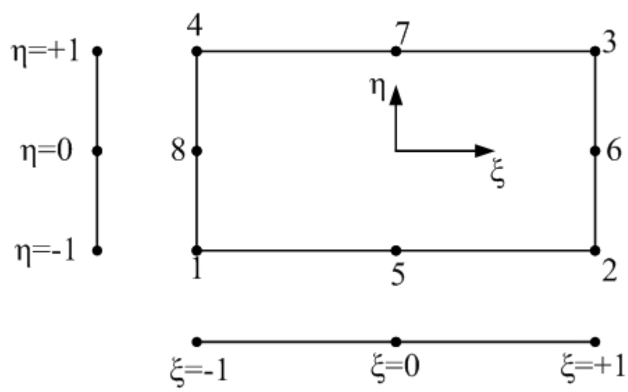


Fig. 3.4.8 Two dimensional eight node rectangular element

Again, at node 5, $(\xi, \eta) = (0, -1)$, and at node 8, $(\xi, \eta) = (-1, 0)$. Hence, at nodes 5 and 8, the term $(\xi + \eta + 1)$ is zero. Using this reasoning, the equation of lines are expressed in Fig. 3.4.9. Thus, the interpolation function associated with node 1 is to be of the form $N_1 = \psi_1(\eta - 1)(\xi - 1)(\xi + \eta + 1)$ where, ψ_1 is unknown constant. As the value of N_1 is 1 at node 1, the magnitude unknown constant ψ_1 will become -1/4. Therefore, the shape function for node 1 will become $N_1 = -\frac{1}{4}(1-\eta)(1-\xi)(\xi+\eta+1)$.

Similarly, ψ_2 will become -1/4 considering the value of N_2 at node 2 as unity and the shape function for node 2 will be $N_2 = \psi_2(\eta - 1)(\xi + 1)(\xi - \eta - 1) = -\frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$. In a similar fashion one can find out other interpolation functions from Fig. 3.4.9 by putting the respective values at various nodes. Thus, the shape function for 8-node rectangular element is given below.

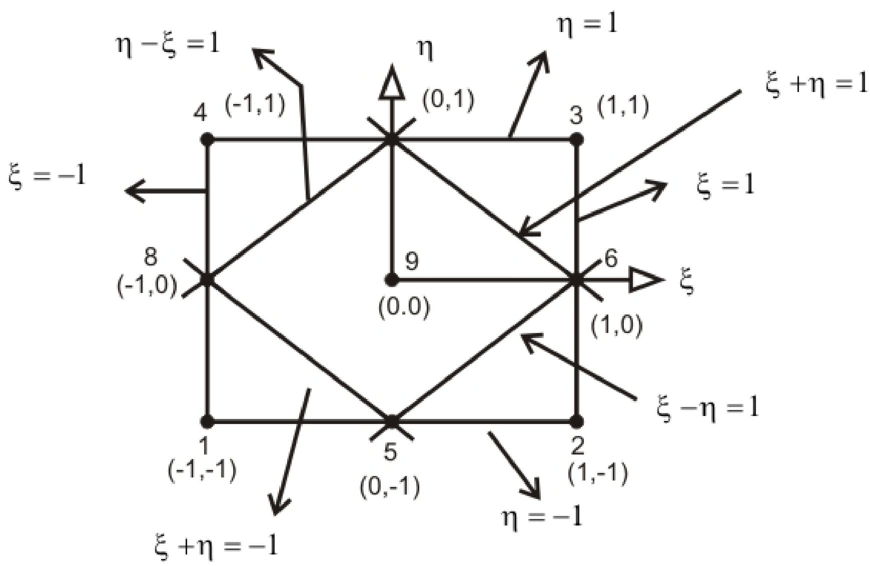


Fig. 3.4.9 Equations of lines for two dimensional eight node element

$$
\begin{aligned}
&N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta), \qquad N_5 = \frac{1}{2}(1-\xi^2)(1-\eta),\\
&N_2 = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta), \qquad N_6 = \frac{1}{2}(1+\xi)(1-\eta^2),\\
&N_3 = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta), \qquad N_7 = \frac{1}{2}(1-\xi^2)(1+\eta),\\
&N_4 = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \text{ and } N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)
\end{aligned}
\tag{3.4.13}
$$

It may be observed that the Lagrange elements have a better degree of completeness in polynomial function compare to serendipity elements. Therefore, Lagrange elements produce comparatively faster and better accuracy.

Lecture 5: Solid Elements

There are two basic families of three-dimensional elements similar to two-dimensional case. Extension of triangular elements will produce tetrahedrons in three dimensions. Similarly, rectangular parallelepipeds are generated on the extension of rectangular elements. Fig. 3.5.1 shows few commonly used solid elements for finite element analysis.

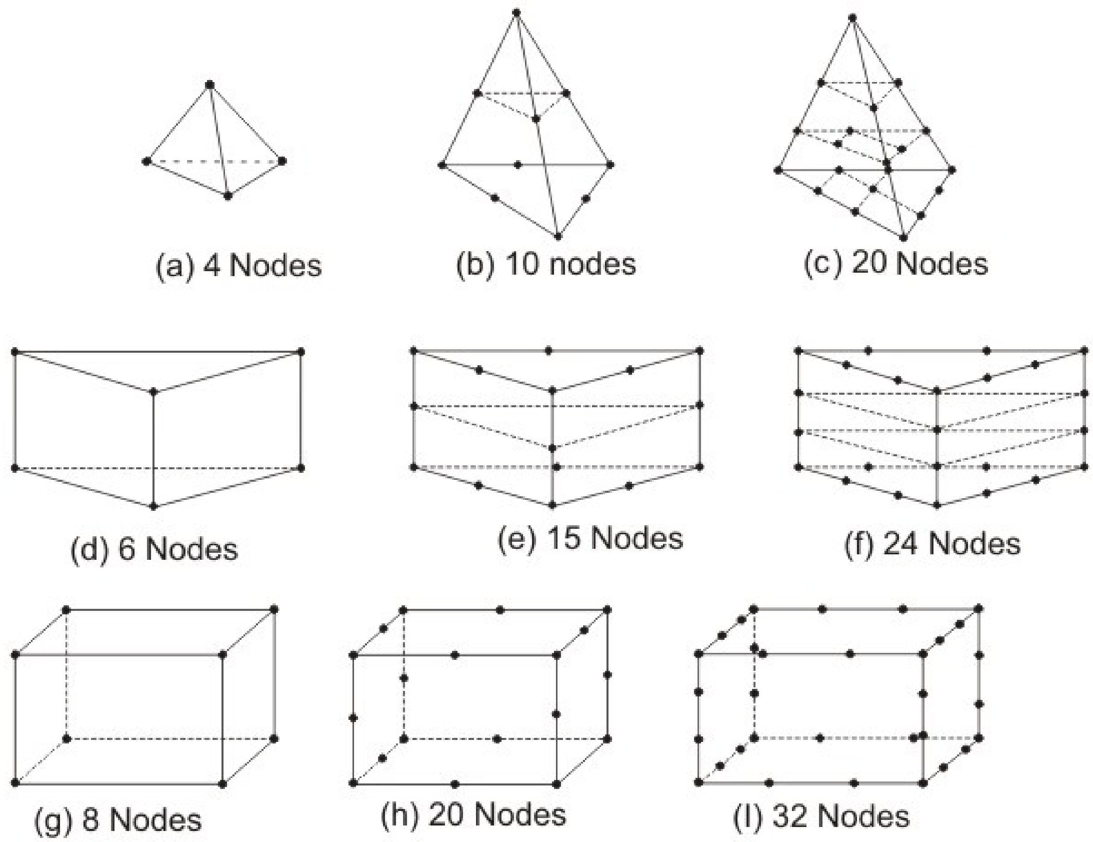


Fig. 3.5.1 Three-dimensional solid elements

Derivation of shape functions for such three dimensional elements in Cartesian coordinates are algebraically quite cumbersome. This is observed while developing shape functions in two dimensions. Therefore, the shape functions for the two basic elements of the tetrahedral and parallelepipeds families will be derived using natural coordinates.

The polynomial expression of the field variable in three dimensions must be complete or incomplete but symmetric to satisfy the geometric isotropy requirements. Completeness and symmetry can be ensured using the Pascal pyramid which is shown in Fig. 3.5.2. It is important to note that each independent variable must be of equal strength in the polynomial.

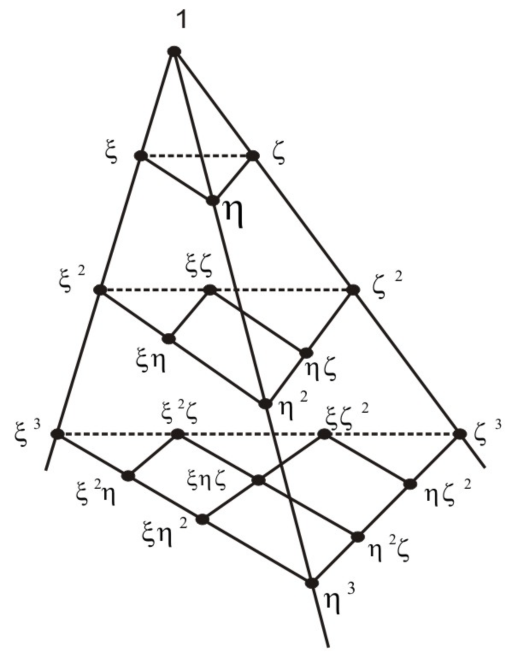


Fig. 3.5.2 Pascal pyramid in three dimensions

The following 3-D quadratic polynomial with complete terms can be applied to an element having 10 nodes.

$$\phi(\xi,\eta,\zeta)=\alpha_0+\alpha_1\xi+\alpha_2\eta+\alpha_3\zeta+\alpha_4\xi^2+\alpha_5\eta^2+\alpha_6\zeta^2+\alpha_7\xi\eta+\alpha_8\eta\zeta+\alpha_9\zeta\xi \quad (3.5.1)$$

However, the geometric isotropy is not an absolute requirement for field variable representation to derive the shape functions.

3.5.1 Tetrahedral Elements

The simplest element of the tetrahedral family is a four node tetrahedron as shown in Fig. 3.5.3. The node numbering has been followed in sequential manner, i.e, in this case anti-clockwise direction. Similar to the area coordinates, the concept of volume coordinates has been introduced here. The coordinates of the nodes are defined both in Cartesian and volume coordinates. Point *P*(*x, y, and z*) as shown in Fig. 3.5.2 is an arbitrary point in the tetrahedron.

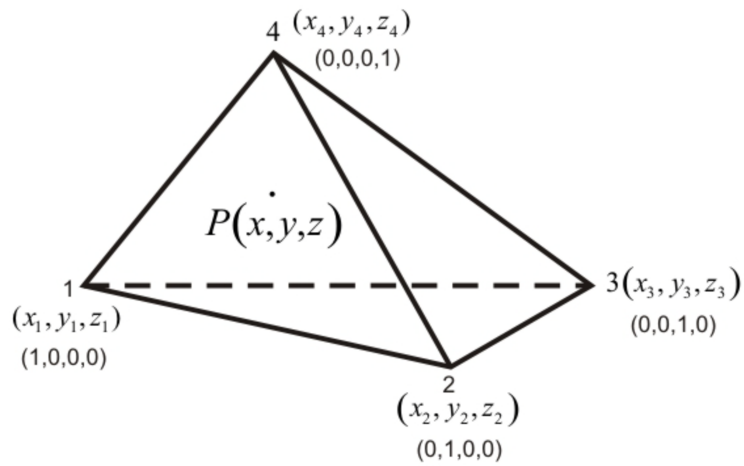


Fig. 3.5.3 Four node tetrahedron element

The linear shape function for this element can be expressed as,

$$\{N\}^T = [L_1 \quad L_2 \quad L_3 \quad L_4] \tag{3.5.2}$$

Here, L_1, L_2, L_3, L_4 are the set of natural coordinates inside the tetrahedron and are defined as follows

$$L_i = \frac{V_i}{V} \tag{3.5.3}$$

Where V_i is the volume of the sub element which is bound by point P and face i and V is the total volume of the element. For example L_1 may be interpreted as the ratio of the volume of the sub element P234 to the total volume of the element 1234. The volume of the element V is given by the determinant of the nodal coordinates as follows:

$$V = \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} \tag{3.5.4}$$

The relationship between the Cartesian and natural coordinates of point P may be expressed as

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \tag{3.5.5}$$

It may be noted that the identity included in the first row ensure the matrix invertible.

$$L_1 + L_2 + L_3 + L_4 = 1 \tag{3.5.6}$$

The inverse relation is given by

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} V_1 & a_1 & b_1 & c_1 \\ V_2 & a_2 & b_2 & c_2 \\ V_3 & a_3 & b_3 & c_3 \\ V_4 & a_4 & b_4 & c_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} \tag{3.5.7}$$

Here, V_i is the volume subtended from face i and terms a_i, b_i, and c_i represent the projected area of face i on the x, y, z coordinate planes respectively and are given as follows:

$$\begin{aligned} a_i &= (z_j y_k - z_k y_j) + (z_k y_l - z_l y_k) + (z_l y_j - z_j y_l) \\ b_i &= (z_j x_k - z_k x_j) + (z_k x_l - z_l x_k) + (z_l x_j - z_j x_l) \\ c_i &= (y_j x_k - y_k x_j) + (y_k x_l - y_l x_k) + (y_l x_j - y_j x_l) \end{aligned} \tag{3.5.8}$$

i, j, k, l will be in cyclic order (i.e., 1→ 2 → 3 → 4 → 1). The volume coordinates fulfil all nodal conditions for interpolation functions. Therefore, the field variable can be expressed in terms of nodal values as

$$\phi(x, y, z) = L_1\phi_1 + L_2\phi_2 + L_3\phi_3 + L_4\phi_4 \tag{3.5.9}$$

Though the shape functions (i.e., the volume coordinates) in terms of global coordinates is algebraically complex but they are straightforward. The partial derivatives of the natural coordinates with respect to the Cartesian coordinates are given by

$$\frac{\partial L_i}{\partial x} = \frac{a_i}{6V}, \qquad \frac{\partial L_i}{\partial y} = \frac{b_i}{6V}, \qquad \frac{\partial L_i}{\partial z} = \frac{c_i}{6V} \tag{3.5.10}$$

Similar to area integral, the general integral taken over the volume of the element is given by,

$$\int_v L_1^p L_2^q L_3^r L_4^s \, dV = \frac{p!q!r!s!}{(p+q+r+s+3)!}.6V \tag{3.5.11}$$

The four node tetrahedral element is a linear function of the Cartesian coordinates. Hence, all the first partial derivatives of the field variable will be constant. The tetrahedral element is a constant strain element as the element exhibits constant gradients of the field variable in the coordinate directions.

Higher order elements of the tetrahedral family are shown in Fig. 3.5.1. The shape functions for such higher order three dimensional elements can readily be derived in volume coordinates, as for higher-order two-dimensional triangular elements. The second element of this family has 10 nodes and a cubic form for the field variable and interpolation functions.

+++++++++++

3.5.2 Brick Elements

Various orders of elements of the parallelepiped family are shown in Fig. 3.5.1. Fig. 3.5.4 shows the eight-node brick element with reference to a global Cartesian coordinate system and then with reference to natural coordinate system. The natural coordinates for the brick element can be relate Cartesian coordinate system by

$$\xi = \frac{x-\overline{x}}{a}, \qquad \eta = \frac{y-\overline{y}}{b} \quad and \quad \zeta = \frac{z-\overline{z}}{c} \tag{3.5.12}$$

Here, 2*a*, 2*b* and 2*c* are the length, height and width of the element. The coordinate of the center of the brick element can be written as follows:

$$\overline{x} = \frac{x_1+x_2}{2}, \quad \overline{y} = \frac{y_1+y_4}{2} \quad and \quad \overline{z} = \frac{z_1+z_5}{2} \tag{3.5.13}$$

Thus, from eq.(3.5.12) and eq.(3.5.13), the nodal values in natural coordinate systems can be derived which is shown in Fig. 3.5.4(b). With the above relations variations of $\xi, \eta \& \zeta$ will be from -1 to +1. Now the interpolation function can be derived in several procedures as done in case of two dimensional rectangular elements. For example, the interpolation function can be derived by inspection in terms of natural coordinate system as follows:

$$\mathrm{N_i}(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i) \tag{3.5.14}$$

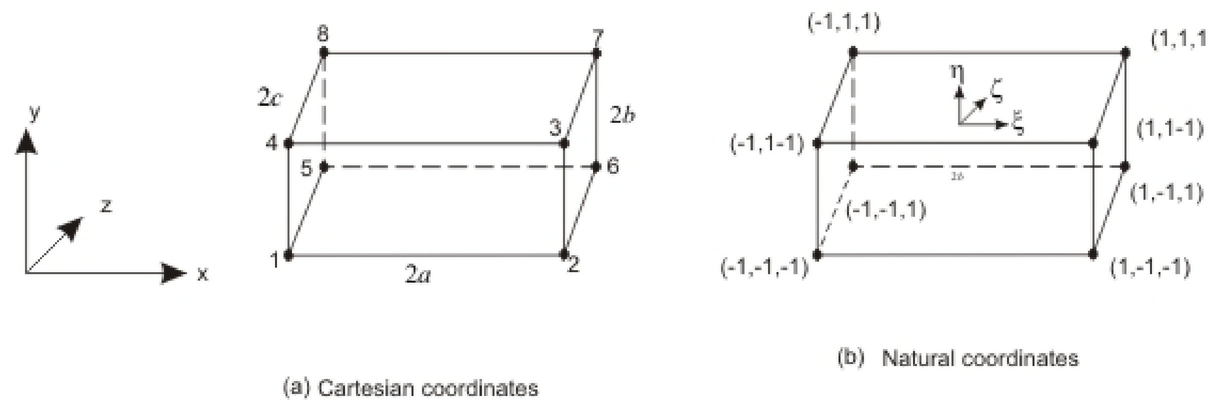


Fig. 3.5.4 Eight node brick element

By using field variable the following terms of the polynomial may be used for deriving the shape function for eight-node brick element.

$$\phi(\xi,\eta,\zeta) = \alpha_0 + \alpha_1\xi + \alpha_2\eta + \alpha_3\zeta + \alpha_4\xi\eta + \alpha_5\eta\zeta + \alpha_6\zeta\xi + \alpha_7\zeta\eta\xi \tag{3.5.14}$$

The above equation is incomplete but symmetric. However, such representations are quite often used and solution convergence is achieved in the finite element analysis. Again, the shape functions for three dimensional 8-node or 27-node brick elements can be derived using Lagrange interpolation function. For this we need to introduce interpolation function in the ζ-direction. Thus, for example, the Lagrange interpolation function for a three dimensional 8 node brick element can be obtained from the product of appropriate interpolation functions in the ξ, η and ζ directions. Therefore, the shape function will become

$$N_i(\xi,\eta,\zeta) = f_i(\xi) f_i(\eta) f_i(\zeta) \text{ Where, } i = 1,2,3, \ldots, \text{n-node} \tag{3.5.15}$$

Thus using the Lagrange interpolation function the shape function at node 1 can be expressed as

$$\begin{aligned} N_1(\xi,\eta,\zeta) &= f_1(\xi) f_1(\eta) f_1(\zeta) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)} \frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} \frac{(\zeta-\zeta_2)}{(\zeta_1-\zeta_2)} \\ &= \frac{(\xi-1)}{-1-(1)} \times \frac{(\eta-1)}{-1-(1)} \times \frac{(\zeta-1)}{-1-(1)} = \frac{1}{4}(1-\xi)(1-\eta)(1-\zeta) \end{aligned} \tag{3.5.16}$$

Using any of the above concepts, the interpolation function for 8-node brick element can be found as follows:

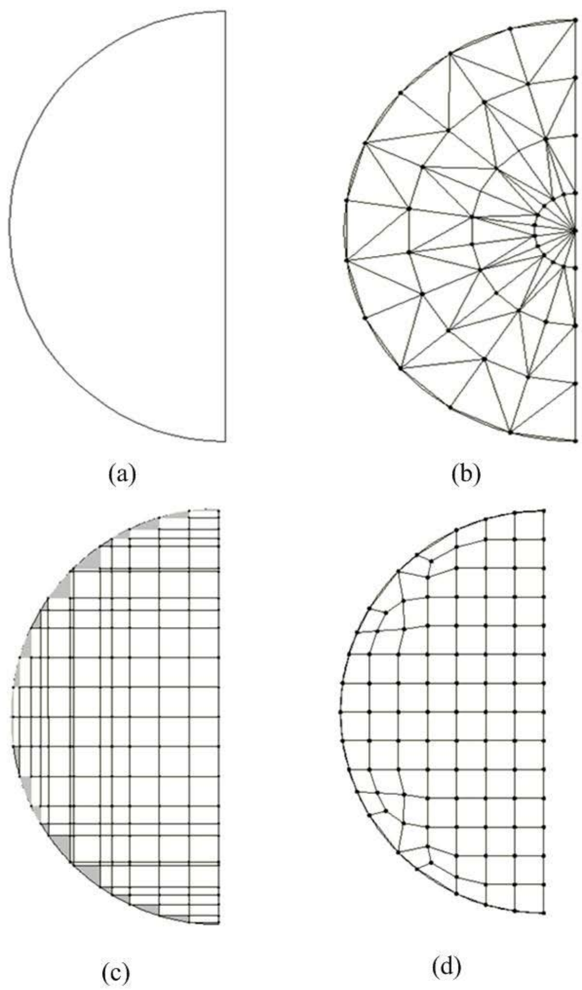
$$\begin{aligned} N_1 &= \frac{1}{4}(1-\xi)(1-\eta)(1-\zeta), \quad N_1 = \frac{1}{4}(1+\xi)(1-\eta)(1-\zeta), \\ N_3 &= \frac{1}{4}(1+\xi)(1+\eta)(1-\zeta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)(1-\zeta), \\ N_5 &= \frac{1}{4}(1-\xi)(1-\eta)(1+\zeta), \quad N_6 = \frac{1}{4}(1+\xi)(1-\eta)(1+\zeta), \\ N_7 &= \frac{1}{4}(1+\xi)(1+\eta)(1+\zeta), \quad N_8 = \frac{1}{4}(1-\xi)(1+\eta)(1+\zeta) \end{aligned} \tag{3.5.17}$$

The shape functions of rectangular parallelepiped elements with higher nodes can be derived in similar manner satisfying all necessary criteria.

Lecture 6: Isoparametric Formulation

3.6.1 Necessity of Isoparametric Formulation

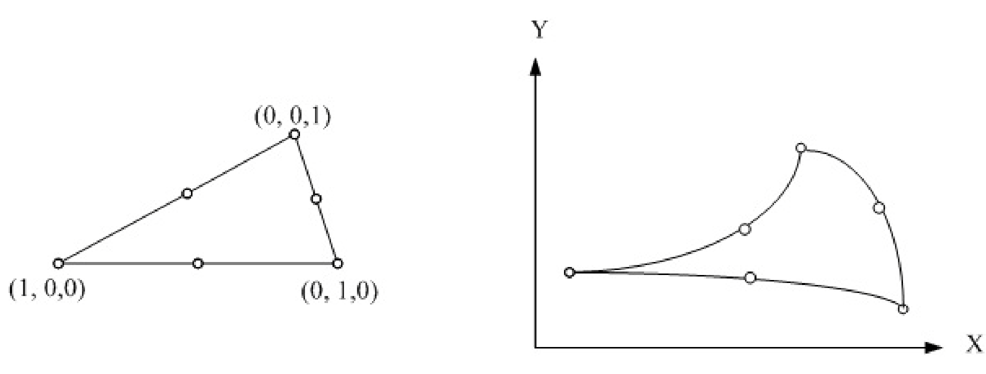
The two or three dimensional elements discussed till now are of regular geometry (e.g. triangular and rectangular element) having straight edge. Hence, for the analysis of any irregular geometry, it is difficult to use such elements directly. For example, the continuum having curve boundary as shown in the Fig. 3.6.1(a) has been discretized into a mesh of finite elements in three ways as shown.

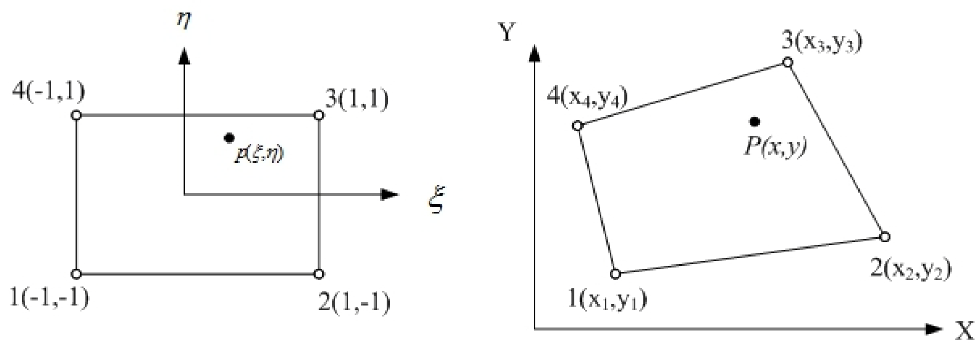


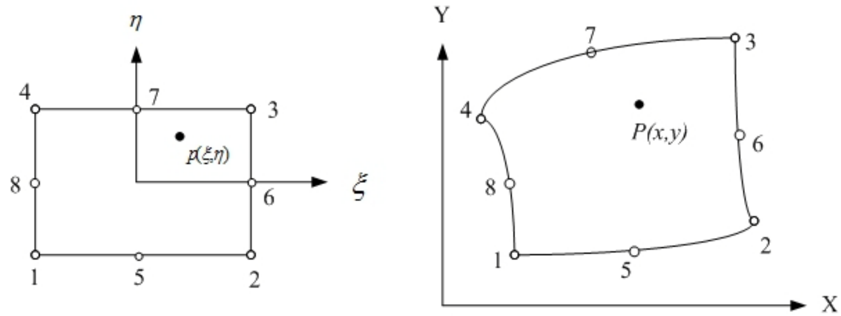
(a) The Continuum to be discritized (b) Discritization using Triangular Elements (c) Discritization using rectangular elements (d) Discritization using a combination of rectangular and quadrilateral elements

Fig 3.6.1 Discretization of a continuum using different elements

Figure 3.6.1(b) presents a possible mesh using triangular elements. Though, triangular elements can suitable approximate the circular boundary of the continuum, but the elements close to the center becomes slender and hence affect the accuracy of finite element solutions. One possible solution to the problem is to reduce the height of each row of elements as we approach to the center. But, unnecessary refining of the continuum generates relatively large number of elements and thus increases computation time. Alternatively, when meshing is done using rectangular elements as shown in Fig 3.6.1(c), the area of continuum excluded from the finite element model is significantly adequate to provide incorrect results. In order to improve the accuracy of the result one can generate mesh using very small elements. But, this will significantly increase the computation time. Another possible way is to use a combination of both rectangular and triangular elements as discussed in section 3.2. But such types of combination may not provide the best solution in terms of accuracy, since different order polynomials are used to represent the field variables for different types of elements. Also the triangular elements may be slender and thus can affect the accuracy. In Fig. 3.6.1(d), the same continuum is discritized with rectangular elements near center and with four-node quadrilateral elements near boundary. This four-node quadrilateral element can be derived from rectangular elements using the concept of mapping. Using the concept of mapping regular triangular, rectangular or solid elements in natural coordinate system (known as parent element) can be transformed into global Cartesian coordinate system having arbitrary shapes (with curved edge or surfaces). Fig. 3.6.2 shows the parent elements in natural coordinate system and the mapped elements in global Cartesian system.




η
4(-1,1)
3(1,1)
$p(\xi,\eta)$
ξ
1(-1,-1)
2(1,-1)
Y
3(x_3,y_3)
4(x_4,y_4)
$P(x,y)$
2(x_2,y_2)
1(x_1,y_1)
X


η
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$p(\xi,\eta)$
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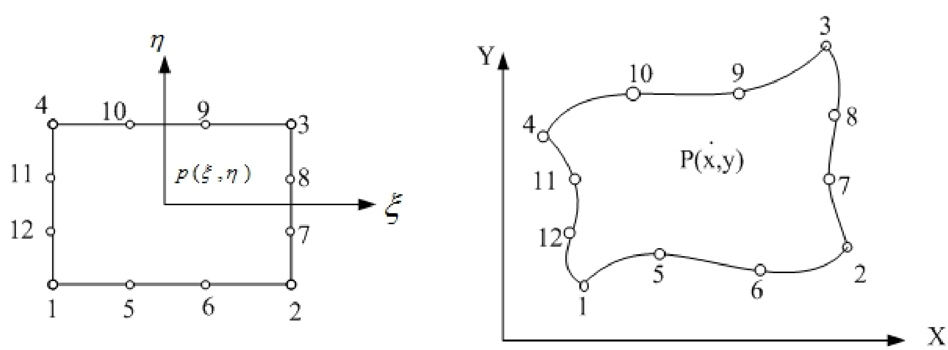
η
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$p(\xi,\eta)$
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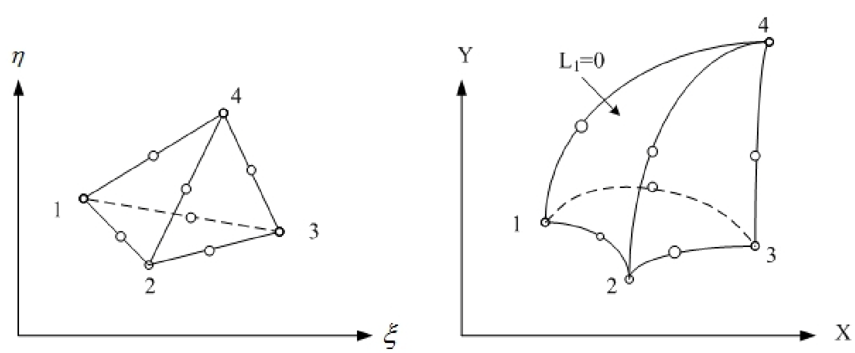
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L_1=0
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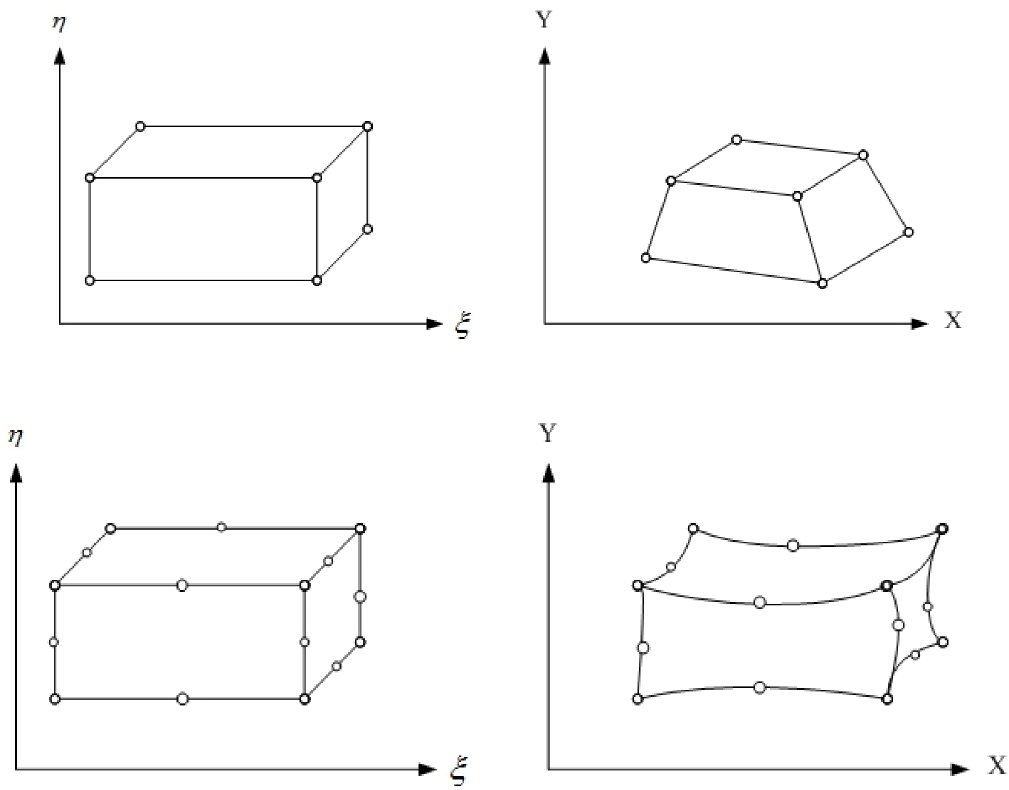

(a) Natural Coordinate System (b) Global Coordinate System

Fig. 3.6.2 Mapping of isoparametric elements in global coordinate system

3.6.2 Coordinate Transformation

The geometry of an element may be expressed in terms of the interpolation functions as follows.

$$x = N_1 x_1 + N_2 x_2 + \ldots + N_n x_n = \sum_{i=1}^{n} N_i x_i$$
$$y = N_1 y_1 + N_2 y_2 + \ldots + N_n y_n = \sum_{i=1}^{n} N_i y_i \qquad (3.6.1)$$
$$z = N_1 z_1 + N_2 z_2 + \ldots + N_n z_n = \sum_{i=1}^{n} N_i z_i$$

Where,

n=No. of Nodes

N_i =Interpolation Functions

x_i, y_i, z_i =Coordinates of Nodal Points of the Element

One can also express the field variable variation in the element as

$$\phi(\xi,\eta,\zeta)=\sum_{i=1}^{n}N_i(\xi,\eta,\zeta)\phi_i \tag{3.6.2}$$

As the same shape functions are used for both the field variable and description of element geometry, the method is known as isoparametric mapping. The element defined by such a method is known as an isoparametric element. This method can be used to transform the natural coordinates of a point to the Cartesian coordinate system and vice versa.

Example 3.6.1

Determine the Cartesian coordinate of the point P (ξ= 0.8, η= 0.9) as shown in Fig. 3.6.3.

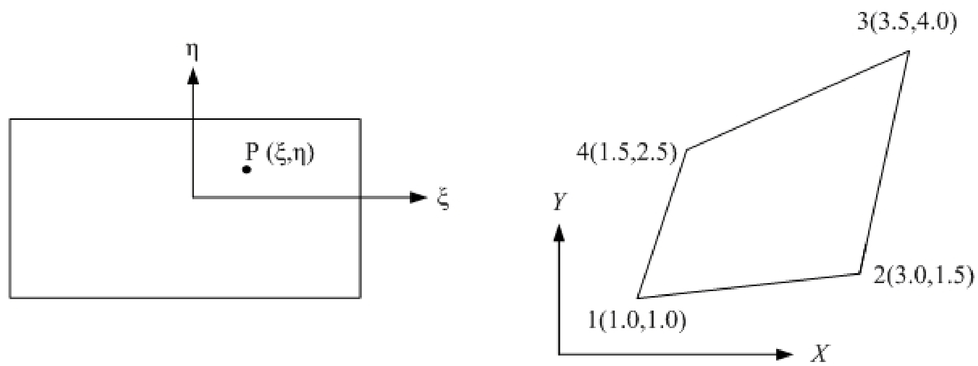

Fig. 3.6.3 Transformation of Coordinates

Solution:

As described above, the relation between two coordinate systems can be represented through their interpolation functions. Therefore, the values of the interpolation function at point P will be

$$N_1=\frac{(1-\xi)(1-n)}{4}=\frac{(1-0.8)(1-0.9)}{4}=0.005$$
$$N_2=\frac{(1+\xi)(1-n)}{4}=\frac{(1+0.8)(1-0.9)}{4}=0.045$$
$$N_3=\frac{(1+\xi)(1+n)}{4}=\frac{(1+0.8)(1+0.9)}{4}=0.855$$
$$N_4=\frac{(1-\xi)(1+n)}{4}=\frac{(1-0.8)(1+0.9)}{4}=0.095$$

Thus the coordinate of point *P* in Cartesian coordinate system can be calculated as

$$x = \sum_{i=1}^{4} N_i x_i = 0.005 \times 1 + 0.045 \times 3 + 0.855 \times 3.5 + 0.095 \times 1.5 = 3.275$$

$$y = \sum_{i=1}^{4} N_i y_i = 0.005 \times 1 + 0.045 \times 1.5 + 0.855 \times 4.0 + 0.095 \times 2.5 = 3.73$$

Thus the coordinate of point P (ξ= 0.8, η= 0.9) in Cartesian coordinate system will be 3.275, 3.73.

Solid isoparametric elements can easily be formulated by the extension of the procedure followed for 2-D elements. Regardless of the number of nodes or possible curvature of edges, the solid element is just like a plane element which is mapped into the space of natural co-ordinates, i.e, $\xi = \pm 1, \eta = \pm 1, \zeta = \pm 1$.

3.6.3 Concept of Jacobian Matrix

A variety of derivatives of the interpolation functions with respect to the global coordinates are necessary to formulate the element stiffness matrices. As the both element geometry and variation of the shape functions are represented in terms of the natural coordinates of the parent element, some additional mathematical obstacle arises. For example, in case of evaluation of the strain vector, the operator matrix is with respect to x and y, but the interpolation function is with ξ *and* η. Therefore, the operator matrix is to be transformed for taking derivative with ξ *and* η. The relationship between two coordinate systems may be computed by using the chain rule of partial differentiation as

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} \quad and \quad \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} \tag{3.6.3}$$

The above equations can be expressed in matrix form as well.

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} \tag{3.6.4}$$

The matrix [*J*] is denoted as Jacobian matrix which is: $\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$. As we know, $x = \sum_{i=1}^{n} N_i x_i$

where, *n* is the number of nodes in an element. Hence, $J_{11} = \dfrac{\partial x}{\partial \xi} = \dfrac{\partial \sum_{i=1}^{n} N_i x_i}{\partial \xi} = \sum_{i=1}^{n} \dfrac{\partial N_i}{\partial \xi} x_i$

Similarly one can calculate the other terms J_{12}, J_{21} and J_{22} of the Jacobian matrix. Hence,

$$[J]=\begin{bmatrix} \sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}y_i \\ \sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}y_i \end{bmatrix} \tag{3.6.5}$$

From eq. (3.6.4), one can write

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \xi} \end{Bmatrix} \tag{3.6.6}$$

Considering $\begin{bmatrix} J_{11}^{*} & J_{12}^{*} \\ J_{21}^{*} & J_{22}^{*} \end{bmatrix}$ are the elements of inverted [J] matrix, we may arise into the following relations.

$$\begin{aligned} \frac{\partial}{\partial x} &= J_{11}^{*}\cdot\frac{\partial}{\partial \xi}+J_{12}^{*}\cdot\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial y} &= J_{21}^{*}\cdot\frac{\partial}{\partial \xi}+J_{22}^{*}\cdot\frac{\partial}{\partial \eta} \end{aligned} \tag{3.6.7}$$

Similarly, for three dimensional case, the following relation exists between the derivative operators in the global and the natural coordinate system.

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} \tag{3.6.8}$$

Where,

$$[J]=\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \tag{3.6.9}$$

[J] is known as the Jacobian Matrix for three dimensional case. Putting eq. (3.6.1) in eq. (3.6.9) and after simplifying one can get

$$[J]=\sum_{i=1}^{n}\begin{bmatrix} \frac{\partial N_i}{\partial \xi}x_i & \frac{\partial N_i}{\partial \xi}y_i & \frac{\partial N_i}{\partial \xi}z_i \\ \frac{\partial N_i}{\partial \eta}x_i & \frac{\partial N_i}{\partial \eta}y_i & \frac{\partial N_i}{\partial \eta}z_i \\ \frac{\partial N_i}{\partial \zeta}x_i & \frac{\partial N_i}{\partial \zeta}y_i & \frac{\partial N_i}{\partial \zeta}z_i \end{bmatrix} \tag{3.6.10}$$

From eq. (3.6.8), one can find the following expression.

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix}=[J]^{-1}\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} \tag{3.6.11}$$

Considering $[J]^{-1}=\begin{bmatrix} J_{11}^{*} & J_{12}^{*} & J_{13}^{*} \\ J_{21}^{*} & J_{22}^{*} & J_{23}^{*} \\ J_{31}^{*} & J_{32}^{*} & J_{33}^{*} \end{bmatrix}$ we can arrived at the following relations.

$$\begin{aligned} \frac{\partial}{\partial x} &= J_{11}^{*}\cdot\frac{\partial}{\partial \xi}+J_{12}^{*}\cdot\frac{\partial}{\partial \eta}+J_{13}^{*}\cdot\frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial y} &= J_{21}^{*}\cdot\frac{\partial}{\partial \xi}+J_{22}^{*}\cdot\frac{\partial}{\partial \eta}+J_{23}^{*}\cdot\frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial z} &= J_{31}^{*}\cdot\frac{\partial}{\partial \xi}+J_{32}^{*}\cdot\frac{\partial}{\partial \eta}+J_{33}^{*}\cdot\frac{\partial}{\partial \zeta} \end{aligned} \tag{3.6.12}$$

Lecture 7: Stiffness Matrix of Isoparametric Elements

3.7.1 Evaluation of Stiffness Matrix of 2-D Isoparametric Elements

For two dimensional plane stress/strain formulation, the strain vector can be represented as

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\end{Bmatrix}=\begin{Bmatrix}J_{11}^{*}\cdot\dfrac{\partial u}{\partial \xi}+J_{12}^{*}\cdot\dfrac{\partial u}{\partial \eta}\\ J_{21}^{*}\cdot\dfrac{\partial v}{\partial \xi}+J_{22}^{*}\cdot\dfrac{\partial v}{\partial \eta}\\ J_{11}^{*}\cdot\dfrac{\partial v}{\partial \xi}+J_{12}^{*}\cdot\dfrac{\partial v}{\partial \eta}+J_{21}^{*}\cdot\dfrac{\partial u}{\partial \xi}+J_{22}^{*}\cdot\dfrac{\partial u}{\partial \eta}\end{Bmatrix} \tag{3.7.1}$$

The above expression can be rewritten in matrix form

$$\{\varepsilon\}=\begin{bmatrix}J_{11}^{*} & J_{12}^{*} & 0 & 0\\ 0 & 0 & J_{21}^{*} & J_{22}^{*}\\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial u}{\partial \xi}\\ \dfrac{\partial u}{\partial \eta}\\ \dfrac{\partial v}{\partial \xi}\\ \dfrac{\partial v}{\partial \eta}\end{Bmatrix} \tag{3.7.2}$$

For an n node element the displacement u can be represented as, $u=\sum_{i=1}^{n}N_i u_i$ and similarly for v & w. Thus,

$$\begin{Bmatrix}\dfrac{\partial u}{\partial \xi}\\ \dfrac{\partial u}{\partial \eta}\\ \dfrac{\partial v}{\partial \xi}\\ \dfrac{\partial v}{\partial \eta}\end{Bmatrix}=\begin{bmatrix}\dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi} & 0 & \cdots & 0\\ \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta} & 0 & \cdots & 0\\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \xi} & \cdots & \dfrac{\partial N_n}{\partial \xi}\\ 0 & \cdots & 0 & \dfrac{\partial N_1}{\partial \eta} & \cdots & \dfrac{\partial N_n}{\partial \eta}\end{bmatrix}\begin{Bmatrix}u_1\\ \vdots\\ u_n\\ v_1\\ \vdots\\ v_n\end{Bmatrix} \tag{3.7.3}$$

As a result, eq. (3.7.2) can be written using eq. (3.7.3) which will be as follows.

$$\{\varepsilon\}=\begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix}\begin{Bmatrix} u_1 \\ : \\ : \\ u_n \\ v_1 \\ : \\ : \\ v_n \end{Bmatrix} \tag{3.7.4}$$

Or,

$$\{\varepsilon\}=[B]\{d\} \tag{3.7.5}$$

Where {d} is the nodal displacement vector and [B] is known as strain displacement relationship matrix and can be obtained as

$$[B]=\begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \tag{3.7.6}$$

It is necessary to transform integrals from Cartesian to the natural coordinates as well for calculation of the elemental stiffness matrix in isoparametric formulation. The differential area relationship can be established from advanced calculus and the elemental area in Cartesian coordinate can be represented in terms of area in natural coordinates as:

$$dA = dx\ dy = |J|\, d\xi\ d\eta \tag{3.7.7}$$

Here $|J|$ is the determinant of the Jacobian matrix. The stiffness matrix for a two dimensional element may be expressed as

$$[k]=\iiint_{\Omega}[B]^T[D][B]d\Omega == t\iint_{A}[B]^T[D][B]dxdy \tag{3.7.8}$$

Here, [B] is the strain-displacement relationship matrix and *t* is the thickness of the element. The above expression in Cartesian coordinate system can be changed to the natural coordinate system as follows to obtain the elemental stiffness matrix

$$[k] = t\int_{-1}^{+1}\int_{-1}^{+1}[B]^T[D][B]|J|d\xi d\eta \tag{3.7.9}$$

Though the isoparametric formulation is mathematically straightforward, the algebraic difficulty is significant.

Example 3.7.1:

Calculate the Jacobian matrix and the strain displacement matrix for four node two dimensional quadrilateral elements corresponding to the gauss point (0.57735, 0.57735) as shown in Fig. 3.6.4.

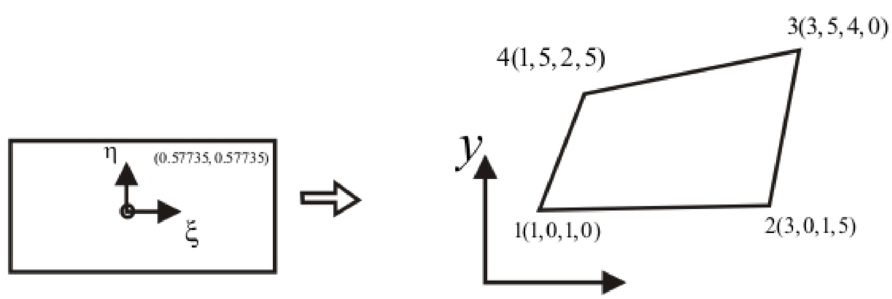

Fig. 3.7.1 Two dimensional quadrilateral element

Solution:

The Jacobian matrix for a four node element is given by,

$$[J] = \begin{bmatrix} \sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \xi}y_i \\ \sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{n}\frac{\partial N_i}{\partial \eta}y_i \end{bmatrix}$$

For the four node element one can find the following relations.

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}, \quad \frac{\partial N_1}{\partial \xi} = -\frac{1-\eta}{4}, \quad \frac{\partial N_1}{\partial \eta} = -\frac{1-\xi}{4}$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4}, \quad \frac{\partial N_2}{\partial \xi} = \frac{1-\eta}{4}, \quad \frac{\partial N_2}{\partial \eta} = -\frac{1+\xi}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4}, \quad \frac{\partial N_3}{\partial \xi} = \frac{1+\eta}{4}, \quad \frac{\partial N_3}{\partial \eta} = \frac{1+\xi}{4}$$

$$N_4 = \frac{(1-\xi)(1+\eta)}{4}, \quad \frac{\partial N_4}{\partial \xi} = -\frac{1+\eta}{4}, \quad \frac{\partial N_4}{\partial \eta} = \frac{1-\xi}{4}$$

Now, for a four node quadrilateral element, the Jacobian matrix will become

$$[J] = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \frac{\partial N_3}{\partial \eta} & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1-\eta}{4} & \frac{1-\eta}{4} & \frac{1+\eta}{4} & -\frac{1+\eta}{4} \\ -\frac{1-\xi}{4} & -\frac{1+\xi}{4} & \frac{1+\xi}{4} & \frac{1-\xi}{4} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

Putting the values of ξ & η as 0.57735 and 0.57735 respectively, one will obtain the following.

$$\frac{\partial N_1}{\partial \xi} = -0.10566 \qquad \frac{\partial N_1}{\partial \eta} = -0.10566$$

$$\frac{\partial N_2}{\partial \xi} = 0.10566 \qquad \frac{\partial N_2}{\partial \eta} = -0.39434$$

$$\frac{\partial N_3}{\partial \xi} = 0.39434 \qquad \frac{\partial N_3}{\partial \eta} = 0.39434$$

$$\frac{\partial N_4}{\partial \xi} = -0.39434 \qquad \frac{\partial N_4}{\partial \eta} = 0.10566$$

Hence, $J_{11} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i = -0.10566 \times 1 + \ 0.10566 \times 3 + 0.39434 \times 3.5 - 0.39434 \times 1.5 = 1.0$

Similarly, J_{12} =0.64632, J_{21} =0.25462 and J_{22} =1.14962.

Hence,

$$J = \begin{bmatrix} 1.00000 & 0.64632 \\ 0.25462 & 1.14962 \end{bmatrix}$$

Thus, the inverse of the Jacobian matrix will become:

$$[J^*] = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} = \begin{bmatrix} 1.1671 & -0.6561 \\ -0.2585 & 1.0152 \end{bmatrix}$$

Hence strain displacement matrix is given by,

$$[B]=\begin{bmatrix} J_{11}^{*} & J_{12}^{*} & 0 & 0 \\ 0 & 0 & J_{21}^{*} & J_{22}^{*} \\ J_{21}^{*} & J_{22}^{*} & J_{11}^{*} & J_{12}^{*} \end{bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} & 0 & \cdots & 0 \\ \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ 0 & \cdots & 0 & \frac{\partial N_1}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix}$$

$$=\begin{bmatrix} 1.1671 & -0.6561 & 0 & 0 \\ 0 & 0 & -0.2585 & 1.0152 \\ -0.2585 & 1.0152 & 1.1671 & -0.6561 \end{bmatrix}\times$$

$$\begin{bmatrix} -0.10566 & 0.10566 & 0.39434 & -0.39434 & 0 & 0 & 0 & 0 \\ -0.10566 & -0.39434 & 0.39434 & 0.10566 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.10566 & 0.10566 & 0.39434 & -0.39434 \\ 0 & 0 & 0 & 0 & -0.10566 & -0.39434 & 0.39434 & 0.10566 \end{bmatrix}$$

$$=\begin{bmatrix} -0.0540 & 0.3820 & 0.2015 & -0.5294 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0800 & -0.4276 & 0.2984 & 0.2092 \\ -0.0800 & -0.4276 & 0.2984 & 0.2092 & -0.0540 & 0.3820 & 0.2015 & -0.5294 \end{bmatrix}$$

3.7.2 Evaluation of Stiffness Matrix of 3-D Isoparametric Elements

Stiffness matrix of 3-D solid isoparametric elements can easily be formulated by the extension of the procedure followed for plane elements. For example, the eight node solid element is analogous to the four node plane element. The strain vector for solid element can be written in the following form.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{Bmatrix} \tag{3.7.10}$$

The above equation can be expressed as

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial x}{\partial z} + \frac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{21}^* & J_{22}^* & J_{23}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* \\ J_{21}^* & J_{22}^* & J_{23}^* & J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* & J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* & 0 & 0 & 0 & J_{11}^* & J_{12}^* & J_{13}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} \\ \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \\ \frac{\partial v}{\partial \zeta} \end{Bmatrix} \tag{3.7.11}$$

For an 8 node brick element u can be represented as, $u = \sum_{i=1}^{8} N_i u_i$ and similarly for v & w.

$$\begin{aligned}
&\frac{\partial u}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} u_i \,,\ \frac{\partial u}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} u_i \ \ \&\ \frac{\partial u}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} u_i \\
&\frac{\partial v}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} v_i \,,\ \frac{\partial v}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} v_i \ \ \&\ \frac{\partial v}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} v_i \\
&\frac{\partial w}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} w_i \,,\ \frac{\partial w}{\partial \eta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} w_i \ \ \&\ \frac{\partial w}{\partial \zeta} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \zeta} w_i
\end{aligned} \tag{3.7.12}$$

Hence eq. (3.7.11) can be rewritten as

$$\{\varepsilon\} = \begin{bmatrix} J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{21}^* & J_{22}^* & J_{23}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* \\ J_{21}^* & J_{22}^* & J_{23}^* & J_{11}^* & J_{12}^* & J_{13}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{31}^* & J_{32}^* & J_{33}^* & J_{21}^* & J_{22}^* & J_{23}^* \\ J_{31}^* & J_{32}^* & J_{33}^* & 0 & 0 & 0 & J_{11}^* & J_{12}^* & J_{13}^* \end{bmatrix} \times \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial \eta} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \xi} & 0 \\ 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} & 0 & \frac{\partial N_i}{\partial \xi} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \tag{3.7.13}$$

Thu, the strain-displacement relationship matrix [B] for 8 node brick element is

$$[B]=\begin{bmatrix} J_{11}^{*} & J_{12}^{*} & J_{13}^{*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{21}^{*} & J_{22}^{*} & J_{23}^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_{31}^{*} & J_{32}^{*} & J_{33}^{*} \\ J_{21}^{*} & J_{22}^{*} & J_{23}^{*} & J_{11}^{*} & J_{12}^{*} & J_{13}^{*} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{31}^{*} & J_{32}^{*} & J_{33}^{*} & J_{21}^{*} & J_{22}^{*} & J_{23}^{*} \\ J_{31}^{*} & J_{32}^{*} & J_{33}^{*} & 0 & 0 & 0 & J_{11}^{*} & J_{12}^{*} & J_{13}^{*} \end{bmatrix} \times \sum_{i=1}^{8} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial \eta} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial \zeta} \\ \frac{\partial N_i}{\partial \eta} & \frac{\partial N_i}{\partial \xi} & 0 \\ 0 & \frac{\partial N_i}{\partial \zeta} & \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} & 0 & \frac{\partial N_i}{\partial \xi} \end{bmatrix} \qquad (3.7.14)$$

The stiffness matrix may be found by using the following expression in natural coordinate system.

$$[\mathrm{k}]=\iiint_{\Omega}[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}][\mathrm{B}]\,\mathrm{d}\Omega=\iiint_{\mathrm{V}}[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}][\mathrm{B}]\,\mathrm{dxdydz}=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}][\mathrm{B}]\,\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\left|\mathrm{J}\right| \qquad (3.7.15)$$

Lecture 8: Numerical Integration: One Dimensional

The integrations, we generally encounter in finite element methods, are quite complicated and it is not possible to find a closed form solutions to those problems. Exact and explicit evaluation of the integral associated to the element matrices and the loading vector is not always possible because of the algebraic complexity of the coefficient of the different equation (i.e., the stiffness influence coefficients, elasticity matrix, loading functions etc.). In the finite element analysis, we face the problem of evaluating the following types of integrations in one, two and three dimensional cases respectively. These are necessary to compute element stiffness and element load vector.

$$\int \phi(\xi)d\xi; \quad \int \phi(\xi,\eta)d\xi d\eta; \quad \int \phi(\xi,\eta,\zeta)d\xi d\eta d\zeta; \tag{3.8.1}$$

Approximate solutions to such problems are possible using certain numerical techniques. Several numerical techniques are available, in mathematics for solving definite integration problems, including, mid-point rule, trapezoidal-rule, Simpson's 1/3rd rule, Simpson's 3/8th rule and Gauss Quadrature formula. Among these, Gauss Quadrature technique is most useful one for solving problems in finite element method and therefore will be discussed in details here.

3.8.1 Gauss Quadrature for One-Dimensional Integrals

The concept of Gauss Quadrature is first illustrated in one dimension in the context of an integral in the form of $I=\int_{-1}^{+1}\phi(\xi)d\xi$ from $\int_{x_1}^{x_2} f(x)dx$. To transform from an arbitrary interval of $x_1 \le x \le x_2$ to an interval of $-1 \le \xi \le 1$, we need to change the integration function from f(x) to $\phi(\xi)$ accordingly. Thus, for a linear variation in one dimension, one can write the following relations.

$$x=\frac{1-\xi}{2}x_1+\frac{1+\xi}{2}x_2=N_1x_1+N_2x_2$$

$$\text{so for } \xi=-1, x=\frac{1-(-1)}{2}x_1+\frac{1-1}{2}x_2=x_1$$

$$\xi=+1, \quad x=x_2$$

$$\therefore I=\int_{x_1}^{x_2} f(x)dx=\int_{-1}^{+1}\phi(\xi)d\xi$$

Numerical integration based on Gauss Quadrature assumes that the function $\phi(\xi)$ will be evaluated over an interval $-1 \le \xi \le 1$. Considering an one-dimensional integral, Gauss Quadrature represents the integral $\phi(\xi)$ in the form of

$$I=\int_{-1}^{+1}\phi(\xi)d\xi \approx \sum_{i=1}^{n} w_i\phi(\xi_i) \approx w_1\phi(\xi_1)+w_2\phi(\xi_2)+\ldots..+w_\eta\phi(\xi_\eta) \tag{3.8.2}$$

Where, the ξ_1, ξ_2, ξ_3, ..., ξ_n represents *n* numbers of points known as Gauss Points and the corresponding coefficients w_1, w_2, w_3, …, w_n are known as weights. The location and weight coefficients of Gauss points are calculated by Legendre polynomials. Hence this method is also sometimes referred as Gauss-Legendre Quadrature method. The summation of these values at *n* sampling points gives the exact solution of a polynomial integrand of an order up to 2*n*-1. For example, considering sampling at two Gauss points we can get exact solution for a polynomial of an order (2×2-1) or 3. The use of more number of Gauss points has no effect on accuracy of results but takes more computation time.

3.8.2 One- Point Formula

Considering *n* = 1, eq.(3.8.2) can be written as

$$\int_{-1}^{1} \phi(\xi)d\xi \approx w_1\phi(\xi_1) \tag{3.8.3}$$

Since there are two parameters w_1 and ξ_1, we need a first order polynomial for $\phi(\xi)$ to evaluate the eq.(3.8.3) exactly. For example, considering, $\phi(\xi) = a_0 + a_1\xi$,

$$\begin{aligned}
&\text{Error} = \int_{-1}^{1} (a_0 + a_1\xi)d\xi - w_1\phi(\xi_1) = 0 \\
&\Rightarrow 2a_0 - w_1(a_0 + a_1\xi_1) = 0 \\
&\Rightarrow a_0(2 - w_1) - w_1a_1\xi_1 = 0
\end{aligned} \tag{3.8.4}$$

Thus, the error will be zero if $w_1 = 2$ and $\xi_1 = 0$. Putting these in eq.(3.8.3), for any general ϕ, we have

$$I = \int_{-1}^{1} \phi(\xi)d\xi = 2\phi(0) \tag{3.8.5}$$

This is exactly similar to the well known midpoint rule.

3.8.3 Two-Point Formula

If we consider n = 2, then the eq.(3.8.2) can be written as

$$\int_{-1}^{1} \phi(\xi)d\xi \approx w_1\phi(\xi_1) + w_2\phi(\xi_2) \tag{3.8.6}$$

This means we have four parameters to evaluate. Hence we need a 3rd order polynomial for $\phi(\xi)$ to exactly evaluate eq.(3.8.6).

Considering, $\phi(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$

$$\text{Error} = \left[\int_{-1}^{1} \left(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3\right)d\xi\right] - \left[w_1\phi(\xi_1) + w_2\phi(\xi_2)\right]$$

$$\Rightarrow 2a_0 + \frac{2}{3}a_2 - w_1\left(a_0 + a_1\xi_1 + a_2\xi_1^{\,2} + a_3\xi_1^{\,3}\right) - w_2\left(a_0 + a_1\xi_2 + a_2\xi_2^{\,2} + a_3\xi_2^{\,3}\right) = 0$$

$$\Rightarrow \left(2 - w_1 - w_2\right)a_0 - \left(w_1\xi_1 + w_2\xi_2\right)a_1 + \left(\frac{2}{3} - w_1\xi_1^{\,2} - w_2\xi_2^{\,2}\right)a_2 - \left(w_1\xi_1^{\,3} + w_2\xi_2^{\,3}\right)a_3 = 0$$

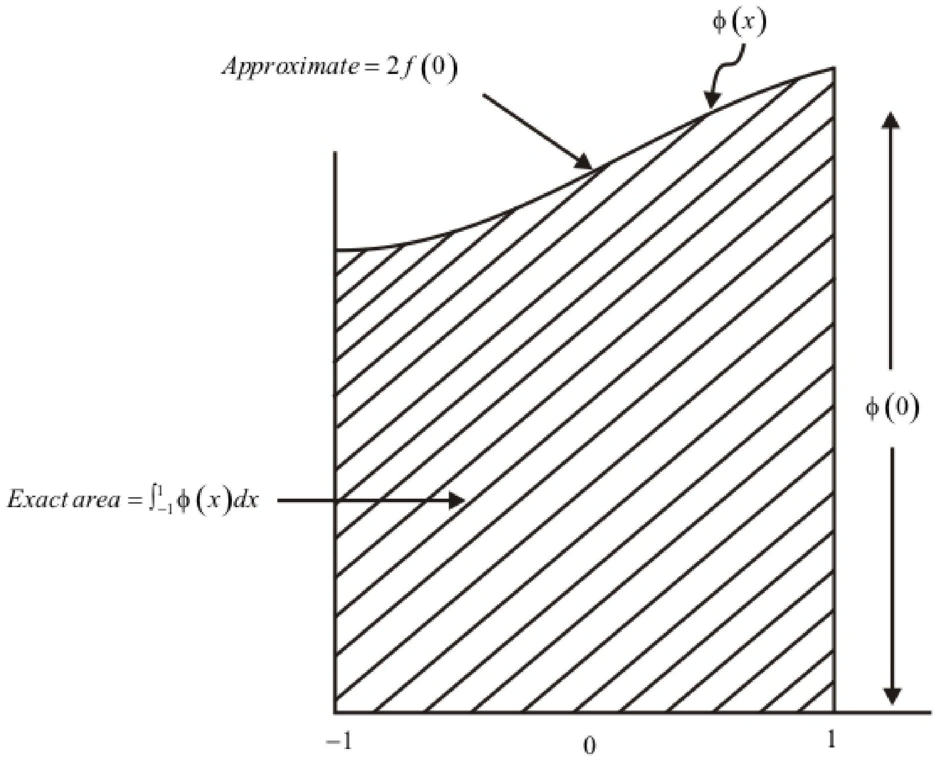

Fig 3.8.1 One-point Gauss Quadrature

Requiring zero error yields

$$\begin{aligned} & w_1 + w_2 = 2 \\ & w_1\xi_1 + w_2\xi_2 = 0 \\ & w_1\xi_1^2 + w_2\xi_2^2 = \frac{2}{3} \\ & w_1\xi_1^3 + w_2\xi_2^3 = 0 \end{aligned} \tag{3.8.7}$$

These nonlinear equations have the unique solution as

$$w_1 = w_2 = 1 \qquad \xi_1 = -\xi_2 = -1/\sqrt{3} = -0.5773502691 \tag{3.8.8}$$

From this solution, we can conclude that *n*-point Gaussian Quadrature will provide an exact solution if $\phi(\xi)$ is a polynomial of order (2*n*-1) or less. Table 3.8.1 gives the values of w_1 and ξ_1 for Gauss Quadrature formulas of orders n = 1 through n = 6. From the table it can be observed that the gauss points are symmetrically placed with respect to origin and those symmetrical points have the same weights. For accuracy in the calculation maximum number digits for gauss point and gauss weights should be taken. The Location and weights given in the Table 3.8.1 must be used when the limits of integration ranges from -1 to 1. Integration limits other than [-1, 1], should be appropriately changed to [-1, 1] before applying these values.

Table 3.8.1 Gauss points and corresponding weights

Number of points, n	**Gauss Point Location,** ξ_1	**Weight,** w_1
1	0.0	2.0
2	±0.5773502692 ($=\pm 1/\sqrt{3}$)	1.0
3	0.0	0.8888888889 (=8/9)
	±0.7745966692 ($=\pm\sqrt{6}$)	0.5555555556 (=5/9)
4	±0.3399810436	0.6521451549
	±0.861363116	0.3478548451
5	0.0	0.5688888889
	±0.5384693101	0.4786286705
	±0.9061798459	0.2369268851
6	±0.2386191861	0.4679139346
	±0.6612093865	0.3607615730
	±0.9324695142	0.1713244924

Example 1:

Evaluate $I = \int_0^1 \left(e^x - \frac{2x}{x^2 - 2} \right) dx$ using one, two and three point gauss Quadrature.

Solution:

Before applying the Gauss Quadrature formula, the existing limits of integration should be changed from [0, 1] to [-1, +1]. Assuming, $\xi = a + bx$, the upper and lower limit can be changed. i.e., at x = 0, ξ = -1 and at x = 1, ξ = +1. Thus, putting these conditions and solving for a & b, we get a = -1 and b = 2. The relation between two coordinate systems will become $\xi = 2x - 1$ and $d\xi = 2dx$.

Therefore the initial equation can be written as

$$I=\int_{-1}^{1}\left(e^{\left(\frac{\xi+1}{2}\right)}-\frac{2\left(\frac{\xi+1}{2}\right)}{\left(\frac{\xi+1}{2}\right)^{2}-2}\right)dx$$

$$\text{Or, } I=\frac{1}{2}\int_{-1}^{1}\left(e^{\frac{(\xi+1)}{2}}-\frac{4(\xi+1)}{(\xi+1)^{2}-8}\right)d\xi$$

<u>Using one point gauss Quadrature:</u>

$w_1 = 2,\ \xi_1 = 0$ and

$I \approx 2\phi(0)$

$$\text{Or } I\approx 2\left(\frac{1}{2}\left(e^{0.5}+\frac{4}{7}\right)\right)=2.22015$$

<u>Using two point gauss Quadrature:</u>

$w_1 = w_2 = 1$

$\xi_1 = -0.5773502692$

$\xi_2 = 0.5773502692$

Putting these values and calculating, $I = 2.39831$

<u>Using three point gauss Quadrature:</u>

$w_1 = 0.555555556$

$\xi_1 = -0.774596669$

$w_2 = 0.888888889$

$\xi_2 = 0.000000000$

$w_3 = 0.555555556$

$\xi_3 = 0.774596669$

and $I = 2.41024$

This may be compared with the exact solution as $I_{exact} = 2.41193$

Lecture 9: Numerical Integration: Two and Three Dimensional

Numerical integrations using Gauss Quadrature method can be extended to two and three dimensional cases in a similar fashion. Such integrations are necessary to perform for the analysis of plane stress/strain problem, plate and shell structures and for the three dimensional stress analysis.

3.9.1 Gauss Quadrature for Two-Dimensional Integrals

For two dimensional integration problems the above mentioned method can be extended by first evaluating the inner integral, keeping η constant, and then evaluating the outer integral. Thus,

$$\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1}\phi(\xi,\eta)\,d\xi d\eta \approx \int_{-1}^{1}\left[\sum_{i=1}^{n} w_i \phi(\xi_i,\eta)\right] d\eta \approx \sum_{i=1}^{n} w_j \left[\sum_{i=1}^{n} w_i \phi(\xi_i,\eta_j)\right]$$

Or,

$$\mathrm{I} \approx \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \phi(\xi_i,\eta_j) \qquad (3.9.1)$$

In a matrix form we can rewrite the above expression as

$$I \approx \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix} \begin{bmatrix} \phi(\xi_1,\eta_1) & \phi(\xi_1,\eta_2) & & \phi(\xi_1,\eta_n) \\ \phi(\xi_2,\eta_1) & \phi(\xi_2,\eta_2) & & \phi(\xi_2,\eta_n) \\ & & \ddots & \\ \phi(\xi_n,\eta_1) & \phi(\xi_n,\eta_2) & & \phi(\xi_n,\eta_n) \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{Bmatrix} \qquad (3.9.2)$$

Example 1:

Evaluate the integral: $\mathrm{I} = \int_{y=c=-4}^{y=d=4}\int_{x=a=2}^{x=b=3} (1-x)^2 (2-y)^2 dxdy$

Solution:

Before applying the Gauss Quadrature formula, the above integral should be converted in terms of ξ and η and the existing limits of y should be changed from [-4,4] to [-1, 1] and that of x is from [2,3] to [-1,1].

$$x = \frac{(b-a)}{2}\xi + \frac{(b+a)}{2} = \frac{(\xi+5)}{2}; \quad dx = \frac{d\xi}{2}$$

$$y = \frac{(d-c)}{2}\eta + \frac{(d+c)}{2} = 4\eta; \quad dy = 4d\eta$$

$$I = 2\int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1}\left(\frac{3+\xi}{2}\right)^2 (2-4\eta)^2\, d\xi d\eta = \int_{\eta=-1}^{\eta=+1}\int_{\xi=-1}^{\xi=+1}\phi(\xi,\eta)d\xi d\eta$$

where $\quad \phi(\xi,\eta) = 2\left(\frac{3+\xi}{2}\right)^2 (2-4\eta)^2 = 2(3+\xi)^2(1-2\eta)^2$

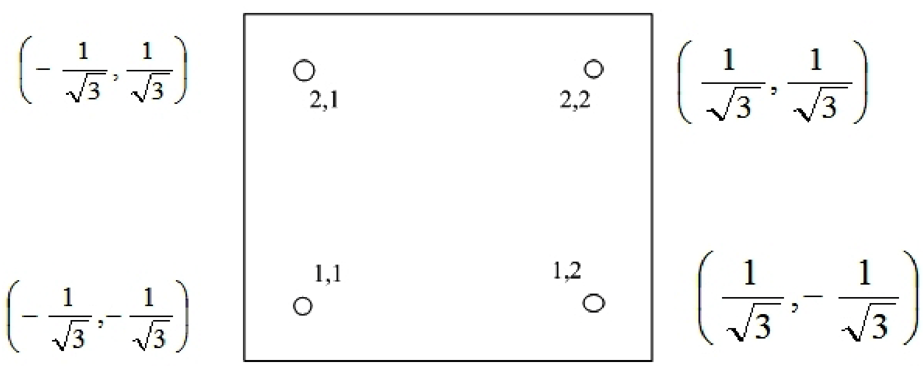

Fig. 3.9.1 Gauss points for two-dimensional integral

$$\xi_1 = -\frac{1}{\sqrt{3}}; \eta_1 = -\frac{1}{\sqrt{3}}; \xi_2 = \frac{1}{\sqrt{3}}; \eta_2 = \frac{1}{\sqrt{3}}$$

$$\phi(\xi_1, \eta_1) = 2\left(3 - \frac{1}{\sqrt{3}}\right)^2 \left(1 + \frac{2}{\sqrt{3}}\right)^2 = 54.49857$$

$$\phi(\xi_2, \eta_1) = \left(\frac{3 + \frac{1}{\sqrt{3}}}{2}\right)^2 \left(2 + \frac{4}{\sqrt{3}}\right)^2 = 118.83018$$

$$\phi(\xi_2, \eta_2) = \left(\frac{3 + \frac{1}{\sqrt{3}}}{2}\right)^2 \left(2 - \frac{4}{\sqrt{3}}\right)^2 = 0.61254$$

$$\phi(\xi_1, \eta_2) = \left(\frac{3 - \frac{1}{\sqrt{3}}}{2}\right)^2 \left(2 - \frac{4}{\sqrt{3}}\right)^2 = 0.28093$$

$$I = \{w_1 \quad w_2\} \begin{bmatrix} \phi(\xi_1, \eta_1) & \phi(\xi_1, \eta_2) \\ \phi(\xi_2, \eta_1) & \phi(\xi_2, \eta_2) \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}$$

$$= \{1 \quad 1\} \begin{bmatrix} 54.49857 & 0.28093 \\ 118.83018 & 0.61254 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

= 174.22222 agrees with the exact value 174.22222

3.9.2 Gauss Quadrature for Three-Dimensional Integrals

In a similar way one can extend the gauss Quadrature for three dimensional problems also and the integral can be expressed by.

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \phi(\xi, \eta, \zeta) d\xi d\eta d\zeta \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} w_i w_j w_k \phi(\xi_i, \eta_j, \zeta_k) \tag{3.9.3}$$

The above equation will produce exact value for a polynomial integrand if the sampling points are selected as described earlier sections.

3.9.3 Numerical Integration of Element Stiffness Matrix

As discussed earlier notes, the element stiffness matrix for three dimensional analyses in natural coordinate system can be written as

$$[k]=\iiint_{\Omega}[B]^T[D][B]d\Omega=\iiint_{V}[B]^T[D][B]dxdydz=\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1}[B]^T[D][B]d\xi d\eta d\zeta|J| \quad (3.9.4)$$

Here, [*B*] and [*D*] are the strain displacement relationship matrix and constitutive matrix respectively and integration is performed over the domain. As the element stiffness matrix will be calculated in natural coordinate system, the strain displacement matrix [B] and Jacobian matrix [J] are functions of ξ,η and ζ. In case of two dimensional isoparametric element, the stiffness matrix will be simplified to

$$[k]=t\int_{-1}^{+1}\int_{-1}^{+1}[B]^T[D][B]d\xi d\eta|J| \quad (3.9.5)$$

This is actually an 8×8 matrix containing the integrals of each element. We do not need to integrate elements below the main diagonal of the stiffness matrix as it is symmetric. Considering, $\phi(\xi,\eta)=t[B]^T[D][B]|J|$, the element stiffness matrix will become after numerical integration as

$$[k]=\sum_{i=1}^{n}\sum_{j=1}^{n}w_i w_j\phi(\xi_i,\eta_j) \quad (3.9.6)$$

Using a 2×2 rule, we get

$$[k]=w_1^2\phi(\xi_1,\eta_1)+w_1w_2\phi(\xi_1,\eta_2)+w_2w_1\phi(\xi_2,\eta_1)+w_2^2\phi(\eta_2,\eta_2) \quad (3.9.7)$$

Where $w_1=w_2=1.0, \xi_1=\eta_1=-0.57735....,$ and $\xi_2=\eta_2=+0.57735....$ Here, w_n is the weight factor at integration point n. A suitable computer program can be written to calculate the element stiffness matrix through the numerical integration. The process of obtaining stiffness matrix using Gauss Quadrature integration will be demonstrated through a numerical example in module 5.

3.10.4 Gauss Quadrature for Triangular Elements

The procedure described for the rectangular element will not be applicable directly. The Gauss Quadrature is extended to include triangular elements in terms of triangular area coordinates.

$$I=\iint_{A}\phi(L_1,L_2,L_3)dA\approx\sum_{i=1}^{n}w_i\phi\left(\left(L_1^i,L_2^i,L_2^i\right)\right) \quad (3.9.8)$$

Where, *L* terms are the triangular area coordinates and the w_i terms are the weights associated with those coordinates. The locations of integration points are shown in Fig. 3.9.2.

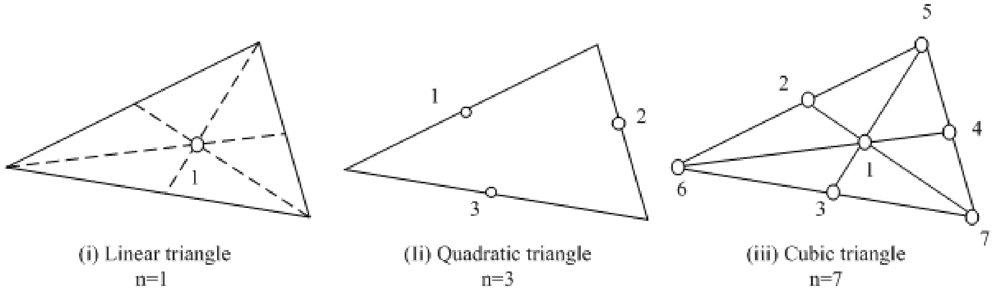

Fig. 3.9.2 Gauss points for triangles

The sampling points and their associated weights are described below:
For sampling point =1 (Linear triangle)

$$w_1 = 1 \qquad L_1^1 = L_2^1 = L_3^1 = \frac{1}{3} \tag{3.9.9}$$

For sampling points =3 (Quadratic triangle)

$$\begin{aligned} w_1 &= \frac{1}{3} & L_1^1 &= L_2^1 = \frac{1}{2},\ L_3^1 = 0 \\ w_2 &= \frac{1}{3} & L_1^2 &= 0,\ L_2^2 = L_3^2 = \frac{1}{2} \\ w_3 &= \frac{1}{3} & L_1^3 &= \frac{1}{2},\ L_2^3 = 0,\ L_3^3 = \frac{1}{2} \end{aligned} \tag{3.9.10}$$

For sampling point = 7 (Cubic triangle)

$$\begin{aligned} w_1 &= \frac{27}{60} & L_1^1 &= L_2^1 = L_3^1 = \frac{1}{3} \\ w_2 &= \frac{8}{60} & L_1^2 &= L_2^2 = \frac{1}{2}, L_3^2 = 0 \\ w_3 &= \frac{8}{60} & L_1^3 &= 0, L_2^3 = L_3^2 = \frac{1}{2} \\ w_4 &= \frac{8}{60} & L_1^4 &= L_3^4 = \frac{1}{2}, L_2^4 = 0 \\ w_5 &= \frac{3}{60} & L_1^5 &= 1, L_2^5 = L_3^5 = 0 \\ w_6 &= \frac{3}{60} & L_1^6 &= L_3^6 = 0, L_2^6 = 1 \\ w_7 &= \frac{3}{60} & L_1^7 &= L_2^7 = 0, L_3^7 = 1 \end{aligned} \tag{3.9.11}$$

3.10.5 Gauss Quadrature for Tetrahedron

The Gauss Quadrature for triangles can be effectively extended to include tetrahedron elements in terms of tetrahedron volume coordinates.

$$I = \iint_A \phi\left(L_1, L_2, L_3, L_4\right) dA \approx \sum_{i=1}^{n} w_i \phi\left(\left(L_1^i, L_2^i, L_3^i, L_4^i\right)\right) \tag{3.9.12}$$

Where, L terms are the volume coordinates and the w_i terms are the weights associated with those coordinates. The locations of Gauss points are shown in Fig. 3.9.3.

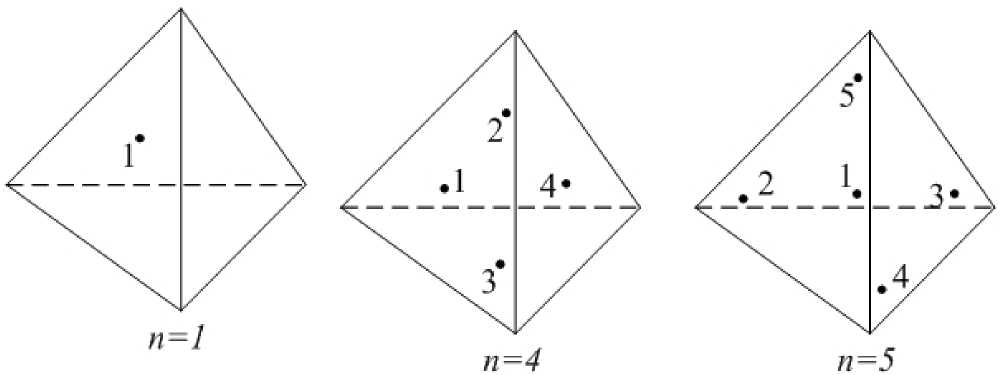

Fig. 3.9.3 Gauss points for tetrahedrons

The sampling points and their associated weights are described below:

For sampling point = 1 (Linear tetrahedron)

$$w_1 = 1 \qquad L_1^1 = L_2^1 = L_3^1 = L_4^1 = \frac{1}{4} \tag{3.9.13}$$

For sampling points = 4 (Quadratic tetrahedron)

$$\begin{aligned}
w_1 &= \frac{1}{4} & L_1^1 &= 0.5854102, L_2^1 = L_3^1 = L_4^1 = 0.1381966 \\
w_2 &= \frac{1}{4} & L_2^2 &= 0.5854102, L_1^2 = L_3^2 = L_4^2 = 0.1381966 \\
w_3 &= \frac{1}{4} & L_3^3 &= 0.5854102, L_1^3 = L_2^3 = L_4^3 = 0.1381966 \\
w_4 &= \frac{1}{4} & L_4^4 &= 0.5854102, L_1^4 = L_2^4 = L_3^4 = 0.1381966
\end{aligned} \tag{3.9.14}$$

For sampling points = 5 (Cubic tetrahedron)

$$\begin{aligned}
&w_1 = -\frac{4}{5} && L_1^1 = L_2^1 = L_3^1 = L_4^1 = \frac{1}{4} \\
&w_2 = \frac{9}{20} && L_1^2 = \frac{1}{3}, L_2^2 = L_3^2 = L_4^2 = \frac{1}{6} \\
&w_3 = \frac{9}{20} && L_2^3 = \frac{1}{3}, L_1^3 = L_3^3 = L_4^3 = \frac{1}{6} \\
&w_4 = \frac{9}{20} && L_3^4 = \frac{1}{3}, L_1^4 = L_2^4 = L_4^4 = \frac{1}{6} \\
&w_5 = \frac{9}{20} && L_4^5 = \frac{1}{3}, L_1^5 = L_2^5 = L_3^4 = \frac{1}{6}
\end{aligned} \tag{3.9.15}$$

Worked out Examples

Example 3.1 Calculation of displacement using area coordinates

The coordinates of a three node triangular element is given below. Calculate the displacement at point P if the displacements of nodes 1, 2 and 3 are 11 mm, 14mm and 17mm respectively using the concepts of area coordinates.

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 4 \\ 1 & 3 & 6 \end{vmatrix} = \frac{1}{2}[(30\text{-}12) - (12\text{-}9) + (8\text{-}15)] = \frac{8}{2} = 4$$

$$A_1 = \frac{1}{2}\begin{bmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 5 & 4 \\ 1 & 3 & 6 \end{vmatrix} = \frac{1}{2}[(30\text{-}12) - (18\text{-}12) + (12\text{-}20)] = \frac{4}{2} = 2$$

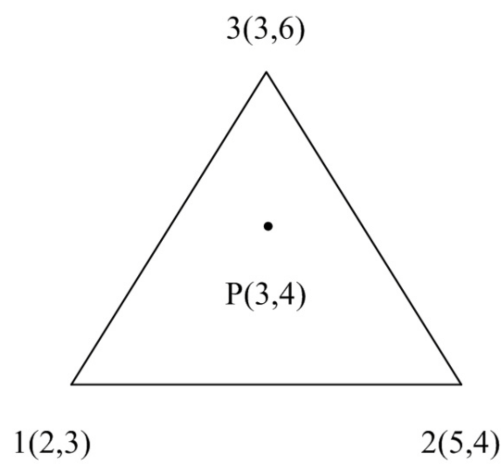

Fig. Ex.3.1 Nodal coordinates of a triangular element

$$A_2 = \frac{1}{2}\begin{bmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{bmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 3 & 6 \\ 1 & 2 & 3 \end{vmatrix} = \frac{1}{2}[(9\text{-}12) - (9\text{-}8) + (18\text{-}12)] = \frac{2}{2} = 1$$

$A_3 = \frac{1}{2} \begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 5 & 4 \end{vmatrix} = \frac{1}{2}$ [(8-15) - (12-20) + (9-8)] = $\frac{2}{2}$ = 1

$N_1 = \frac{A_1}{A} = \frac{2}{4}$ = 0.5

$N_2 = \frac{A_2}{A} = \frac{1}{4}$ = 0.25

$N_3 = \frac{A_3}{A} = \frac{1}{4}$ = 0.25

u = $N_1 u_1 + N_2 u_2 + N_3 u_3$
= 0.5 x 11 + 0.25 x 14 + 0.25 x 17 = 13.25 mm

Example 3.2 Derivation of shape function of four node triangular element

Derive the shape function of a four node triangular element.

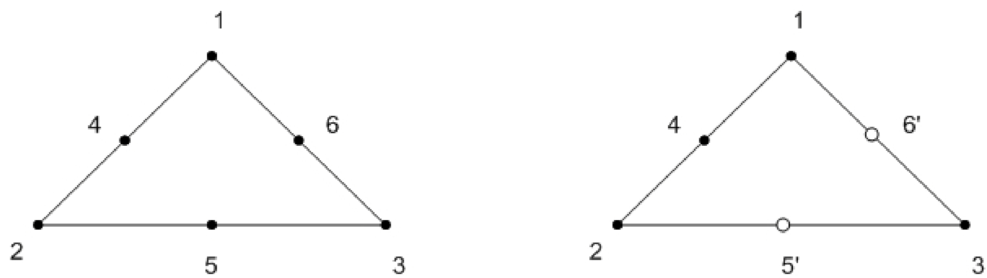

Fig. Ex.3.2 Degrading for four node element

The procedure for four node triangular element is the same as five node triangular element to derive its interpolation functions. Here, node 5 and 6 are omitted and therefore displacements in these nodes can be expressed in terms of the displacements at their corner nodes. Hence,

$$u'_5 = \frac{u_2 + u_3}{2} \quad \text{and} \quad u'_6 = \frac{u_1 + u_3}{2} \tag{3.11.1}$$

Substituting the values of u'_5 and u'_6 in eq.(3.3.8), the following relations can be obtained.

$$\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 \frac{(u_2 + u_3)}{2} + N_6 \frac{(u_3 + u_1)}{2} \\ &= \left(N_1 + \frac{N_6}{2}\right) u_1 + \left(N_2 + \frac{N_5}{2}\right) u_2 + \left(N_3 + \frac{N_5 + N_6}{2}\right) u_3 + N_4 u_4 \end{aligned} \tag{3.11.2}$$

Now, the displacement at any point inside the four node element can be expressed by its nodal displacement with help of shape function.

$$u = N'_1 u_1 + N'_2 u_2 + N'_3 u_3 + N'_4 u_4 \tag{3.11.3}$$

Comparing eq. (3.11.2) and eq. (3.11.3), one can find the following relations.

$$\begin{aligned} N'_1 &= N_1 + \frac{N_6}{2} = L_1 (2L_1 - 1) + \frac{4L_3 L_1}{2} = L_1 (1 - 2L_2) \\ N'_2 &= N_2 + \frac{N_5}{2} = L_2 (2L_2 - 1) + \frac{4L_2 L_3}{2} = L_2 (1 - 2L_1) \\ N'_3 &= N_3 + \frac{N_5 + N_6}{2} = L_3 (2L_3 - 1) + \frac{4L_2 L_3 + 4L_3 L_1}{2} = L_3 \\ N'_4 &= N_4 = 4L_1 L_2 \end{aligned} \tag{3.11.4}$$

Thus, the shape functions for the four node triangular element are

$$\begin{aligned}
N_1' &= L_1\left(1-2L_2\right)\\
N_2' &= L_2\left(1-2L_1\right)\\
N_3' &= L_3\\
N_4' &= 4L_1L_2
\end{aligned} \tag{3.11.5}$$

Example 3.3 Numerical integration for two dimensional problems

Evaluate the integral: $I = \int_{-2}^{3}\left(x^2+11x-32\right)dx$ using one, two and three point gauss Quadrature. Also, find the exact solution for comparison of accuracy.

Solution:

The existing limits of integration should be changed from [-2, +3] to [-1, +1]. Assuming, $\xi = a + bx$, the upper and lower limit can be changed. i.e., at $x_1 = -2, \xi_1 = -1$ and at $x_2 = 3, \xi_2 = +1$. Thus, putting these limits and solving for *a* & *b*, we get a = -0.2 and b = 0.4. The relation between two coordinate systems will become:

$$\xi = -0.2 + 0.4\text{x} \text{ or } \text{x} = \frac{5\xi+1}{2} \text{ and } \text{dx} = 2.5\text{d}\xi$$

Thus, the initial equation can be written as

$$I = \int_{-2}^{3}\left(x^2+11x-32\right)dx = 2.5\int_{-1}^{+1}\left[\left(\frac{5\xi+1}{2}\right)^2 + 11\left(\frac{5\xi+1}{2}\right) - 32\right]d\xi$$

(i) <u>Exact Solution:</u>

$$I = \int_{-2}^{3}\left(x^2+11x-32\right)dx$$

$$= \left[\frac{x^3}{3} + \frac{11x^2}{2} - 32x\right]_{-2}^{3}$$

$$= \left[9 + \frac{99}{2} - 96\right] - \left[-\frac{8}{3} + 22 + 64\right]$$

$$= -37.5 - 83.33333 = -120.83333$$

Thus, I_{exact} = -120.83333

(ii) <u>One Point Formula:</u>

$$I = \int_{-1}^{+1} \phi(\xi)\, d\xi = w_1 \phi(\xi_1)$$

For one point formula in Gauss Quadrature integration, $w_1 = 2,\ \xi_1 = 0$. Thus,

$$I_1 = 2 \times 2.5 \left[\left(\frac{5 \times 0 + 1}{2} \right)^2 + 11 \left(\frac{5 \times 0 + 1}{2} \right) - 32 \right]$$

$$= 5 \left[\frac{1}{4} + \frac{11}{2} - 32 \right] = -131.25$$

Thus, % of error = (120.83333-131.25)×100/120.83333 = 8.62%

(iii) <u>Two Point Formula:</u>

Here, for two point formula in Gauss Quadrature integration,

$$\mathrm{w}_1 = \mathrm{w}_2 = 1.0 \text{ and } \xi_1 = -\xi_2 = -\frac{1}{\sqrt{3}}. \text{ Thus,}$$

$$I_2 = w_1 \phi(\xi_1) + w_2 \phi(\xi_2)$$

$$1.0 \times 2.5 \times \left[\left(\frac{\frac{-5}{\sqrt{3}} + 1}{2} \right)^2 + 11 \left(\frac{\frac{-5}{\sqrt{3}} + 1}{2} \right) - 32 \right] + 1.0 \times 2.5 \times \left[\left(\frac{\frac{5}{\sqrt{3}} + 1}{2} \right)^2 + 11 \left(\frac{\frac{5}{\sqrt{3}} + 1}{2} \right) - 32 \right]$$

$$= (0.88996 - 10.37713 - 32) \times 2.5 + (3.77671 + 21.3771 - 32) \times 2.5$$

$$= -48.3333 \times 2.5$$

$$= -120.83325$$

Thus, % of error = (120.83333-120.83325)×100/120.83333 = 6.62×10^{-05}

(iv) <u>Three Point Formula:</u>

Here, for three point formula in Gauss Quadrature integration,

$$\mathrm{w}_1 = 0.8889, \qquad \xi_1 = 0.0$$

$$w_2 = 0.5556, \qquad \xi_2 = +0.7746$$

$$w_3 = 0.5556, \qquad \xi_3 = -0.7746$$

Thus,

$$I_3 = w_1 \phi(\xi_1) + w_2 \phi(\xi_2) + w_3 \phi(\xi_3)$$

$$I_3 = 0.8889\times 2.5\times\left[\left(\frac{5\times 0+1}{2}\right)^2 + 11\times\frac{5\times 0+1}{2} - 32\right]$$
$$+0.5556\times 2.5\times\left[\left(\frac{5\times 0.7746+1}{2}\right)^2 + 11\times\frac{5\times 0.7746+1}{2} - 32\right]$$
$$+0.5556\times 2.5\times\left[\left(\frac{-5\times 0.7746+1}{2}\right)^2 + 11\times\frac{-5\times 0.7746+1}{2} - 32\right]$$
$$I_3 = 0.8889\times 2.5\times[0.25+5.5-32]$$
$$+0.5556\times 2.5\times[5.9365+26.8015-32]$$
$$+0.5556\times 2.5\times[2.0635-15.8015-32]$$
$$= 2.5\times(-23.3336+0.4100-25.4120)$$
$$= -2.5\times 48.3356 = -120.839$$

Thus, % of error = (120.83333-120.839)×100/120.83333 = 4.69×10^{-03}. However, difference of results will approach to zero, if few more digits after decimal points are taken in calculation.

Example 3.4 Numerical integration for three dimensional problems

Evaluate the integral: $\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(1-2\xi)^2(1-\eta)^2(3\zeta-2)^2\,d\xi d\eta d\zeta$

Solution:

Using two point gauss Quadrature formula for the evaluation of three dimensional integration, we have the following sampling points and weights.

$$w_1 = w_2 = 1$$
$$\xi_1 = -0.5773502692$$
$$\xi_2 = 0.5773502692$$
$$\eta_1 = -0.5773502692$$
$$\eta_2 = 0.5773502692$$
$$\zeta_1 = -0.5773502692$$
$$\zeta_2 = 0.5773502692$$

Putting the above values, in $\phi(\xi,\eta,\zeta) = (1-2\xi)^2(1-\eta)^2(3\zeta-2)^2$ one can find the following values in 8 (i.e., 2 × 2 × 2) sampling points.

$\phi(\xi_1, \eta_1, \zeta_1) = 160.8886$

$\phi(\xi_1, \eta_1, \zeta_2) = 0.8293$

$\phi(\xi_1, \eta_2, \zeta_1) = \ 11.5513$

$\phi(\xi_1, \eta_2, \zeta_2) = 0.0595$

$\phi(\xi_2, \eta_1, \zeta_1) = 0.8293$

$\phi(\xi_2, \eta_1, \zeta_2) = 0.0043$

$\phi(\xi_2, \eta_2, \zeta_1) = \ 0.0595$

$\phi(\xi_2, \eta_2, \zeta_2) = 0.0003$

Now, $I = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} w_i w_j w_k \phi(\xi_i, \eta_j, \zeta_k)$

Thus, $I = w_1 w_1 w_1 \phi(\xi_1, \eta_1, \zeta_1) + w_1 w_1 w_2 \phi(\xi_1, \eta_1, \zeta_2) + \ldots + w_2 w_2 w_2 \phi(\xi_2, \eta_2, \zeta_2)$ = 174.222, where as I_{exact} = 174.222.