Lecture 1: Virtual Work and Variational Principle

2.1.1 Introduction

Finite element formulation can be constructed from governing differential equations over a domain. This can be formulated by various ways like Virtual work method, Variational method, Weighted Residual Method etc.

2.1.2 Principle of Virtual Work

The principle of virtual work is a very useful approach for solving varieties of structural mechanics problem. When the force and displacement are unrelated to the cause and effect relation, the work is called virtual work. Therefore, the virtual work may be caused by true force moving through imaginary displacements or vice versa. Thus, the principle of virtual work can be divided into two categories: (a) principle of virtual forces and (b) principle of virtual displacements. The principle of virtual forces establishes the compatibility conditions. The principle of virtual displacements establishes the conditions of equilibrium and is used in the displacement model of the finite element technique.

The external virtual work is the work done by real load moving through imaginary displacements in a structure. These loads include both the load distributed over the entire surface and volume. Thus, the virtual work done by the external force is:

$$\delta W_E = \int_\Gamma \{\delta u \quad \delta v \quad \delta w\} \begin{Bmatrix} F_{\Gamma x} \\ F_{\Gamma y} \\ F_{\Gamma z} \end{Bmatrix} d\Gamma + \int_\Omega \{\delta u \quad \delta v \quad \delta w\} \begin{Bmatrix} F_{\Omega x} \\ F_{\Omega y} \\ F_{\Omega z} \end{Bmatrix} d\Omega \tag{2.1.1}$$

Where, δu, δv and δw are the components of the virtual displacements in x, y and z direction respectively. $F_{\Gamma x}$, $F_{\Gamma y}$ and $F_{\Gamma z}$ are the surface forces and $F_{\Omega x}$, $F_{\Omega y}$ and $F_{\Omega z}$ are the body forces in x, y and z direction respectively. In the above equation, the integration is carried out over the entire surface in the first term and over the entire volume in the second term. The above expression can be rewritten as:

$$\delta W_E = \int_\Gamma \delta \{d\}^T \{F_\Gamma\} d\Gamma + \int_\Omega \delta \{d\}^T \{F_\Omega\} d\Omega \tag{2.1.2}$$

Here, $\{d\}^T = \{u \quad v \quad w\}$. For the three dimensional stress-strain condition, there are six components of stresses ($\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$) and six components of strains in virtual displacement fields ($\delta\varepsilon_x, \delta\varepsilon_y, \delta\varepsilon_z, \delta\gamma_{xy}, \delta\gamma_{yz}, \delta\gamma_{zx}$). Therefore, the virtual internal work can be expressed as follows:

$$\delta U = \int_{\Omega} \left\{ \delta\varepsilon_x \quad \delta\varepsilon_y \quad \delta\varepsilon_z \quad \delta\gamma_{xy} \quad \delta\gamma_{yz} \quad \delta\gamma_{zx} \right\} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} d\Omega \tag{2.1.3}$$

Or

$$\delta U = \int_{\Omega} \delta \{\varepsilon\}^T \{\sigma\} d\Omega \tag{2.1.4}$$

According to principle of virtual work, the work done by external forces due to the virtual displacement of a structure in equilibrium is equal to the work done by the internal forces for the virtual internal displacement. Therefore, $\delta W_E = \delta U$ Thus eqs. (2.1.2) and (2.1.4) can be made equal and can be related as follows:

$$\int_{\Gamma} \delta \{d\}^T \{F_\Gamma\} d\Gamma + \int_{\Omega} \delta \{d\}^T \{F_\Omega\} d\Omega = \int_{\Omega} \delta \{\varepsilon\}^T \{\sigma\} d\Omega \tag{2.1.5}$$

2.1.3 Variational Principle

Variational formulation is the generalized method of formulating the element stiffness matrix and load vector using the variational principle of solid mechanics. The strain energy in a structural body is given by the relation

$$U = \frac{1}{2} \iiint_{\Omega} \{\varepsilon\}^T \{\sigma\} d\Omega \tag{2.1.6}$$

For a 3D structural problem, stress has six components: $\{\sigma\}^T = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}\}$. Similarly, there are six components of strains: $\{\varepsilon\}^T = \{\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}\}$. Now the strain-displacement relationship can be expressed as $\{\varepsilon\} = [B]\{d\}$, where {*d*} is the displacement vector in x, y and z directions and [*B*] is called as the strain displacement relationship matrix. Again, the stress can be represented in terms of its constitutive relationship matrix: $\{\sigma\} = [D]\{\varepsilon\}$. Here $[D]$ is called as the constituent relationship matrix. Using the above relationship in the strain energy equation one can arrive

$$U = \frac{1}{2} \iiint_{\Omega} \left[[B]\{d\} \right]^T [D]\{B\}\{d\} d\Omega \tag{2.1.7}$$

Applying the variational principle one can express

$$\{F\} = \frac{\partial U}{\partial \{d\}} = \iiint_{\Omega} [B]^T [D][B] d\Omega \{d\} \tag{2.1.8}$$

Now, from the relationship of $\{F\} = [K]\{d\}$, one can arrive at the element stiffness matrix as:

$$[K] = \iiint_{\Omega} [B]^T [D][B] d\Omega \tag{2.1.9}$$

Thus, by the use of variational principle, the stiffness matrix of a structural element can be obtained as expressed in the above equation.

2.1.4 Weighted Residual Method

Virtual work and Variational method are applicable and adequate for most of the problems. However, in some cases functional analogous to potential energy cannot be written because of not having clear physical meaning. For some applications, such as in fluid mechanics problem, functional needed for a variational approach cannot be expressed. For some types of fluid flow problems, only differential equations and boundary conditions are available. For Such problems weighted residual method can be used for obtaining the solutions. Approximate solutions of differential equation satisfy only part of conditions of the problem. For example a differential equation may be satisfied only at few points, rather than at each. The strategy used in weighted residual method is to first take an approximate solution and then its validity is assessed. The different methods in weighted Residual Method are

- Collocation method
- Least square method
- Method of moment
- Galerkin method

The mathematical statement of a physical problem can be defined as:

In domain Ω,

$$\mathrm{Du} - \mathrm{f} = 0 \tag{2.1.10}$$

Where,

D is the differential operator

u = u(x) = dependent variables such as displacement, pressure, velocity, potential function

x = independent variables such as coordinates of a point

f = a function of x which may be constant or zero

If $\bar{\mathrm{u}}$ is an approximate solution then residual in domain Ω,

$$\mathrm{R} = \mathrm{D}\bar{\mathrm{u}} - \mathrm{f} \tag{2.1.11}$$

According to the weighted residual method, the weak form of above equation will become

$$\int_{\Omega} w_i \; R \; d\Omega = 0 \quad \text{for } i = 1, 2, 3, ..., n$$

or (2.1.12)

$$\int_{\Omega} w_i \left(D\bar{u} - f \right) d\Omega = 0$$

Where weighting function $w_i = w_i(x)$ is chosen from the approximate basis function used for constructing approximated solution $\bar{u}$.

Lecture 2: Galerkin Method

2.2.1 Introduction

Galerkin method is the most widely used among the various weighted residual methods. Galerkin method incorporates differential equations in their weak form, i.e., before starting integration by parts it is in strong form and after by parts it will be in weak form, so that they are satisfied over a domain in an integral. Thus, in case of Galerkin method, the equations are satisfied over a domain in an integral or average sense, rather than at every point. The solution of the equations must satisfy the boundary conditions. There are two types of boundary conditions:

- Essential or kinematic boundary condition
- Non essential or natural boundary condition

For example, in case of a beam problem ($EI\frac{\partial^4 y}{\partial x^4} - q = 0$) differential equation is of forth order. As a result, displacement and slope will be essential boundary condition where as moment and shear will be non-essential boundary condition.

2.2.2 Galerkin Method for 2D Elasticity Problem

For a two dimensional elasticity problem, equation of equilibrium can be expressed as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_{\Omega x} = 0 \qquad (2.2.1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + F_{\Omega y} = 0 \qquad (2.2.2)$$

Where, $F_{\Omega x}$ and $F_{\Omega y}$ are the body forces in X and Y direction respectively. Let assume, $\Gamma_{\Omega x}$ and $\Gamma_{\Omega y}$ are surface forces in X and Y direction and α as angle made by normal to surface with X– axis (Fig. 2.2.1). Therefore, force equilibrium of element can be written as:

$$F_{\Gamma x}(PQ)t = \sigma_x(OP)t + \tau_{xy}(OQ)t$$

$$F_{\Gamma x} = \sigma_x \frac{OP}{PQ} + \tau_{xy}\frac{OQ}{PQ} = \sigma_x \cos\alpha + \tau_{xy}\sin\alpha = \sigma_x \cos\alpha + \tau_{xy}\mathrm{Cos}(90-\alpha)$$

$$\text{Thus, } F_{\Gamma x} = \sigma_x \ell + \tau_{xy} m \qquad (2.2.3)$$

Where, ℓ and m are direction cosines of normal to the surface. Similarly,

$$F_{\Gamma y} = \tau_{xy}\ell + \sigma_y m \qquad (2.2.4)$$

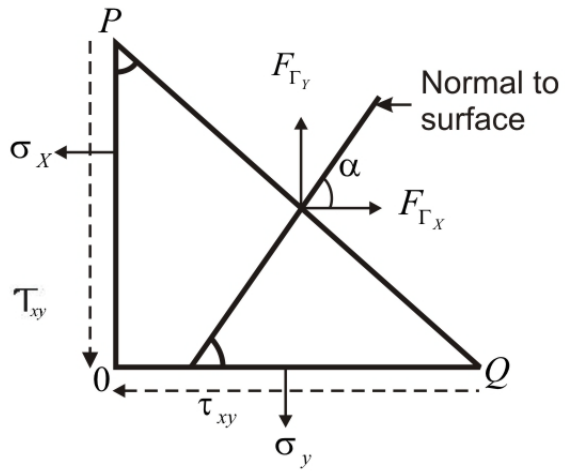

Fig. 2.2.1 Elemental stress in 2D

Adopting Galerkin's approach

$$\left[\iint\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+F_{\Gamma x}\right)\delta u+\iint\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+F_{\Gamma x}\right)\delta v\right]dxdy=0 \qquad (2.2.5)$$

Where δu and δv are weighting functions i.e elemental displacements in X and Y directions respectively. Now one can expand above equation by using Green's Theorem.

Green Theorem states that if $\phi(x,y)$ and $\psi(x,y)$ are continuous functions then their first and second partial derivatives are also continuous. Therefore,

$$\iint\left[\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y}\right]dxdy=-\iint\phi\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]dxdy+\int\phi\left[\frac{\partial\psi}{\partial x}\ell+\frac{\partial\psi}{\partial y}m\right]ds \qquad (2.2.6)$$

Assuming, $\phi=\sigma_x;\frac{\partial\psi}{\partial x}=\delta u;\ \frac{\partial\psi}{\partial y}=0$ one can rewrite with the use of above relation as

$$\iint\frac{\partial\sigma_x}{\partial x}\delta u\ dx\ dy=-\iint\sigma_x\frac{\partial(\delta u)}{\partial x}dx\ dy+\int\sigma_x\ell\ \delta u\ ds \qquad (2.2.7)$$

Similarly, assuming $\phi=\sigma_y;\ \frac{\partial\psi}{\partial x}=0$ and $\frac{\partial\psi}{\partial y}=\delta v$

$$\iint\frac{\partial\sigma_y}{\partial y}\delta v\ dx\ dy=-\iint\sigma_y\frac{\partial(\delta v)}{\partial y}dx\ dy+\int\sigma_y m\ \delta v\ ds \qquad (2.2.8)$$

Again, assuming $\phi = \tau_{xy};\ \frac{\partial \psi}{\partial x} = \delta v; \frac{\partial \psi}{\partial y} = 0$

$$\iint \frac{\partial \tau_{xy}}{\partial y} \delta v \ dx \ dy = -\iint \tau_{xy} \frac{\partial (\delta v)}{\partial x} dx \ dy + \int \tau_{xy} \ell \ \delta v \ ds \qquad (2.2.9)$$

And assuming, $\phi = \tau_{xy};\ \frac{\partial \psi}{\partial x} = 0;\ \frac{\partial \psi}{\partial y} = \delta u$

$$\iint \frac{\partial \tau_{xy}}{\partial y} \delta u \ dx \ dy = -\iint \tau_{xy} \frac{\partial (\delta u)}{\partial y} dx \ dy + \int \tau_{xy} m \ \delta u \ ds$$

Putting values of eqs. (2.2.7), (2.2.8) and (2.2.9), in eq. (2.2.5), one can get the following relation:

$$-\iint \left[\sigma_x \frac{\partial}{\partial x}(\delta u) + \sigma_y \frac{\partial}{\partial y}(\delta v) + \tau_{xy} \frac{\partial}{\partial x}(\delta v) + \tau_{xy} \frac{\partial}{\partial y}(\delta u) \right] dx \ dy$$
$$+ \int \left[\sigma_x \ell \delta u + \sigma_y m \delta v + \tau_{xy} \ell \delta v + \tau_{xy} m \delta u \right] ds + \iint F_{\Omega x} \delta u \ dx \ dy + \iint F_{\Omega y} \delta v \ dx \ dy = 0 \qquad (2.2.10)$$

Rearranging the terms of above expression, the following relations are obtained.

$$-\iint \left[\sigma_x \frac{\partial}{\partial x}(\delta u) + \sigma_y \frac{\partial}{\partial y}(\delta v) + \tau_{xy} \frac{\partial}{\partial x}(\delta v) + \tau_{xy} \frac{\partial}{\partial y}(\delta u) \right] dx \ dy + \iint \left(F_{\Omega x} \delta u + F_{\Omega y} \delta v \right) dx \ dy$$
$$+ \int \left(\sigma_x \ell + \tau_{xy} m \right) \delta u ds + \int \left(\tau_{xy} \ell + \sigma_y m \right) \delta v ds = 0 \qquad (2.2.11)$$

Here, $F_{\Omega x}$ and $F_{\Omega y}$ are the body forces and δu & δv are virtual displacements in X and Y directions respectively.

Considering first term of eq. (2.2.11), virtual displacement δu is given to the element of unit thickness. Dotted position in Fig. 2.2.2 shows the virtual displacement. Thus, work done by σ_x:

$$\sigma_x dy \left[\delta u + \frac{\partial}{\partial x}(\delta u) dx \right] - \sigma_x dy \delta u = \sigma_x \frac{\partial}{\partial x}(\delta u) dx dy \qquad (2.2.12)$$

Similarly, considering second term of eq. (2.2.11), virtual work done by body forces is

$$\iint \left(F_{\Omega x} \delta u + F_{\Omega y} \delta v \right) dx \ dy$$

Putting eqs. (2.2.3) & (2.2.4) in third term of eq. (2.2.11) we get the virtual work done by surface forces as:

$$\int F_{\Gamma x}\delta u ds + \int F_{\Gamma y}\delta v ds$$

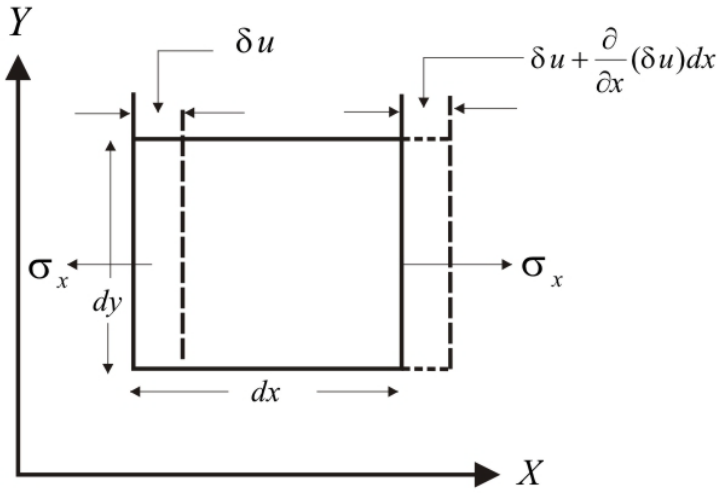

Fig. 2.2.2 Element subjected to stresses

Due to virtual displacement δu, change in strain $\delta \in_x$ is given by:

$$\delta \in_x = \frac{\left[\delta u + \frac{\partial}{\partial x}(\delta u)dx\right] - \delta u}{dx} = \frac{\partial}{\partial x}(\delta u) \qquad (2.2.13)$$

The virtual work done by σ_x is $\sigma_x.\delta \in_x .dxdy$. Similarly all the individual term in the first term of eq. (2.2.11) can be derived from eq. (2.2.13) which will be as follows:

$$\iint \sigma_x \frac{\partial}{\partial x}(\delta u)dxdy = \iint \sigma_x \delta \in_x dxdy$$

$$\iint \sigma_y \frac{\partial}{\partial y}(\delta v)dxdy = \iint \sigma_y \delta \in_y dxdy \qquad (2.2.14)$$

$$\iint \tau_{xy}\left\{\frac{\partial}{\partial x}(\delta v) + \frac{\partial}{\partial y}(\delta u)\right\} = \iint \tau_{xy}\delta\gamma_{xy}dxdy$$

Now, the work done by internal forces will be

$$\delta U = \iint \left(\sigma_x \delta \in_x + \sigma_y \delta \in_y + \tau_{xy} \delta \gamma_{xy} \right) dxdy \tag{2.2.15}$$

If external work done is represented by W_E and U is the internal work done then,

$$-\delta U + \delta w_E = 0 \text{ or } \delta U = \delta w_E \tag{2.2.16}$$

Thus in elasticity problems, Galerkin's method turns out to be the principle of virtual work, which can be stated that "A Deformable body is said to be in equilibrium, if the total work done by external forces is equal to the total work done by internal forces." The work done above is virtual as either forces or deformations are also virtual. Thus, Galerkin's approach can be followed in all problems involving solution of a set of equations subjected to specified boundary values.

2.2.3 Galerkin Method for 2D Fluid Flow Problem

Let consider the two dimensional incompressible fluid equation which can be expressed by pressure variable only as follows.

$$\nabla^2 p = 0 \tag{2.2.17}$$

Where p is the pressure inside the fluid domain. The above equation can be expressed in 2D form as:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

or

$$p_{,ii} = 0 \tag{2.2.18}$$

Applying weighted residual method, the weak form of the above equation will become

$$\int_\Omega w_i \; p_{,ii} \; d\Omega = 0 \tag{2.2.19}$$

Integrating by parts of the above expression, the following relation can be obtained.

$$\int_\Gamma w_i \; p_{,i} \; d\Gamma - \int_\Omega w_{i,i} \; p_{,i} \; d\,\Omega = 0$$

$$\text{or } \int_\Omega w_{i,i} \; p_{,i} \; d\,\Omega = \int_\Gamma w_i \; p_{,i} \; d\Gamma \tag{2.2.20}$$

If the nodal pressure and interpolation functions are denoted by $\overline{p}$ and N respectively, then the pressure at any point inside the fluid domain can be expressed as

$$p = [N]\{\overline{p}\}$$

Similarly, the weighted function can also be written with the help of interpolation function as

$$w = [N]\{\overline{w}\}$$

Thus, $p_{i,i} = [L]\{p\} = [L][N]\{\bar{p}\} = [B]\{\bar{p}\}$, where, $[L] = \left[\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right]$ = differential operator.

Similarly, $w_{i,i} = [L]\{W\} = [L][N]\{\bar{w}\} = [B]\{\bar{w}\}$

$$\text{Thus,} \int_{\Omega} w_{i,i}\ p_{,i}\ d\Omega = \int [\bar{w}]^T [B]^T [B][\bar{p}] d\Omega \tag{2.2.21}$$

$$\int_{\Gamma} w_i\ p_{,i}\ d\Gamma = \int_{\Gamma} \{\bar{w}\}^T [N]^T \frac{\partial p}{\partial n} d\Gamma \tag{2.2.22}$$

Here, Γdenotes the surface of the fluid domain and *n* represents the direction normal to the surface. Thus, from eq. (2.2.20), one can write the expression as:

$$\text{Thus,} \int_{\Omega} \{\bar{w}\}^T [B]^T [B]\{\bar{p}\} d\Omega = \int_{\Gamma} \{\bar{w}\}^T [N]^T \frac{\partial p}{\partial n} d\Gamma$$

$$\text{Or,}\quad [G]\{\bar{p}\} = \{S\} \tag{2.2.23}$$

Where,

$$[G] = \int_{\Omega} [B]^T [B] d\Omega = \int_{\Omega} \left(\frac{\partial}{\partial x}[N]^T \frac{\partial}{\partial x}[N] + \frac{\partial}{\partial y}[N]^T \frac{\partial}{\partial y}[N] \right) d\Omega$$

$$\text{and}\ \{S\} = \int_{\Gamma} [N]^T \frac{\partial p}{\partial n} d\Gamma \tag{2.2.27}$$

Here, *n* is the direction normal to the surface. Thus, solving the above equation with the prescribed boundary conditions, one can find out the pressure distribution inside the fluid domain by the use of finite element technique.

Lecture 3: Finite Element Method: Displacement Approach

2.3.1 Choice of Displacement Function

Displacement function is the beginning point for the structural analysis by finite element method. This function represents the variation of the displacement within the element. On the basis of the problem to be solved, the displacement function needs to be approximated in the form of either linear or higher-order function. A convenient way to express it is by the use of polynomial expressions.

2.3.1.1 Convergence criteria

The convergence of the finite element solution can be achieved if the following three conditions are fulfilled by the assumed displacement function.

a. The displacement function must be continuous within the elements. This can be ensured by choosing a suitable polynomial. For example, for an *n* degrees of polynomial, displacement function in I dimensional problem can be chosen as:

$$u = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + ... + \alpha_n x^n \qquad (2.3.1)$$

b. The displacement function must be capable of rigid body displacements of the element. The constant terms used in the polynomial (α_0 to α_n) ensure this condition.

c. The displacement function must include the constant strains states of the element. As element becomes infinitely small, strain should be constant in the element. Hence, the displacement function should include terms for representing constant strain states.

2.3.1.2 Compatibility

Displacement should be compatible between adjacent elements. There should not be any discontinuity or overlapping while deformed. The adjacent elements must deform without causing openings, overlaps or discontinuous between the elements.

Elements which satisfy all the three convergence requirements and compatibility condition are called Compatible or Conforming elements.

2.3.1.3 Geometric invariance

Displacement shape should not change with a change in local coordinate system. This can be achieved if polynomial is balanced in case all terms cannot be completed. This ‘balanced’ representation can be achieved with the help of Pascal triangle in case of two-dimensional polynomial. For example, for a polynomial having four terms, the invariance can be obtained if the following expression is selected from the Pascal triangle.

$$u = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy \qquad (2.3.2)$$

The geometric invariance can be ensured by the selection of the corresponding order of terms on either side of the axis of symmetry.

$$1$$

$$x \quad y$$

$$x^2 \quad xy \quad y^2$$

$$x^3 \quad x^2y \quad xy^2 \quad y^3$$

$$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$$

Fig. 2.3.1 Pascal's Triangle

2.3.2 Shape Function

In finite element analysis, the variations of displacement within an element are expressed by its nodal displacement ($u = \sum N_i u_i$) with the help of interpolation function since the true variation of displacement inside the element is not known. Here, u is the displacement at any point inside the element and u_i are the nodal displacements. This interpolating function is generally a polynomial with n degree which automatically provides a single-valued and continuous field. In finite element literature, this interpolation function (N_i) is referred to "Shape function" as well. For linear interpolation, n will be 1 and for quadratic interpolation n will become 2 and so on. There are two types of interpolation functions namely (i) Lagrange interpolation and (ii) Hermitian interpolation. Lagrange interpolation function is widely used in practice. Here the assumed function takes on the same values as the given function at specified points. In case of Hermitian interpolation function, the slopes of the function also take the same values as the given function at specified points. The derivation of shape function for varieties of elements will be discussed in subsequent lectures.

2.3.3 Degree of Continuity

Let consider ϕ as an interpolation function in a piecewise fashion over finite element mesh. While such interpolation function ϕ can be ensured to vary smoothly within the element, the transition between adjacent elements may not be smooth. The term C^m is considered to define the continuity of a piecewise displacement. A function C^m is continuous if its derivative up to and including degree m are inter-element continuous. For example, for one dimensional problem, $\phi = \phi(x)$ is C^0 continuous if ϕ is continuous, but $\phi_{,x}$ is not. Similarly, $\phi = \phi(x)$ is C^1 continuous if ϕ and $\phi_{,x}$ are continuous, but $\phi_{,xx}$ is not. In general, C^0 element is used to model plane and solid body and C^1 element is used to model beam, plate and shell like structure, where inter-element continuity of slope is necessary to ensure. Let assume a linear function for bar like element: $\phi_1 = \alpha_0 + \alpha_1 x$ This function is C^0 continuous as

$\phi_{1,x}$ is discontinuous. If the interpolation function is considered as $\phi_2 = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ then $\phi_{2,x} = \alpha_1 + 2\alpha_2 x$ is also continuous but $\phi_{2,xx} = 2\alpha_2$ is discontinuous. As a result, this function ϕ_2 will become C^1 continuous.

2.3.4 Isoparametric Elements

If the shape functions (N_i) used to represent the variation of geometry of the element are the same as the shape functions (N'_i) used to represent the variation of the displacement then the elements are called isoparametric elements. For example, the coordinates (x,y) inside the element are defined by the shape functions (N_i) and displacement (u,v) inside the element are defined by the shape functions (N'_i) as below.

$$\begin{aligned} x &= N_i x_i & u &= N'_i u_i \\ y &= N_i y_i & v &= N'_i v_i \end{aligned} \qquad (2.3.3)$$

If $N_i = N'_i$, then the element is called isroparametric. Fig. 2.3.2(a) shows the two dimensional 8 node isoparametric element.

If the geometry of element is defined by shape functions of order higher than that for representing the variation of displacements, then the elements are called superparametric (Fig. 2.3.2(b)).

If the geometry of element is defined by shape functions of order lower than that for representing the variation of displacements then the elements are called subparametric (Fig. 2.3.2(c)).

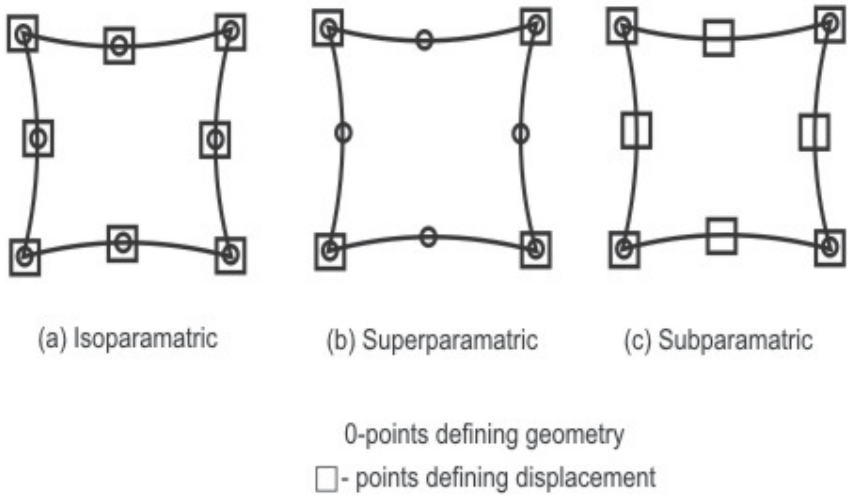

Fig. 2.3.2 Shape functions for geometry and displacements

2.3.5 Various Elements

Selection of the order of the polynomial depends on the type of elements. For example, in case of one dimensional element having single degrees of freedom with two nodes, the displacement function can be chosen as $u = \alpha_0 + \alpha_1 x$. However, if the same has two degrees of freedom at each node, then the chosen displacement function should be $u = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$. Various types of elements used in finite element analysis are given below:

1. <u>One dimensional elements.</u>
   (a) Two node element
   (b) Three node element

(a) 2 node element (b) 3 node element

Fig. 2.3.3 One dimensional elements

2. <u>Two dimensional elements</u>
   (a) Triangular element
   (b) Rectangular element
   (c) Quadrilateral element
   (d) Quadrilateral formed by two triangles
   (e) Quadrilateral formed by four triangles

Few of the elements with number of nodes are shown in Fig. 2.3.4.

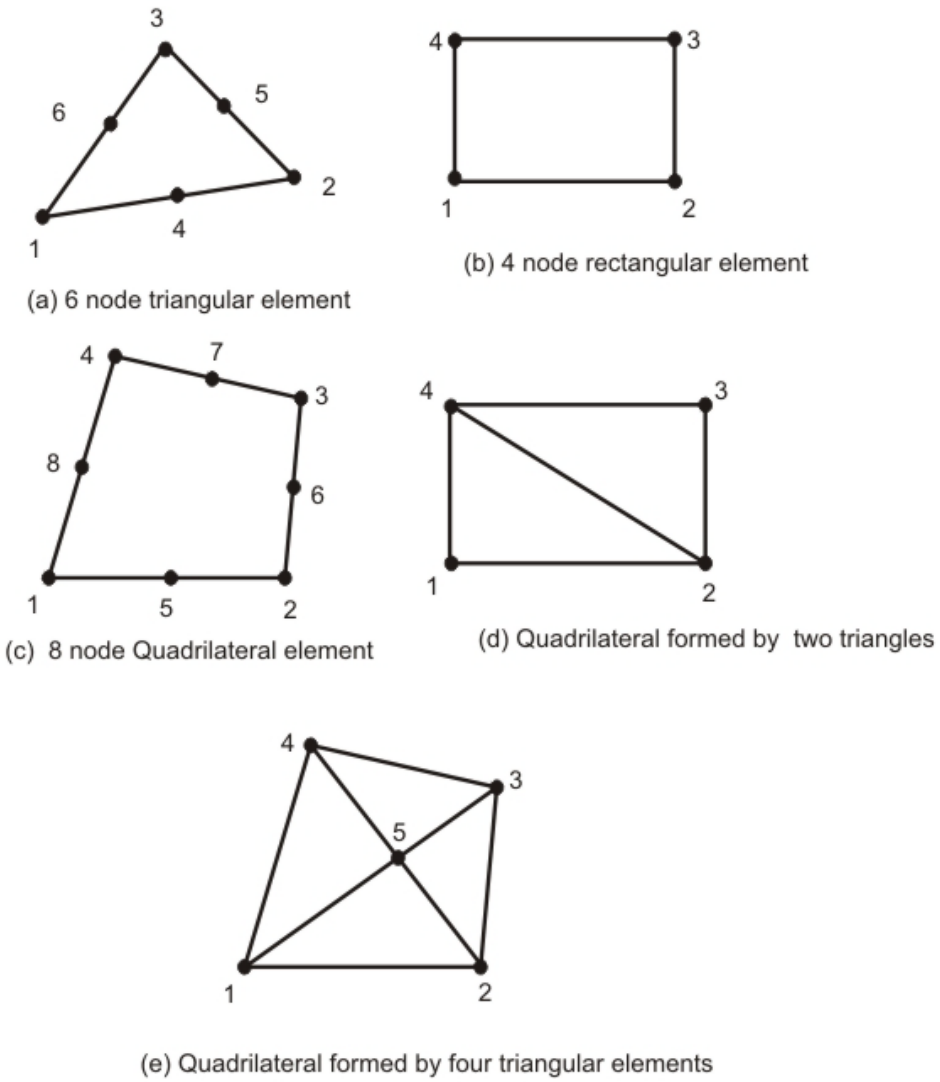

Fig. 2.3.4 Two dimensional elements

3. Three dimensional elements.
   (a) Tetrahedron
   (b) Rectangular brick element

Few of the three dimensional solid elements are shown in Fig. 2.3.5.

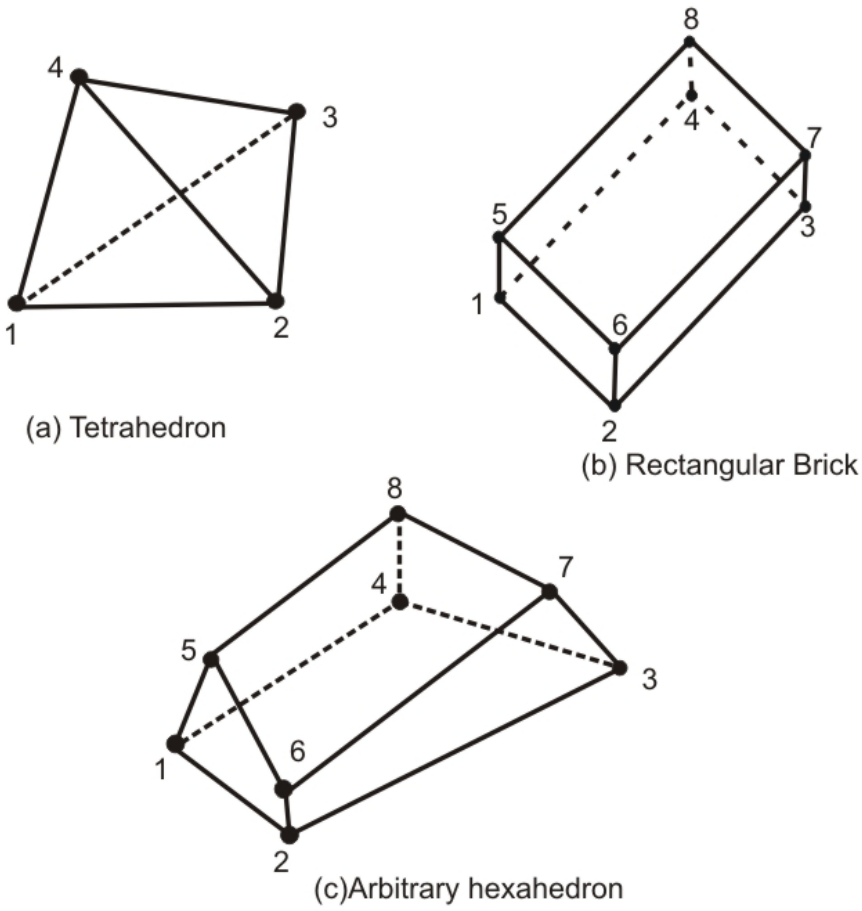

Fig. 2.3.5 Three dimensional elements

Lecture 4: Stiffness Matrix and Boundary Conditions

2.4.1 Element Stiffness Matrix

The stiffness matrix of a structural system can be derived by various methods like variational principle, Galerkin method etc. The derivation of an element stiffness matrix has already been discussed in earlier lecture. The stiffness matrix is an inherent property of the structure. Element stiffness is obtained with respect to its axes and then transformed this stiffness to structure axes. The properties of stiffness matrix are as follows:

- Stiffness matrix is symmetric and square.
- In stiffness matrix, all diagonal elements are positive.
- Stiffness matrix is be positive definite

2.4.2 Global Stiffness Matrix

A structural system is an assemblage of number of elements. These elements are interconnected together to form the whole structure. Therefore, the element stiffness of all the elements are first need to be calculated and then assembled together in systematic manner. It may be noted that the stiffness at a joint is obtained by adding the stiffness of all elements meeting at that joint.

To start with, the degrees of freedom of the structure are numbered first. This numbering will start from 1 to *n* where *n* is the total degrees of freedom. These numberings are referred to as degrees of freedom corresponding to global degrees of freedom. The element stiffness matrix of each element should be placed in their proper position in the overall stiffness matrix. The following steps may be performed to calculate the global stiffness matrix of the whole structure.

a. Initialize global stiffness matrix $[K]$ as zero. The size of global stiffness matrix will be equal to the total degrees of freedom of the structure.
b. Compute individual element properties and calculate local stiffness matrix $[k]$ of that element.
c. Add local stiffness matrix $[k]$ to global stiffness matrix $[K]$ using proper locations
d. Repeat the Step b. and c. till all local stiffness matrices are placed globally.

The steps to be followed in the computer program are shown in the form of flow chart in Fig. 2.4.1 for assembling the local stiffness matrix to global stiffness matrix.

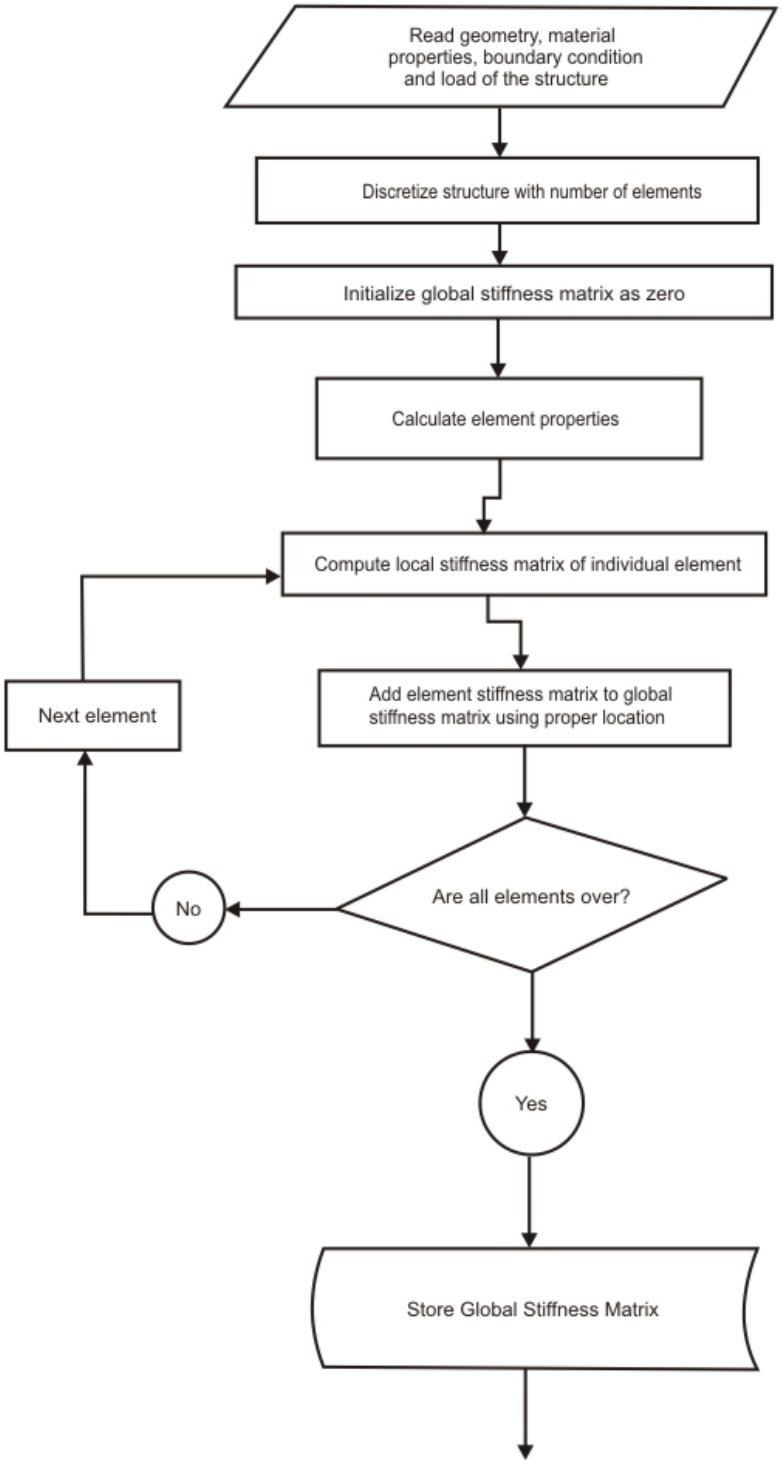

Fig. 2.4.1 Assemble of stiffness matrix from local to global

2.4.3 Boundary Conditions

Under this section, procedure to include the effect of boundary condition in the stiffness matrix for the finite element analysis will be discussed. The solution cannot be obtained unless support conditions are included in the stiffness matrix. This is because, if all the nodes of the structure are included in displacement vector, the stiffness matrix becomes singular and cannot be solved if the structure is not supported amply, and it cannot resist the applied loads. A solution cannot be achieved until the boundary conditions *i.e.*, the known displacements are introduced.

In finite element analysis, the partitioning of the global matrix is carried out in a systematic way for the hand calculation as well as for the development of computer codes. In partitioning, normally the equilibrium equations can be partitioned by rearranging corresponding rows and columns, so that prescribed displacements are grouped together. For example, let consider the equation of equilibrium is expressed in compact form as:

$$\{F\} = [K]\{d\} \tag{2.4.1}$$

Where,

$[K]$ is the global stiffness matrix,

$\{d\}$ is the displacement vector consisting of global degrees of freedom, and

$\{F\}$ is the load vector corresponding to degrees of freedom.

By the method of partitioning the above equation can be partitioned in the following manner.

$$\begin{Bmatrix} \{F_\alpha\} \\ \{F_\beta\} \end{Bmatrix} = \begin{bmatrix} [K_{\alpha\alpha}] & [K_{\alpha\beta}] \\ [K_{\beta\alpha}] & [K_{\beta\beta}] \end{bmatrix} \begin{Bmatrix} \{d_\alpha\} \\ \{d_\beta\} \end{Bmatrix} \tag{2.4.2}$$

Where, subscripts α refers to the displacements free to move and β refers to the prescribed support displacements. As the prescribed displacements $\{d_\beta\}$ are known, eq. (2.4.2) may be written in expanded form as:

$$\{F_\alpha\} = [K_{\alpha\alpha}]\{d_\alpha\} + [K_{\alpha\beta}]\{d_\beta\} \tag{2.4.3}$$

Thus it is possible to obtain the free displacement of the structure $\{d_\alpha\}$ as

$$\{d_\alpha\} = [K_{\alpha\alpha}]^{-1} \left\{ \{F_\alpha\} - [K_{\alpha\beta}]\{d_\beta\} \right\} \tag{2.4.4}$$

If the displacements at supports $\{d_\beta\}$ are zero, then the above equation can be simplified to the following expression.

$$\{d_\alpha\} = [K_{\alpha\alpha}]^{-1} \{F_\alpha\} \tag{2.4.5}$$

Thus, by rearranging assembled matrix, the portion corresponding to the unknown displacements in eq. (2.4.4) can be taken out for the solution purpose. This is possible as the known displacements $\{d_\beta\}$ are restrained, i.e., displacements are zero. If the support has some known displacements, then eq. (2.4.4) can be used to find the solution. If the few supports of the structures yield, then the above method may be modified by partitioning the stiffness matrix into three parts as shown below:

$$\begin{Bmatrix}\{F_\alpha\}\\ \{F_\beta\}\\ \{F_\gamma\}\end{Bmatrix}=\begin{bmatrix}[K_{\alpha\alpha}] & [K_{\alpha\beta}] & [K_{\alpha\gamma}]\\ [K_{\beta\alpha}] & [K_{\beta\beta}] & [K_{\beta\gamma}]\\ [K_{\gamma\alpha}] & [K_{\gamma\beta}] & [K_{\gamma\gamma}]\end{bmatrix}\begin{Bmatrix}\{d_\alpha\}\\ \{d_\beta\}\\ \{d_\gamma\}\end{Bmatrix} \quad (2.4.6)$$

Here, α refers to unknown displacement; β refers to known displacement ($\neq 0$) and γ refers to zero displacement. Thus, the above equation can be separated and solved independently to find required unknown results as shown below.

$$\{F_\alpha\}=[K_{\alpha\alpha}]\{d_\alpha\}+[K_{\alpha\beta}]\{d_\beta\}+[K_{\alpha\gamma}]\{d_\gamma\}$$
$$\text{or, } [K_{\alpha\alpha}]\{d_\alpha\}=\{F_\alpha\}-[K_{\alpha\beta}]\{d_\beta\} \text{ as } \{d_\gamma\}=\{0\}$$
$$\text{Thus, } \{d_\alpha\}=[K_{\alpha\alpha}]^{-1}\left\{\{F_\alpha\}-[K_{\alpha\beta}]\{d_\beta\}\right\} \quad (2.4.7)$$

For computer programming, several techniques are available for handling boundary conditions. One of the approaches is to make the diagonal element of stiffness matrix corresponding to zero displacement as unity and corresponding all off-diagonal elements as zero. For example, let consider a 3×3 stiffness matrix with following force-displacement relationship.

$$\begin{Bmatrix}F_1\\ F_2\\ F_3\end{Bmatrix}=\begin{bmatrix}k_{11} & k_{12} & k_{13}\\ k_{21} & k_{22} & k_{23}\\ k_{31} & k_{32} & k_{33}\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix} \quad (2.4.8)$$

Now, if the third node has zero displacement (i.e., $d_3 = 0$) then the matrix will be modified as follows to incorporate the boundary condition.

$$\begin{Bmatrix}F_1\\ F_2\\ 0\end{Bmatrix}=\begin{bmatrix}k_{11} & k_{12} & 0\\ k_{21} & k_{22} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix} \quad (2.4.9)$$

Thus, while inverting whole matrix, d_3 will become zero automatically.

To incorporate known support displacement in computer programming following procedure may be adopted. Considering the displacement d_2 has known value of δ, 1st row of eq. (2.4.8) can be written as:

$$F_1 = k_{11}\times d_1 + k_{12}\times d_2 + k_{13}\times d_3 \quad (2.4.10)$$

Or

$$F_1 - k_{12}\times\delta = k_{11}\times d_1 + k_{13}\times d_3 \quad (2.4.11)$$

Now the 2nd row of eq. (2.4.8) has to become:

$$\{\delta\} = \{d_2\} \tag{2.4.12}$$

Similarly 3rd row will be:

$$F_3 - k_{32} \times \delta = k_{31} \times d_1 + k_{33} \times d_3 \tag{2.4.13}$$

Thus above three equations can be written in a combined form as

$$\begin{Bmatrix} F_1 - k_{12}\delta \\ \delta \\ F_{32} - \delta \end{Bmatrix} = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & 1 & 0 \\ k_{31} & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{2.4.14}$$

Another approach may also be followed to take care the known restrained displacements by assigning a higher value δ (say $\delta = 10^{20}$) in the diagonal element corresponding to that displacement.

$$\begin{Bmatrix} F_1 \\ \delta \times 10^{20} \times k_{22} \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} \times 10^{20} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{2.4.15}$$

$$\therefore \delta \times 10^{20} \times k_{22} = k_{21} d_1 + k_{22} \times 10^{20} \times d_2 + k_{23} \times d_3$$

As d_3 is corresponding to zero displacement, the above equation can be simplified to the following.

$$\therefore \delta \times 10^{20} \times k_{22} = k_{21} d_1 + k_{22} \times 10^{20} \times d_2$$

$$\text{or } \delta \times 10^{20} \times k_{22} = k_{22} \times 10^{20} \times d_2$$

$$\Rightarrow d_2 = \delta \rightarrow \text{known displacement is ensured}$$

If the overall stiffness matrix is to be formed in half band form then the numbering of nodes should be such that the bandwidth is minimum. For this the labels are put in a systematic manner irrespective of whether the joint displacements are unknowns or restraints. However, if the unknown displacements are labeled first then the matrix operations can be restricted up to unknown displacement labels and beyond that the overall stiffness matrix may be ignored.