Introduction to Galerkin and Finite Element Methods

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Outline

- Overview of Differential Equations
- The Galerkin Method
- The Finite Element Method

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Overview of Differential Equations

There are two types of differential equations:

- *Ordinary differential equations (ODEs)* are differential equations where the solution has one independent variable. An example is
$$y'(t) = t^2 y(t)$$
- *Partial differential equations (PDEs)* are differential equations where the solution has many independent variables. An example is
$$\frac{\partial u}{\partial x}(x, y) + \frac{\partial u}{\partial y}(x, y) = x^2 + xy$$

Differential equations usually have an associated domain with initial conditions (called *boundary conditions*).

Overview of Differential Equations

PDEs have many applications to real world problems.

- Helmholtz's Equation: Used in electrodynamics

$$-\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) - ku(x,y) = f(x,y)$$

- Heat Equation: Governs distribution of heat

$$\frac{\partial u}{\partial t}(x,y,t) - \frac{\partial^2 u}{\partial x^2}(x,y,t) - \frac{\partial^2 u}{\partial y^2}(x,y,t) = f(x,y,t)$$

- Burgers' Equation: Used for traffic flow

$$\frac{\partial u}{\partial t}(x,t) + u(x,t)\frac{\partial u}{\partial x}(x,t) = 0$$

- Beam Equation: Used in elasticity of materials

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial^4 u}{\partial x^4}(x,t) = 0$$

Overview of Differential Equations

- An *exact solution* to a differential equation is a function that, when substituted into the differential equation, results in a true statement.
- A *numerical solution* to a differential equation is an approximation to an exact solution.

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- The Finite Element Method

The Galerkin Method is very popular for finding numerical solutions to differential equations.

The idea is to approximate the solution to a differential equation by very nice and simple functions.

The Galerkin Method

1. Identify the differential equation to solve, along with its domain and boundary conditions.
2. Identify the vector space in which to look for a solution, called the *solution space*.
3. Rewrite the differential equation in a special way, know as the *weak formulation*.
4. Decide what type of functions are to be used to approximate the solution.
5. Rewrite the weak formulation to reflect these approximating functions.
6. Solve the resulting weak formulation for an approximate solution.

1. Identify the Differential Equation

We will use Helmholtz's Equation in one dimension (an ODE).

$$
\begin{aligned}
-u''(x) - 3u(x) &= cos(x) \quad \text{for } x \in [0,2] \\
u(0) &= 0 \\
u'(2) &= 1
\end{aligned}
$$

2. Identify the Solution Space

Based upon the given differential equation and domain, we use for our solution space the set of all smooth functions with domain $[0, 2]$ that are 0 for $x = 0$. We will call this vector space V.

Note that we want the solution of the differential equation to come from this set. (But it also should satisfy the condition $u'(2) = 1$.)

3. Find the Weak Formulation

Let $v \in V$.

$$\begin{aligned}
-u''(x) - 3u(x) &= cos(x) \\
-u''(x)v(x) - 3u(x)v(x) &= cos(x)v(x) \\
\int_0^2 -u''(x)v(x)dx - \int_0^2 3u(x)v(x)dx &= \int_0^2 cos(x)v(x)dx \\
-v(2) + \int_0^2 u'(x)v'(x) - 3u(x)v(x)dx &= \int_0^2 cos(x)v(x)dx \\
\int_0^2 u'(x)v'(x) - 3u(x)v(x)dx &= \int_0^2 cos(x)v(x)dx + v(2)
\end{aligned}$$

The last line is the weak formulation.

4. Develop Approximating Functions

To approximate the solution, we use a subspace of V.

Consider $\{x, x^2, x^3\}$. We use we use this as a basis for our approximating subspace.

Let $V_3 = span\{x, x^2, x^3\}$. We will also let u_3 denote our approximate solution (in V_3). As such, there exists $\{\alpha_i\}, i = 1, 2, 3$ such that

$$u_3(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$$

5. Update Weak Formulation

To approximate the solution, we replace our solution space V with our approximating solution space V_3. The weak formulation becomes:

$$\begin{aligned}\int_0^2 u'(x)v'(x) - 3u(x)v(x)dx &= \int_0^2 cos(x)v(x)dx + v(2)\\ \int_0^2 u_3'(x)v_3'(x) - 3u_3(x)v_3(x)dx &= \int_0^2 cos(x)v_3(x)dx + v_3(2)\end{aligned}$$

for all $v_3 \in V_3$.

6. Solve the Approximate Weak Formulation

To solve this for our approximation, we start by using

$$u_3(x) = \sum_{i=1}^{3} \alpha_i x^i \quad \text{and} \quad u_3'(x) = \sum_{i=1}^{3} \alpha_i i x^{i-1}$$

This gives us

$$\int_0^2 u_3'(x)v_3'(x) - 3u_3(x)v_3(x)dx = \int_0^2 cos(x)v_3(x)dx + v_3(2)$$

$$\int_0^2 [\sum_{i=1}^{3} \alpha_i i x^{i-1}]v_3'(x) - 3[\sum_{i=1}^{3} \alpha_i x^i]v_3(x)dx =$$

$$\int_0^2 cos(x)v_3(x)dx + v_3(2)$$

6. Solve the Approximate Weak Formulation

The approximate weak formulation can be written as

$$\sum_{i=1}^{3}[\alpha_i \int_0^2 ix^{i-1}v_3'(x) - 3x^i v_3(x)dx] = \int_0^2 cos(x)v_3(x)dx + v_3(2)$$

Since this still holds for all $v_3 \in V_3$, we get three equations by picking three choices for $v_3 \in V_3$: x, x^2, and x^3 (the basis for our approximation space V_3).

6. Solve the Approximate Weak Formulation

$$\sum_{i=1}^{3}[\alpha_i \int_0^2 ix^{i-1} - 3x^i x dx] = \int_0^2 cos(x)x \; dx + 2$$

$$\sum_{i=1}^{3}[\alpha_i \int_0^2 2ix^{i-1}x - 3x^i x^2 dx] = \int_0^2 cos(x)x^2 dx + 4$$

$$\sum_{i=1}^{3}[\alpha_i \int_0^2 3ix^{i-1}x^2 - 3x^i x^3 dx] = \int_0^2 cos(x)x^3 dx + 8$$

The resulting linear system is

$$\begin{aligned} -6\alpha_1 - 8\alpha_2 - 11.2\alpha_3 &= 2.402448 \\ -8\alpha_1 - 8.533333\alpha_2 - 8\alpha_3 &= 4.154008 \\ -11.2\alpha_1 - 8\alpha_2 + 2.742857\alpha_3 &= 7.865929 \end{aligned}$$

Solving gives $u_3(x) = -.350567x - .402926x^2 + .261104x^3$.

Ways to improve accuracy of our approximation

1. Higher degree polynomials
2. Use other functions for a basis

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The Finite Element Method is a Galerkin Method that uses piecewise functions to approximate the solution of a differential equation.

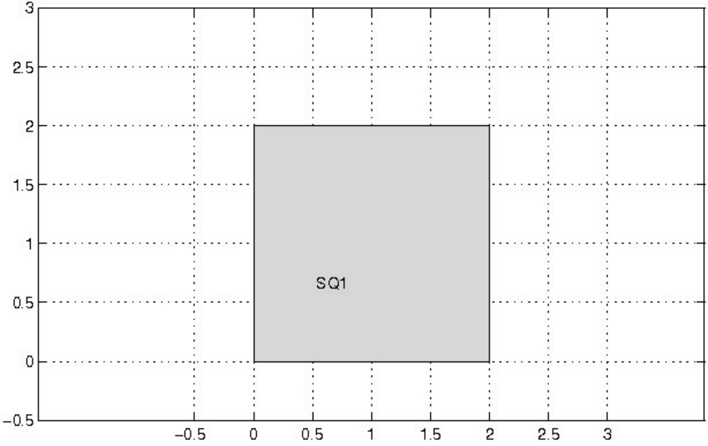
We divide the domain into geometric regions called *elements*. We then form an approximate solution on each of these elements.

As an example, consider Poisson's Equation in 2 variables:

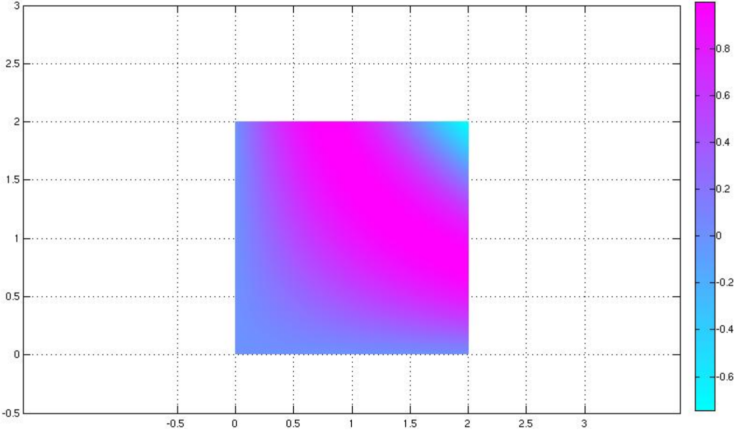
$$\begin{aligned} -\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial^2 u}{\partial y^2}(x,y) &= y^2 sin(xy) + x^2 sin(xy) \\ u(0,y) &= 0 \\ u(x,0) &= 0 \\ u(2,y) &= sin(2y) \\ u(x,2) &= sin(2x) \end{aligned}$$

The solution is given by $u(x,y) = sin(xy)$.

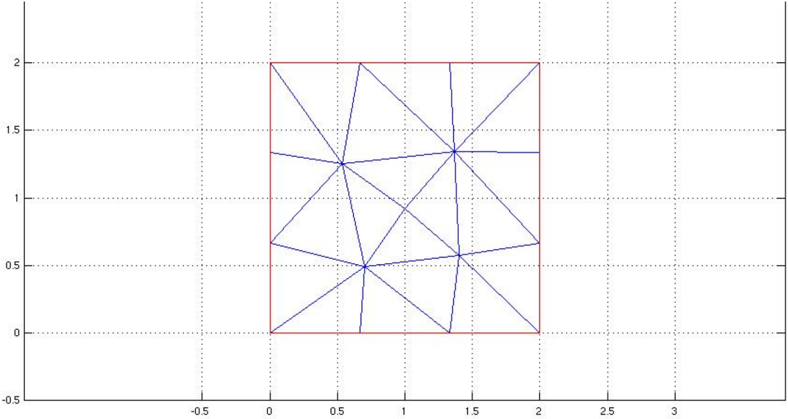
Domain



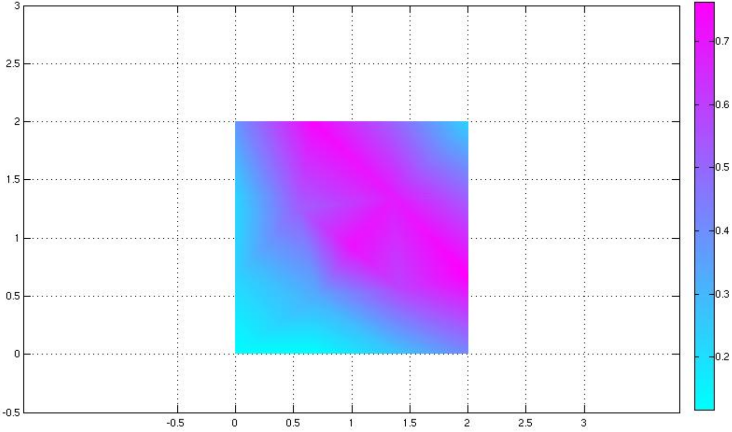
Exact Solution



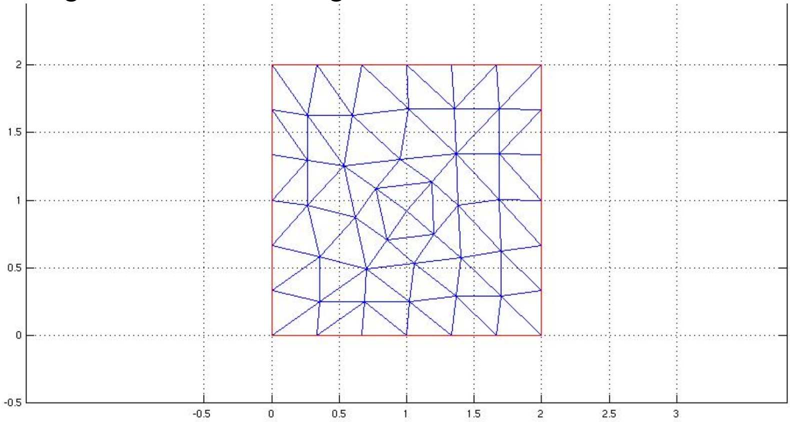
Triangular Mesh - 20 Triangles



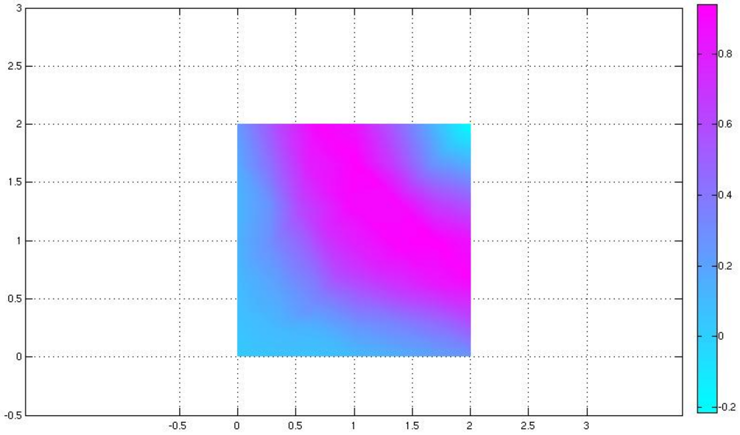
Approximate Solution with 20 Triangles



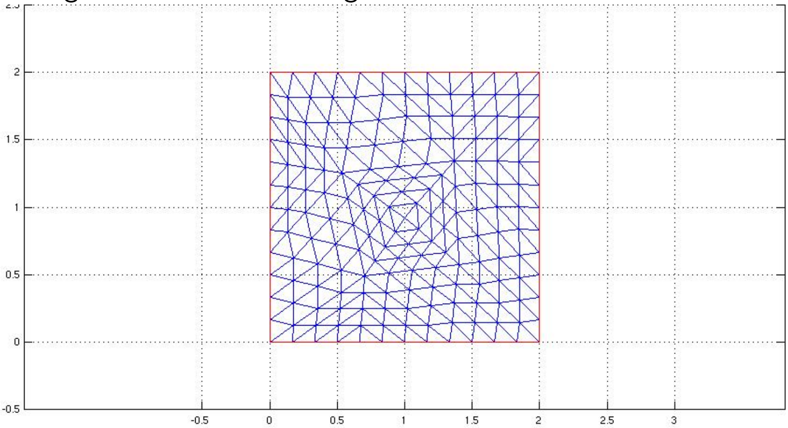
Triangular Mesh - 80 Triangles



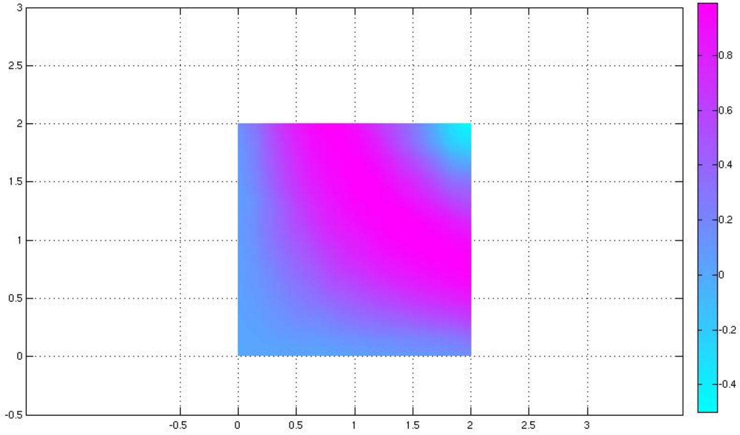
Approximate Solution with 80 Triangles



Triangular Mesh - 320 Triangles



Approximate Solution with 320 Triangles



References

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