The Natural Coordinate System and Its Applications in Atmospheric Physics

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In this lecture the momentum equation for a viscousless fluid on a rotating planet is analyzed in the frame of the natural coordinate system and special flow regimes are described. The natural coordinate system is introduced by means of basic concepts of differential geometry that are applied to parametric curves in the three-dimensional space. The Frenet-Serret formulas are derived and used to project the momentum equation in the scalar components. The geostrophic, the cyclostrophic and the inertial regimes are discussed in detail.

1. Introduction

The physical approach to the reality largely benefits from the mathematical language. Physical laws and concepts are traduced in mathematical relations because the logical and completely objective mathematical frame is a proper environment where physical models can be developed. Furthermore mathematics gives very powerful tools for the exploration of the physical models, both for diagnostic and prognostic purposes. It is well known that the choice of a suitable reference frame, and a coordinate system, makes easier the mathematical analysis of a physical model. In particular the solution of the differential equations that link the causes to the effects are reduced to special equations, which have already been deeply studied and whose solutions are known. In this lecture, the natural coordinate system is defined for the geophysical fluid dynamics purposes. This coordinate system allows the analysis of the momentum equation for a fluid parcel on a rotating planet and it makes easy the study of the motion properties for special, but common, flow regimes.

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2. The momentum equation

Let's consider the momentum equation for a unit volume of fluid whose density is ρ and moving with speed $\boldsymbol{v}$ on a slowly and uniformly rotating planet.

$$\frac{d\boldsymbol{v}}{dt} = -2\boldsymbol{\Omega} \times \boldsymbol{v} - \frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{g} \tag{2.1}$$

Here $\boldsymbol{\Omega}$ is the planet angular velocity vector, p is the pressure field and $\boldsymbol{g}$ is the gravity acceleration vector. Viscosity has been neglected, so the corresponding term does not appear in the 2.1. Centrifugal acceleration ($\boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{r}$) due to planet rotation has considered negligible; in some special cases it can be included in the gravity term. Further details on the momentum equation can be found in literature: Batchelor (1994), (Crisciani 2005, cap. 6), Dutton (1995), Gill (1982), Haltiner & Martin (1957), Holton (1972), Meyer (1982), Pedlosky (1987). This vectorial form of the momentum equation links the causes of the motion of a viscousless fluid to its acceleration and it is not the result of the application of any specific coordinate system . When a coordinate system is selected for the space, the equation (2.1) is equivalent to a set of three scalar equations, one for each of the vector components. In the following we will project the vectorial equation (2.1) in scalar components by means of the the natural coordinate system.

3. Frenet-Serret formulas and the natural coordinate system

The natural coordinate system is defined by means of the trajectory of a fluid parcel. From the geometrical point of view, the trajectory of the parcel is a curve in the three-dimensional space that connects all the positions of the fluid parcel. In particular if $\boldsymbol{r}(t)$ is the vector giving the parcel position at the time t, then the trajectory of the parcel is the function:

$$\boldsymbol{r} : [t_0, t_1] \subseteq R \longrightarrow R^3 \quad \text{and} \quad \boldsymbol{r} \in C^2[t_0, t_1] \tag{3.1}$$

that is the trajectory is a parametric curve which is differentiable and whose derivatives are continuous at least up to the second order on the parameter interval $[t_0, t_1]$. The unit vector $\boldsymbol{\tau}$ that is tangent to the trajectory in each point is defined by means of the fluid velocity vector $\boldsymbol{v} := \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt}$, in fact:

$$\boldsymbol{\tau} := \frac{\boldsymbol{v}}{\parallel \boldsymbol{v} \parallel} \quad \text{if} \quad \boldsymbol{v} \neq 0 \tag{3.2}$$

and it is one of the basic vectors of the natural coordinate system. Since $\boldsymbol{v}$ is a function of the parameter t, that describes the trajectory in the space, $\boldsymbol{\tau}$ itself is a function of the same parameter, so generally $\boldsymbol{\tau}$ is not a constant vector along the trajectory.

Definition 1 (Length of the trajectory).
According to the geometrical meaning of the trajectory of a fluid parcel having velocity $\boldsymbol{v}(t)$, $\boldsymbol{v} : [t_0, t_1] \subseteq R \longrightarrow R^3$, where t is the parameter of the curve $\boldsymbol{r}(t)$, for example the time, the length of the trajectory $l(t)$ is defined as follow:

$$l(t) := \int_{t_0}^{t} (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} \mathrm{d}t' \qquad t \in [t_0, t_1] \tag{3.3}$$

It is worth to note that $l(t) \geqslant 0 \quad \forall t \in [t_0, t_1]$, that is the length of the trajectory is a non negative, monotonic increasing, real function of the parameter t. Furthermore:

$$\dot{l} = \frac{dl}{dt} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} = v(t) \tag{3.4}$$

Corollary 1.
If the parameter of the trajectory is its length, then the velocity vector is the unit vector, $\boldsymbol{\tau}$, tangent to the curve.

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dl} = \frac{d\boldsymbol{r}}{dt}\frac{dt}{dl} = \frac{\boldsymbol{v}}{v} = \boldsymbol{\tau} \tag{3.5}$$

Where the (3.4) has been used to express $\frac{dt}{dl}$. The length of a curve in the space is usually referred as the *natural parameter* of the curve. The reason for that is easy to understand from the property given by 3.5.

The second unit vector of the natural coordinate system is a consequence of an important property of $\boldsymbol{\tau}$, that is shown in the following lemma.

Lemma 1.
Given an euclidean space and a unit vector $\boldsymbol{\tau}$, and $\boldsymbol{\tau}$ is function od a parameter l, $\boldsymbol{\tau} \in C^1[a, b]$ then $\dot{\boldsymbol{\tau}} \perp \boldsymbol{\tau}$.

Proof.
Since:

$$\boldsymbol{\tau} : [a, b] \subseteq R \longrightarrow R^3 \qquad \boldsymbol{\tau} \in C^1[a, b] \qquad \| \boldsymbol{\tau}(l) \| = 1 \quad \forall l \in [a, b]$$

then

$$\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 1 \quad \Rightarrow \quad \frac{d(\boldsymbol{\tau} \cdot \boldsymbol{\tau})}{dl} = 0 \quad \Rightarrow \quad \dot{\boldsymbol{\tau}} \cdot \boldsymbol{\tau} = 0 \quad \forall l \in [a, b] \tag{3.6}$$

This result follows form the the definition of the modulus of a vector in an euclidean space and because of the linearity of the scalar product with respect the differentiation operator $\frac{d}{dl}$.

Definition 2 (Unit vector normal to the trajectory).
Given a trajectory $\boldsymbol{r}(t)$, $\boldsymbol{n}$ the unit vector normal to trajectory is defined as follow:

$$\boldsymbol{n} := \frac{\dot{\boldsymbol{\tau}}}{\| \dot{\boldsymbol{\tau}} \|} \qquad \| \dot{\boldsymbol{\tau}} \| \neq 0 \tag{3.7}$$

Where $\dot{\boldsymbol{\tau}}$ is the unit vector tangent to the trajectory in the point $\boldsymbol{r}(t)$.

Thanks to the vector product operation, it is straightforward to define the unit vector that completes the base for the three-dimensional space where the trajectory is defined. So naturally follows:

Definition 3 (The binormal unit vector).
Given a trajectory of a fluid parcel $\boldsymbol{r}(t)$, for each point where the unit vectors tangent ($\boldsymbol{\tau}$) and normal ($\boldsymbol{n}$) to the trajectory exist, then the **binormal** *unit vector is defined by means of the vector product:*

$$\boldsymbol{b} := \boldsymbol{\tau} \times \boldsymbol{n} \tag{3.8}$$

The unitary of the vector $\boldsymbol{b}$ is a direct consequence of the fact that $\boldsymbol{\tau}$ and $\boldsymbol{n}$ are unit vectors and $\boldsymbol{\tau} \perp \boldsymbol{n}$.

Corollary 2 (The moving trihedral).
When a trajectory of a fluid parcel is given and the unit vectors $\boldsymbol{\tau}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are defined, then they represent a right hand three-dimensional reference frame, so a base for the the space. They are called the moving trihedral of the curve, furthermore $\boldsymbol{\tau}$ and $\boldsymbol{n}$ lay† on the osculating plane‡ of the curve. $\boldsymbol{n}$ is always pointing towards the concavity of the curve††.

Definition 4 (Radius of curvature).
When a trajectory $\boldsymbol{r}(l)$ is given and the parameter of the curve is the length of the trajectory l, then $\| \boldsymbol{v} \| = 1$, because of the COROLLARY 1, and $\dot{\boldsymbol{v}} = k(l)\boldsymbol{n}$, as a consequence of DEFINITION 2. The physical dimension of k is the inverse of a length and it is called the

† See exercise A.1

‡ The osculating plane of a curve is the limit plane that contains the two tangent unit vectors $\boldsymbol{\tau}(t)$ and $\boldsymbol{\tau}(t + \Delta t)$ as $\Delta t \longrightarrow 0$, if the plane exists.

†† See exercise A.2

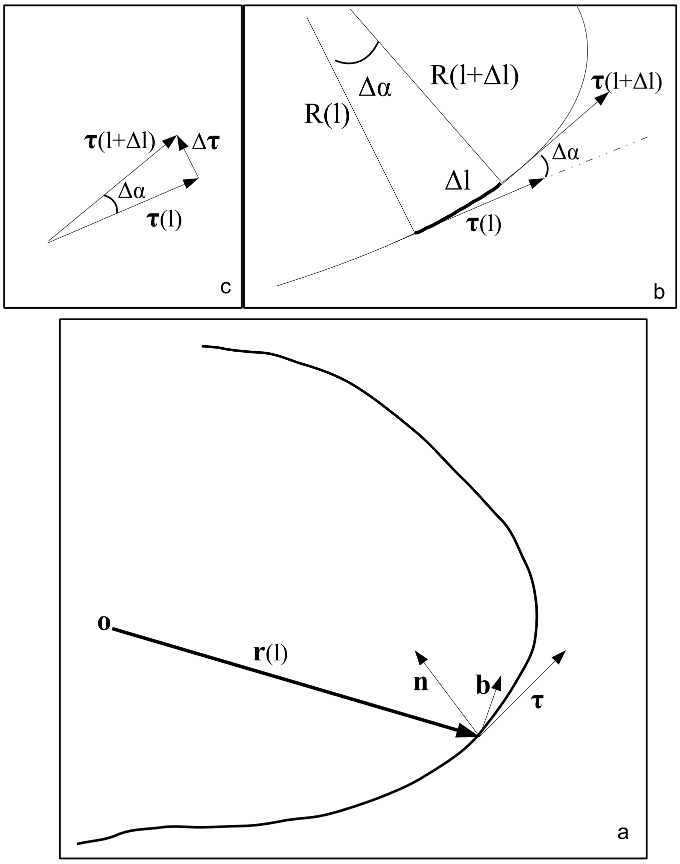

FIGURE 1. a) The trajectory of the fluid parcel with the position vector $\boldsymbol{r}(l)$ and the moving trihedral. b) The radius of curvature R and the unit vector tangent to the trajectory $\boldsymbol{\tau}$ for two different, but very close, position along the trajectory. c) The difference vector $\Delta\boldsymbol{\tau}$ ad its relation with the radius of curvature $\Delta\alpha = \frac{\Delta l}{R(l)}$.

curvature *of the trajectory in the position* $\boldsymbol{r}(l)$*. The* **radius of curvature** *is defined as* $R(l) := 1/k(l) \quad if \quad k(l) \neq 0$.

Note that $k = \| \dot{\boldsymbol{\tau}} \|$, so $k(l) \geqslant 0 \quad \forall\ l$ where the normal unit vector exists. As a consequence, the radius of curvature in not negative too.

The geometrical interpretation of the radius of curvature is straightforward by means of the figure 1. In fact:

$$k(l) = \| \frac{d\boldsymbol{\tau}}{dl} \| = \lim_{\Delta l \to 0} \frac{\| \Delta\boldsymbol{\tau} \|}{\Delta l} = \lim_{\Delta l \to 0} \frac{\frac{\Delta l}{R(l)}}{\Delta l} = \frac{1}{R(l)} \tag{3.9}$$

If $k \to 0$ then $R \to \infty$. Note that the radius of curvature is always positive because $\boldsymbol{n}$ points towards the concavity of the trajectory.

Lemma 2 (Frenet-Serret formulas).
Given the trajectory $\boldsymbol{r}(l)$*, where* l *is the natural parameter, if it is possible to define the moving trihedral,* $(\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b})$*, then the Frenet-Serret formulas hold:*

$$\dot{\boldsymbol{\tau}} = k\boldsymbol{n} \tag{3.10}$$

$$\dot{\boldsymbol{n}} = -k\boldsymbol{\tau} - \chi\boldsymbol{b} \tag{3.11}$$

$$\dot{\boldsymbol{b}} = \chi\boldsymbol{n} \tag{3.12}$$

Where k *is the curvature and* χ *is a real number called the* **torsion** *of the trajectory.*

Proof.
The identity (3.10) follows from the definition of radius of curvature. To demonstrate the (3.12), consider that $\boldsymbol{b}$ is a unit vector (see DEFINITION 3), then, as a consequence of LEMMA 1, $\dot{\boldsymbol{b}} \perp \boldsymbol{b}$. Furthermore:

$$\dot{\boldsymbol{b}} = (\boldsymbol{\tau} \dot{\times} \boldsymbol{n}) = (\dot{\boldsymbol{\tau}} \times \boldsymbol{n}) + (\boldsymbol{\tau} \times \dot{\boldsymbol{n}}) = (k\boldsymbol{n} \times \boldsymbol{n}) + (\boldsymbol{\tau} \times \dot{\boldsymbol{n}}) = \boldsymbol{\tau} \times \dot{\boldsymbol{n}} = \chi\boldsymbol{n} \tag{3.13}$$

where the derivatives are made with respect the natural parameter l. In fact the vector $\boldsymbol{\tau} \times \dot{\boldsymbol{n}}$ is perpendicular to $\boldsymbol{\tau}$ and keeping in mind that $\dot{\boldsymbol{b}} \perp \boldsymbol{b}$, in the three-dimensional space, the only possibility is $\dot{\boldsymbol{b}} \parallel \boldsymbol{n}$. The proof of the identity (3.11) comes from the definition of moving trihedral (COROLLARY 2). In fact, $\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{\tau}$, so:

$$\begin{aligned} \dot{\boldsymbol{n}} &= \dot{\boldsymbol{b}} \times \boldsymbol{\tau} + \boldsymbol{b} \times \dot{\boldsymbol{\tau}} = (\chi\boldsymbol{n} \times \boldsymbol{\tau}) + (\boldsymbol{b} \times k\boldsymbol{n}) = -\chi(\boldsymbol{\tau} \times \boldsymbol{n}) - k(\boldsymbol{n} \times \boldsymbol{b}) \\ &= -\chi\boldsymbol{b} - k\boldsymbol{\tau} \end{aligned} \tag{3.14}$$

The Frenet-Serret formulas are important because they tell us how the moving trihedral is transformed along the trajectory.

$$\boldsymbol{\tau}(l + \Delta l) = \boldsymbol{\tau}(l) + \dot{\boldsymbol{\tau}}\Delta l + o(\Delta l^2) \tag{3.15}$$

$$\boldsymbol{n}(l + \Delta l) = \boldsymbol{n}(l) + \dot{\boldsymbol{n}}\Delta l + o(\Delta l^2) \tag{3.16}$$

$$\boldsymbol{b}(l + \Delta l) = \boldsymbol{b}(l) + \dot{\boldsymbol{b}}\Delta l + o(\Delta l^2) \tag{3.17}$$

And by means of the Frenet-Serret formulas:

$$\begin{aligned} \boldsymbol{\tau}(l + \Delta l) &= \boldsymbol{\tau}(l) + k\Delta l\ \boldsymbol{n} + o(\Delta l^2) \\ \boldsymbol{n}(l + \Delta l) &= \boldsymbol{n}(l) - k\Delta l\ \boldsymbol{\tau} - \chi\Delta l\ \boldsymbol{b} + o(\Delta l^2) \\ \boldsymbol{b}(l + \Delta l) &= \boldsymbol{b}(l) + \chi\Delta l\ \boldsymbol{n} + o(\Delta l^2) \end{aligned}$$

According to the definition of the curvature, the torsion of the trajectory and the geometrical meaning of Δl, both the terms $k\Delta l$ and $\chi\Delta l$ are interpreted as small angles of rotation of the moving trihedral when it moves along the trajectory by the infinitesimal length Δl. The Frenet-Serret formulas can be rewritten by means of the matrix notation and follow:

$$\overset{\mathbf{A}}{\begin{bmatrix} 0 & k & 0 \\ -k & 0 & -\chi \\ 0 & \chi & 0 \end{bmatrix}} \overset{\mathbf{B}}{\begin{bmatrix} \boldsymbol{\tau} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}} = \overset{\dot{\mathbf{B}}}{\begin{bmatrix} \dot{\boldsymbol{\tau}} \\ \dot{\boldsymbol{n}} \\ \dot{\boldsymbol{b}} \end{bmatrix}} \tag{3.18}$$

Note that the matrix $\mathbf{A}$ is antisymetric.

Observation

The moving trihedral $(\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b})$ is not defined if $\dot{\boldsymbol{\tau}} = 0$, furthermore if the concavity of the curve changes along the path, then the vector $\boldsymbol{b}$ changes abruptly its versus, see figure 2. These features of the moving trihedral are disadvantages that limit its application as reference frame for the study of the momentum equation. To fix the problem, restrictions are imposed to the unit normal vector $\boldsymbol{n}$ and the natural coordinate system is defined.

DEFINITION 5 (THE NATURAL COORDINATE SYSTEM).
Given a trajectory $\boldsymbol{r}(t)$, $\quad \boldsymbol{r} : [a,b] \subseteq R \longrightarrow R^3 \quad \boldsymbol{r} \in C^2[a,b]$, the natural coordinate system is defined as follow:

$$\boldsymbol{\tau} := \frac{\dot{\boldsymbol{r}}}{\parallel \dot{\boldsymbol{r}} \parallel} \qquad \parallel \dot{\boldsymbol{r}} \parallel \neq 0 \tag{3.19}$$

$$\begin{cases} \boldsymbol{n} := \frac{\dot{\boldsymbol{\tau}}}{\|\dot{\boldsymbol{\tau}}\|} & \textit{if} \quad \parallel \dot{\boldsymbol{\tau}} \parallel \neq 0 \quad \textit{and} \quad \dot{\boldsymbol{\tau}} \quad \textit{points to the left of} \quad \boldsymbol{\tau} \\ \boldsymbol{n} := -\frac{\dot{\boldsymbol{\tau}}}{\|\dot{\boldsymbol{\tau}}\|} & \textit{if} \quad \parallel \dot{\boldsymbol{\tau}} \parallel \neq 0 \quad \textit{and} \quad \dot{\boldsymbol{\tau}} \quad \textit{points to the right of} \quad \boldsymbol{\tau} \end{cases} \tag{3.20}$$

If for a specific t, $\dot{\boldsymbol{\tau}}(t) = 0$ then $\boldsymbol{n}$ is defined as the unit vectors normal to $\boldsymbol{\tau}$ and pointing to its left that gives continuity to the $\boldsymbol{n}(t)$ function.

$$\boldsymbol{b} := \boldsymbol{\tau} \times \boldsymbol{n} \quad \textit{when} \quad \boldsymbol{\tau} \quad \textit{and} \quad \boldsymbol{n} \quad \textit{are defined} \tag{3.21}$$

Since in the natural coordinate system $\boldsymbol{n}$ is always pointing to the left of $\boldsymbol{\tau}$, then the curvature k, as defined in the DEFINITION 4, is not sufficient to describe the trajectory curvature in the natural coordinate systems, so it is necessary to give a sign to the curvature k as follow:

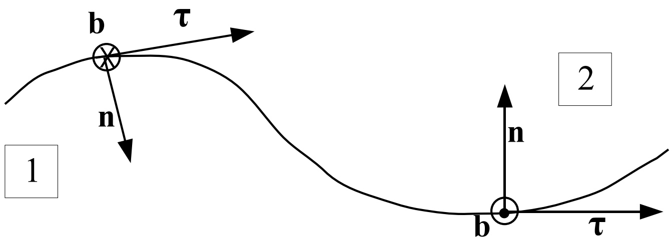

FIGURE 2. The trajectory of the fluid parcel and the disadvantages of the moving trihedral. The normal unit vector $\boldsymbol{n}$ is always pointing towards the concavity of the curve, while the tangent unit vector $\boldsymbol{\tau}$ is along the motion, so when the curvature changes the resulting unit vector $\boldsymbol{b} = \boldsymbol{\tau} \times \boldsymbol{n}$ experiences abrupt inversions. From position 1 to position 2 $\boldsymbol{b}$ experiences an inversion.

COROLLARY 3 (THE SIGN OF THE RARIUS OF CURVATURE).
In the natural coordinate system, given a trajectory $\boldsymbol{r}(l)$, l is the natural parameter of the curve, the curvature is the real number:

$$k := \dot{\boldsymbol{\tau}} \cdot \boldsymbol{n} \qquad \textit{where} \qquad \dot{\boldsymbol{\tau}} = \frac{d\boldsymbol{\tau}}{dl} \tag{3.22}$$

So the curvature k is positive when the concavity of the trajectory is towards the left with respect $\boldsymbol{\tau}$, while it is negative when the concavity is on the opposite side. Consequently the sign of k is hold by the radius of curvature according its definition.

4. The momentum equations in natural coordinates

Let's consider a fluid parcel moving along its trajectory $\boldsymbol{r}(t)$, t is the time and it is the parameter of the curve. Consider the momentum equation (2.1) in the natural coordinate system $(\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b})$. The right hand side of the equation (2.1) is the parcel acceleration that, according with the (3.2), it is:

$$\boldsymbol{v} = v\boldsymbol{\tau} \tag{4.1}$$

and computing of the derivative of $\boldsymbol{v}$ with respect the time, keeping in mind that $\dot{\boldsymbol{v}}$ can be expressed as a function of the natural parameter of the curve l, we get an expression for the acceleration that allows the use of the Frenet-Serret formulas for the $\dot{\boldsymbol{\tau}}$. In fact.

$$\begin{aligned} \dot{\boldsymbol{v}} &= \frac{d\boldsymbol{v}}{dt} = \frac{d\boldsymbol{v}}{dl}\frac{dl}{dt} = v\frac{d\boldsymbol{v}}{dl} = \\ &= v\frac{d(v\boldsymbol{\tau})}{dl} = v(\frac{dv}{dl}\boldsymbol{\tau} + v\frac{d\boldsymbol{\tau}}{dl}) = v(\frac{dv}{dl}\boldsymbol{\tau} + vk\boldsymbol{n}) = \\ &= v\frac{dv}{dl}\boldsymbol{\tau} + v^2k\boldsymbol{n} = \frac{dv}{dt}\boldsymbol{\tau} + v^2k\boldsymbol{n} \end{aligned} \tag{4.2}$$

That is:

$$\dot{\boldsymbol{v}} = \frac{d\boldsymbol{v}}{dt} = \frac{dv}{dt}\boldsymbol{\tau} + v^2 k\boldsymbol{n} \tag{4.3}$$

$$\dot{\boldsymbol{v}} = \frac{d\boldsymbol{v}}{dt} = \frac{dv}{dt}\boldsymbol{\tau} + \frac{v^2}{R}\boldsymbol{n} \qquad R \neq 0 \tag{4.4}$$

The (4.3) expresses the acceleration in natural coordinates by means of the curvature, while the (4.4) make use of the radius of curvature of the trajectory, when it exists.

So in the natural coordinate system the acceleration of a fluid parcel has two contributions. One is along the trajectory tangent direction ($\boldsymbol{\tau}$) and it is related to the velocity modulus rate of change. The second one is perpendicular to the trajectory ($\boldsymbol{n}$) and it is associated with the centripetal force acting on the parcel. Note that k is positive (so is R) if the trajectory is turning to the left of $\boldsymbol{\tau}$, while k (and R too) is negative for right curved trajectories. No contribution of the acceleration is along the $\boldsymbol{b}$ unit vector direction.

The right hand side of the equation (2.1) is projected on the natural coordinate system easily. In general, the gravity acceleration contributes along all the directions ($\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b}$), but if $\boldsymbol{b}$ is normal to the planet surface, or better it is parallel to vertical direction, then $\boldsymbol{g}$ has $\boldsymbol{b}$ component only. This is the typical case of horizontal motions.

The pressure gradient has three components in general:

$$\boldsymbol{\nabla} p = \frac{\partial p}{\partial l}\boldsymbol{\tau} + \frac{\partial p}{\partial n}\boldsymbol{n} + \frac{\partial p}{\partial z}\boldsymbol{b} \tag{4.5}$$

Here l denotes the length of the trajectory, that is the variable describing the displacement along the curve, n is the spacial variable describing the displacement along the direction of the normal unit vector $\boldsymbol{n}$ and z is the variable used for the displacements along the binormal $\boldsymbol{b}$.

The contribution of the Coriolis force is $-2\boldsymbol{\Omega} \times \boldsymbol{v}$ and $-\boldsymbol{\Omega} \times \boldsymbol{v} \perp \boldsymbol{v}$ because of the vectorial product properties As a consequence, the Coriolis acceleration does not act along the trajectory, but operates normally to it, so it does not produce any work and it modifies only the curvature and the torsion of the trajectory.

So:

$$-2\boldsymbol{\Omega} \times \boldsymbol{v} = -a_n\boldsymbol{n} - a_b\boldsymbol{b} \tag{4.6}$$

Where a_n and a_b are the components of the $2\boldsymbol{\Omega} \times \boldsymbol{v}$ to the $\boldsymbol{n}$ and $\boldsymbol{b}$ directions respectively.

A special case is that of an horizontal trajectory, that is a trajectory whose osculating plane is the horizontal plane, which is normal to the vertical direction with respect to the

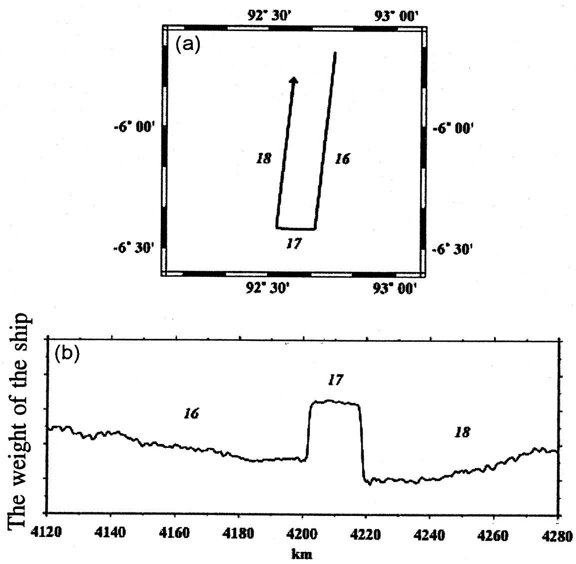

FIGURE 3. Measurements of gravity on a ship sailing in the Pacific ocean. The route is reported (a) on a longitude-latitude box, while the gravity data are plotted as a function of the distance covered by the ship (b) in arbitrary units. The parts of the route parallel to the meridians, 16 and 18 in (a) show a light gravity with respect to the measurements made during the westward part of the route, 17, due to the Coriolis contribution to the acceleration. Figure taken from Persson (2001)

planet surface. In that case the component of the planet angular velocity vector along $\boldsymbol{b}$ is $\boldsymbol{\Omega}\sin\phi$ and ϕ is the latitude of the point $\boldsymbol{r}(t)$ along the trajectory, so the vector $-2\boldsymbol{\Omega}\times\boldsymbol{v}$ becomes:

$$-2\boldsymbol{\Omega}\times\boldsymbol{v} = -2\begin{bmatrix} \boldsymbol{\tau} & \boldsymbol{n} & \boldsymbol{b} \\ \Omega_\tau & \Omega_n & \Omega\sin\phi \\ v & 0 & 0 \end{bmatrix} = -2v\Omega\sin\phi\boldsymbol{n} + 2\Omega_n v\boldsymbol{b} \qquad (4.7)$$

The Ω_n and Ω_τ are the contribution of $\boldsymbol{\Omega}$ along the $\boldsymbol{n}$ and $\boldsymbol{\tau}$ directions respectively, while $\Omega\sin\phi$ is the contribution along the vertical, that is along $\boldsymbol{b}$. The $2\Omega\sin\phi$ therm is often summarized in the Coriolis parameter, that is defined as follow:

$$f := 2\Omega\sin\phi \qquad (4.8)$$

The $2\Omega_n v\boldsymbol{b}$ is along the vertical and, according to the previous considerations on the gravity acceleration it modifies the intensity of $\boldsymbol{g}$. This is known as the Eötvös effect (Persson 2001), and it is usually several orders of magnitude weaker than $\boldsymbol{g}$, anyway it is possible to measure it, see figure 3.

According the these considerations, the momentum equation (2.1) is projected in the three scalar components in the natural coordinate system:

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial l} + g_\tau \qquad \text{equation for} \quad \boldsymbol{\tau} \tag{4.9}$$

$$\frac{v^2}{R} = v^2 k = -a_n - \frac{1}{\rho}\frac{\partial p}{\partial n} + g_n \qquad \text{equation for} \quad \boldsymbol{n} \tag{4.10}$$

$$0 = -a_b - \frac{1}{\rho}\frac{\partial p}{\partial z} + g_b \qquad \text{equation for} \quad \boldsymbol{b} \tag{4.11}$$

If the motion is horizontal then equations (4.9), (4.10), (4.11) by means of the Coriolis parameter (4.8) and the (4.7) become:

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial l} \qquad \text{equation for} \quad \boldsymbol{\tau} \tag{4.12}$$

$$\frac{v^2}{R} = v^2 k = -fv - \frac{1}{\rho}\frac{\partial p}{\partial n} \qquad \text{equation for} \quad \boldsymbol{n} \tag{4.13}$$

$$0 = 2\Omega_n v - \frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad \text{equation for} \quad \boldsymbol{b} \tag{4.14}$$

The gravity acceleration is negative because $\boldsymbol{b}$ is pointing upward.

Considering stationary solutions of the equation of motion, that is $\frac{d\boldsymbol{v}}{dt} = 0$, it is possible to get simple models that are of useful application. According to this assumption, the $\boldsymbol{\tau}$ component of the fluid velocity is constant, see equation (4.12), so the important information on the fluid velocity comes from the equation (4.13). From the study of that equation and making some additional assumptions it is possible to define three special flow regimes: the inertial motion, the cyclostrophic motion and the geostrophic motion.

5. The inertial motion

The inertial motion occurs when:

$$\frac{\partial p}{\partial l} = 0 \qquad \text{and} \qquad \frac{\partial p}{\partial n} = 0 \tag{5.1}$$

that is there are not horizontal pressure gradients. This motion is typical of oceanic currents when there is an atmospheric forcing on the water surface pressure that vanish in a short time. For example the passage of a rapid frontal system over a sea basin that creates temporary pressure gradients in the water, which are removed rapidly when the front is over. When the gradients disappear the water gets the equilibrium state through the inertial motion. In the atmosphere the inertial motion is almost never observed because the synoptic, mesoscale and microscale forcings generate always pressure gradients, but in principle it exists.

By means of the equations (4.12) and (4.13) the constant value of the fluid velocity in the inertial motion is:

$$v = \frac{f}{k} \quad k \neq 0 \qquad \text{or similarly} \qquad v = -fR \tag{5.2}$$

and since $v \geqslant 0$ this means that

$$fR \leqslant 0 \tag{5.3}$$

In the northern hemisphere $f > 0$ because $\phi > 0$, see (4.8), so $R < 0$ and the motion is anticyclonic, clockwise motion. While in the southern hemisphere $\phi < 0$ and consequntly $f < 0$, so $R > 0$ and the motion is cyclonic, counterclockwise motion.

In the inertial motion there is the balance between the Coriolis force and the centrifugal force. For small scale motions where f can be considered constant, the trajectory of the fluid parcel is a circle because the radius of curvature R is constant too, as a consequence of the stationarity of the motion. It is possible to compute the period T of the circular motion:

$$T = \left| \frac{2\pi R}{v} \right| \tag{5.4}$$

and by means of (5.2) in the (5.4) and recalling that $\Omega = \frac{2\pi}{D}$, where D in the rotation period of the planet,

$$T = \frac{D}{2 \mid \sin\phi \mid} \tag{5.5}$$

It is suggested to compute the radius of curvature of the inertial motion, and the related period of the motion, for typical atmospheric and oceanic horizontal velocities at mid latitudes.

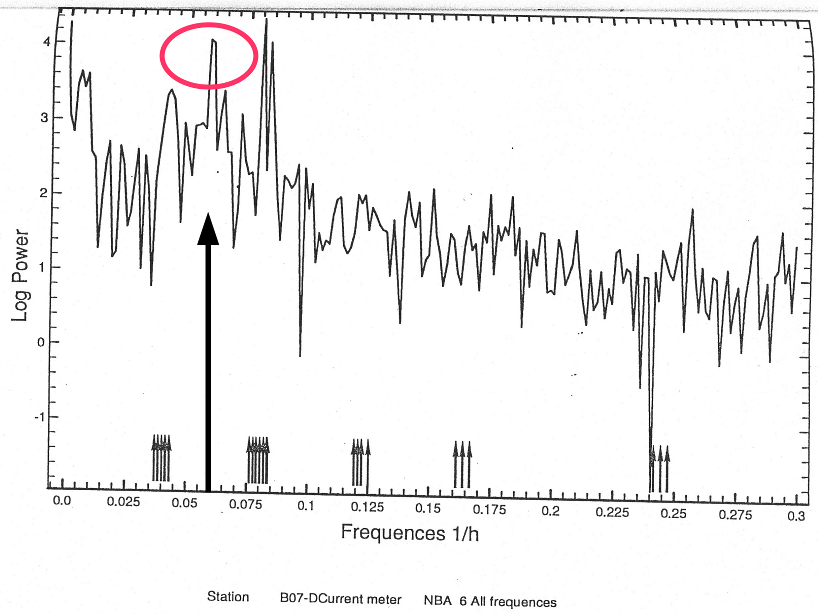

FIGURE 4. Power spectra of the North-South component of the horizontal vector for Adriatic sea currents of the time series measurements made in the frame of the PRISMA project by Osservatorio Geofisico Sperimentale (Trieste ? ITALY), during the period [1995-07-12; 1995-08-18], by a NBA 6 current meter moored 23 m below the sea surface, at the geographical position $44.382^\circ\ N, 13.189^\circ\ E$, where the bottom depth is 50 m, (Giaiotti & Ursella 1996). Tide contributions to the currents are marked with small arrows and the peak of the power corresponding to the typical frequency of the inertial motion ($1/17\ h^{-1}$) is marked with a big arrow and a circle.

In figure 4 is presented the power spectra of the North-South component of an horizontal vector sea current time series. The current was measured in the Adriatic sea at about half the bottom depth of the basin (Giaiotti & Ursella 1996). The geographical position of the Adriatic basin together with its characteristic extended shape along the North-South direction, make the inertial currents to be a significant component of the Adriatic currents. According to the (5.5) and the mid latitudes position of the northern part of the basin, to which the measures here reported refer to (Giaiotti & Ursella 1996), the rotation period of the inertial currents is about 17 hours. In figure 4 the tide contributions to the currents are marked with arrows, furthermore it is evident the peak of the power corresponding to the typical frequency of the inertial motion ($1/17\ h^{-1}$).

6. The cyclostrophic motion

The cyclostrophic motion occurs when:

$$\frac{\partial p}{\partial l} = 0 \qquad \text{and} \qquad kv^2 \gg fv \tag{6.1}$$

that is there are not pressure gradients along the direction of motion, while the pressure gradient is normal to the trajectory and it is balanced by the centrifugal force; the Coriolis force is negligible. So the basic equation for the cyclostrophic motion is:

$$\frac{v^2}{R} = v^2 k = -\frac{1}{\rho}\frac{\partial p}{\partial n} \tag{6.2}$$

To characterize this flow regime it is useful to introduce the Rossby number, which is an adimensional parameter defined as follow:

$$\Re := \frac{v}{R\,f} = \frac{v\,k}{f} \tag{6.3}$$

The Rossby number expresses the ratio between the centrifugal force and the Coriolis force, so the condition for the cyclostrophic flow regime can be expressed as follow:

$$\Re \gg 1 \tag{6.4}$$

It is worth to note that since $v^2 \geqslant 0$ and $\rho > 0$ always along the trajectory, then follows that:

$$R\frac{\partial p}{\partial n} \leqslant 0 \tag{6.5}$$

then the flow is allowed to be cyclonic, counterclockwise, or anticyclonic, clockwise, but in any case the radius of curvature is pointing toward the low pressure area.

Example of this motion is the tornado. In a tornado wind speed easily reach 50 ms^{-1} and the typical radius of the vortex is of the order of 100 m or less, lets say 10 m. As an exercise, the student is suggested to compute the difference of atmospheric pressure between the core of the tornado, the center of the air parcel trajectory, and the outer part of the vortex, where the pressure is almost the unperturbed atmosphere pressure. The result will give the answer to question concerning the upward aspiration of stuff by tornadoes. The principal cause for damages during a tornado event is the strong horizontal wind speed.

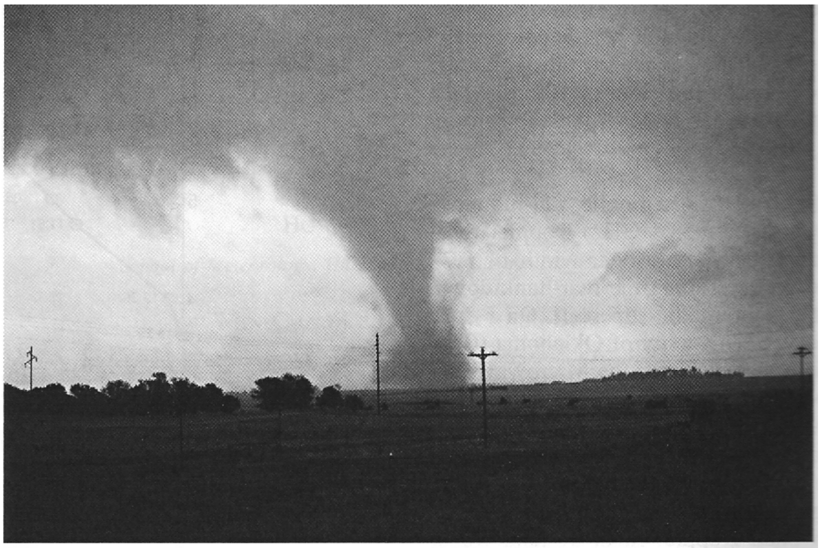

FIGURE 5. Picture of the tornado. The tornado occurred in Bassett, Nebraska, on June 1999. Figure taken form the Bluestein *et al.* (2003) paper.

Take your notes here below

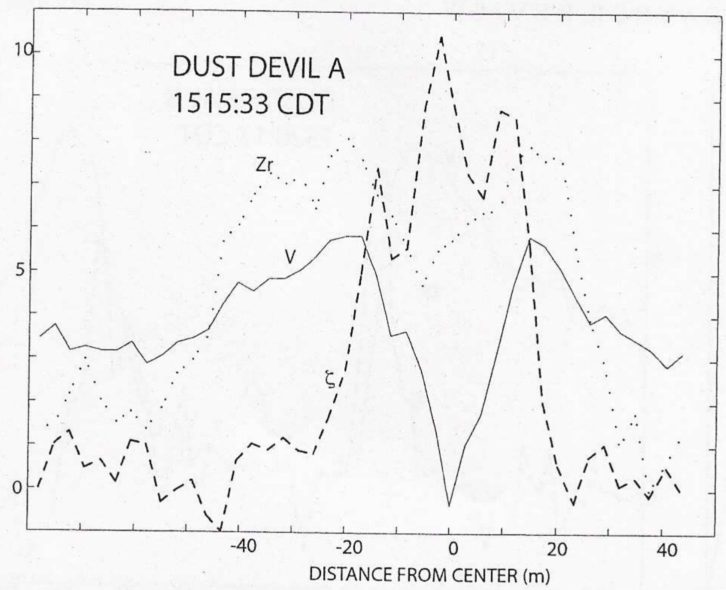

FIG. 9. Approximate vortex-relative azimuthal wind component V (solid line; m s^{-1}), vorticity ζ (dashed line; $\times 10$ s^{-1}), and relative radar reflectivity $Zr = (Z - Z_{noise})/C$, where Z is the reflectivity factor (dBZ), Z_{noise} (dBZ) is a subjectively determined approximate noise floor, and C is a dimensionless compression factor chosen subjectively so that the range of relative reflectivity remains within the scale of the figure, in dust devil A, at 1515:33 CDT, as a function of distance from the center of the vortex (m). Negative (positive) distances are measured to the left (right) of the vortex center, as viewed by the radar. The vorticity at the center is estimated as the average of the vorticities computed just to the right and left of the center.

FIGURE 6. Radial profiles for a dust devil occurred in Texsas on May 1999 and observed by means of a mobile Doppler radar. The solid black line reports the tangential velocity of the vortex in ms^{-1}. The vertical component of the vorticity is plotted in dashed line and it is expressed in units of $\times 10\, s^{-1}$. The dotted line represents the relative radar reflectivity; this variable is not relevant in the context of this lecture so it is not describe in this caption. Picture taken from the Bluestein & Weiss (2004) paper.

Take your notes here below

7. The geostrophic motion

The geostrophic motion occurs when:

$$\frac{\partial p}{\partial l} = 0 \qquad \text{and} \qquad fv \gg kv^2 \tag{7.1}$$

that is there are not pressure gradients along the direction of motion, while there pressure normal to the trajectory is balanced by the Coriolis force and the centrifugal force is negligible.

In this case the Rossby number becomes:

$$\Re \ll 1 \tag{7.2}$$

So the basic equation for the geostrophic motion is:

$$0 = -fv - \frac{1}{\rho}\frac{\partial p}{\partial n} \tag{7.3}$$

It is worth to note that since $v \geqslant 0$ and $\rho > 0$ always along the trajectory, then follows that:

$$\frac{1}{f}\frac{\partial p}{\partial n} \leqslant 0 \tag{7.4}$$

the flow is allowed to be cyclonic, counterclockwise, around the low pressure areas in the northern hemisphere because $f > 0$, and anticyclonic, clockwise, around the high pressure areas. In the southern hemisphere $f < 0$ then the air mass motion is revesed: cyclonic around the highs and and anticyclonic around the lows.

Example of this motion is the wind regime in mid latitude cyclonic and anticyclonic synoptic systems.

Appendix A. Exercises

A.1. *Exercise 1*

Consider a trajectory $\boldsymbol{r}(t)$ and consider a point of the trajectory where the unit vectors $\boldsymbol{\tau}$ and $\boldsymbol{n}$ exist. Demonstrate that the osculating plane exists and the vectors $\boldsymbol{\tau}$ and $\boldsymbol{n}$ lay on the plane.

A.2. *Exercise 2*

Consider a trajectory $\boldsymbol{r}(t)$ and consider a point of the trajectory where the unit vectors $\boldsymbol{n}$ exists. Demonstrate that the $\boldsymbol{n}$ points always towards the concavity of the trajectory.

A.3. *Exercise 3*

Compute the radius of curvature for an inertial motion for an oceanic current of about $25\ cm\,s^{-1}$ at mid latitudes and compare it with those possible at $15°N$, $15°S$ and $65°N$.

Appendix B. Historical notes

B.1. *Jean Frédéric Frenet*

Born: 1816 in Périgueux, France

Died: 1900 in Périgueux, France

Jean Frédéric Frenet was a mathematician, astronomer, and meteorologist. He is best known for being an (independent) co-discoverer of the Frenet-Serret formulas. He wrote six out of the nine formulas, which at that time were not expressed in vector notation, nor using linear algebra. These formulas are important in the theory of space curves (differential geometry), and they were presented in his doctoral thesis at Toulouse in 1847. That year he became a professor at Toulouse, and one year later, 1848, he became professor of mathematics at Lyon. He also was director of an astronomical observatory at Lyon. Four years later, in 1852, he published the Frenet formulas in the Journal de mathmatiques pures at appliques. In 1856 his calculus primer was first published, which ran through seven editions, the last one published posthumously in 1917.

B.2. *Joseph Serret*

Born: 1819 in Paris, France

Died: 1885 in Versailles, France

Joseph Serret graduated from the École Polytechnique in Paris in 1840. He became an entrance examiner for the Ècole Polytechnique in 1848. In 1861 he became professor of celestial mechanics at Collge de France, then two years later he was appointed to the chair of differential and integral calculus at the Sorbonne. Serret joined the Bureau des Longitudes in 1873.

Serret did important work in differential geometry. Together with Bonnet and Bertrand he made major advances in this topic. The fundamental formulae in the theory of space curves are the Frenet-Serret formulae.

In 1860 Serret succeeded Poinsot in the Acadmie des Sciences. In 1871 he retired to Versailles as his health began to deteriorate.

Serret also worked in number theory, calculus and mechanics. He edited the works of Lagrange which were published in 14 volumes between 1867 and 1892. He also edited the 5th edition of Monge in 1850.

B.3. *Roland Eötvös*

Born: 1848 in Budapest, Hungary

Died: 1919 Hungary

Roland von Eötvös, was born in 1848, the year of the Hungarian revolution. He first studied law, but soon switched over to physics and went abroad to study in Heidelberg and Ksönigsberg. After his doctorate he became a university professor in Budapest very soon, and played a leading part in Hungarian science for almost half a century. He gained international recognition first by his innovative work on capillarity, then by his refined experimental methods and extensive field studies in gravity. He died in 1919, but his last and probably most important paper, written together with his colleagues D. Pekr and J. Fekete, was published only in 1922.

Eötvös' classic experiment concerning the proportionality of inertial and gravitating masses, performed at first in 1889, has again become the focus of scientific interest in the 1980's, due to the possibility of the existence of a Fifth Force. Inspired by the beauty of the Newtonian system, he experimentally investigated the proportionality of inertial and gravitating masses in 1889, and reported his results in the Proceedings of the Hungarian Academy in 1890.

In his life he was a well-known and respected personality of the Hungarian and the international scientific community. His name is often related to the establishment of a new sience, the Geophysics. His name denotes a physical unit

Most people were fascinated by his colorful personality. He often said, he was prouder of his sport successes than his scientific descoveries. The mountaineering "Hungarian professor" was so popular in South Tirol, that a peak in the Dolomites was named after him. He was an enthusiastic Photographer. He made hundreds of stereoscopic slides on his mountaineering and scientific expeditions.

In 1873 he became associate member of the Academy, then full member in 1883, and in 1889 he was elected president. Amongst his offices he became minister of religion and education for seven months in 1894. In his inaugural speech as minister he addressed the ministerial staff as follows:

"*We must strive, gentlemen, to make the field of public education a true garden of flowers. To achieve this aim we must first create order in the garden, so that every plant has its place. It is also necessary that each one receives the right nourishment, the soil* *and air that will allow its full development. In short, we have just two things we must do here, to make order and then to help. And gentlemen, I would like us to give more and more assistance and show more tolerance in our regulations.*”

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