9 Numerical Integration in Natural Coordinates

9.1 Purpose

Our purpose is to show how we can integrate the element stiffness matrices numerically using the Gauss-Lagrandre integration technique.

In general, we will not integrate element stiffness matrices directly because, in most cases, there is no simple closed form integral. Since shape functions are given in natural coordinates, we need to map our real coordinates to the natural coordinates. We will then demonstrate integration in the natural coordinate system. We derive the mapping first, then implement numerical integration. Integration for both 1D and 2D will be discussed.

9.2 Change of Variables

From calculus, the change of variables for integration is:

$$\int_a^b f(x)dx = \int_{\zeta_1}^{\zeta_2} f(g(\zeta))\left[\frac{dg(\zeta)}{d\zeta}\right]d\zeta \tag{9-1}$$

where:

$$x = g(\zeta) \qquad \text{and} \qquad g(\zeta_1) = a, \quad g(\zeta_2) = b$$

Here we use our 1-D example as the example to illustrate how the integration is done. We want to integrate the element stiffness matrix (which was computed analytically using both the real and natural coordinates at the end of Lecture 6). The relation between the stiffness matrix in real coordinates and the natural coordinates is given by:

$$\left[\boldsymbol{K^e}\right] = AE\int_0^{L_e} \lfloor \boldsymbol{B}(x)\rfloor^{\boldsymbol{T}} \lfloor \boldsymbol{B}(x)\rfloor dx = AE\int_{-1}^{1} \lfloor \boldsymbol{B}(\zeta)\rfloor^{\boldsymbol{T}} \lfloor \boldsymbol{B}(\zeta)\rfloor \left(\frac{dx}{d\zeta}\right)d\zeta \tag{9-2}$$

We have already derived the shape functions and their derivatives in terms of the natural coordinate, but we still need to define the relationship between x and ζ. We do this by using the same shape functions we used for interpolating the displacement. For a linear (two-noded) one-dimensional element:

$$x = N_I x_I + N_J x_J \tag{9-3}$$

where the shape functions are

$$N_I = \frac{1}{2}(1-\zeta) \qquad \text{and} \qquad N_J = \frac{1}{2}(1+\zeta)$$

Then:

$$\frac{dx}{d\zeta} = \frac{dN_I}{d\zeta} x_I + \frac{dN_J}{d\zeta} x_J \tag{9-4}$$

Substituting the shape functions gives

$$\frac{dx}{d\zeta} = \left(\frac{1}{2}(-1)\right) x_I + \left(\frac{1}{2}(1)\right) x_J = \frac{L_e}{2} \tag{9-5}$$

To integrate the element stiffness matrix of (9-2), we first need to obtain an expression for the derivative (with respect to x) of the shape functions expressed in natural coordinates. We will illustrate this using the first shape function:

$$\frac{dN_I(\zeta)}{d\zeta} = \frac{dN_I(\zeta)}{dx} \frac{dx}{d\zeta} \tag{9-6}$$

We can evaluate the left hand side and the second term on the right hand side of 9-6:

$$\left(\frac{1}{2}(-1)\right) = \frac{dN_I(\zeta)}{dx} \frac{L_e}{2} \tag{9-7}$$

or:

$$\frac{dN_I(\zeta)}{dx} = \left[\frac{L_e}{2}\right]^{-1} \left(\frac{-1}{2}\right) = -\frac{1}{L_e} \tag{9-8}$$

We will look at the first term in the matrix:

$$AE \int_{-1}^{1} B_I(\zeta) B_I(\zeta) \left(\frac{dx}{d\zeta}\right) d\zeta = AE \int_{-1}^{1} \left(\frac{-1}{L_e}\right)\left(\frac{-1}{L_e}\right) \frac{L_e}{2} d\zeta = \frac{AE}{L_e} \tag{9-9}$$

This is the same value we obtained in Lecture 6. Although a simple example, the same concept applies to more complicated integrals.

Question 1: What are the possible advantages of changing variables in calculating an integral?

9.3 Gauss-Legendre Integration

Numerical integration involves calculating a function at selected locations, multiplying the value by a weight, and then summing to obtain the integral, *i.e.*

$$\int_{-1}^{1} f(\zeta) d\zeta \approx \sum_{i=1}^{N_{gp}} w_i f(\zeta_i) \tag{9-10}$$

Let us illustrate. Figure 9-1 shows a linear function over the interval -1 to 1.

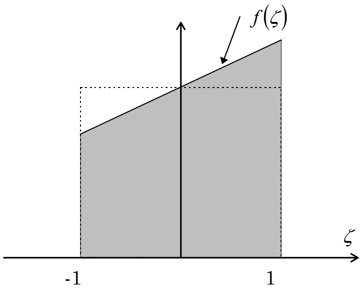

Figure 9-1: Integration of linear function

Clearly, the linear function can be evaluated exactly using one point:

$$\int_{-1}^{1} f(\zeta)d\zeta = \sum_{i=1}^{1} w_i f(\zeta_i) = 2f(0)$$

That is, if we evaluate the function at $\zeta = 0$ and multiply that value by 2, we will obtain the exact integral.

If the function is of higher order than linear, the one-point numerical integration is only approximate. We can increase the accuracy by selecting more integration points. Many numerical integration schemes use evenly spaced evaluation locations such as the Simpson's formula and the Newton-Cotes formula. The *Gauss-Legendre integration* (also called as *Gaussian quadrature*), however, evaluates the integral at cleverly picked locations and weights leading to high accuracy at low cost. This is illustrated in Figure 9-2.

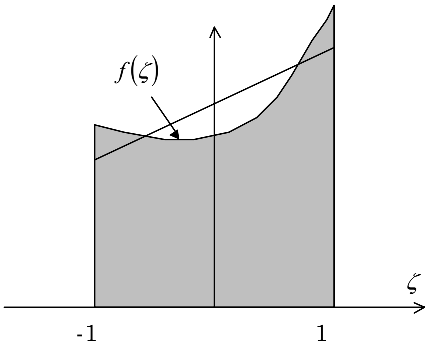

Figure 9-2: Sketch showing integration of higher order curve

We will derive the equations for Guass-Legendre integration for a cubic polynomial. To do this, we let both the weights and the locations at which we will evaluate the function be unknown. We also want to know how many points we need in order to obtain an exact integration. Let us assume that our integrand, a cubic polynomial, can be expressed as

$$f(\zeta) = a + b\zeta + c\zeta^2 + d\zeta^3 \tag{9-11}$$

where *a, b, c* and *d* are arbitrary constants which in general are non-zero. Assume we would need *n* Gauss points to achieve an exact integration. The exact integration would be

$$\int_{-1}^{1} f(\zeta)d\zeta = a\zeta\Big|_{-1}^{1} + \frac{1}{2}b\zeta^2\Big|_{-1}^{1} + \frac{1}{3}c\zeta^3\Big|_{-1}^{1} + \frac{1}{4}d\zeta^4\Big|_{-1}^{1} = 2a + 0 + \frac{2}{3}c + 0 \qquad (9\text{-}12)$$

A numerical integration using *n* Gauss points, called *n*-point Gaussian integral, would give

$$\int_{-1}^{1} f(\zeta)d\zeta = \sum_{i=1}^{n} w_i f(\zeta_i) = \sum_{i=1}^{n} w_i f(\zeta_i) = \sum_{i=1}^{n} w_i \left(a + b\zeta_i + c\zeta_i^2 + d\zeta_i^3\right) \qquad (9\text{-}13)$$

We require that (9-13) be exact for any polynomial up to degree 3. That is, for any coefficients *a, b, c*, and *d*, (9-11) and (9-13) should give the same result. This would lead to following requirements for the *n* pairs of weights (w_i) and the locations (ζ_i):

$$2 = \sum_{i=1}^{n} w_i$$

$$0 = \sum_{i=1}^{n} w_i\zeta_i$$

$$\frac{2}{3} = \sum_{i=1}^{n} w_i\zeta_i^2$$

$$0 = \sum_{i=1}^{n} w_i\zeta_i^3$$

This is a set of 4 equations for 2*n* unknowns. Hence, we should be able to solve for *n* = 2. That is, 2 Gaussian points will be capable of providing exact numerical integration for any polynomial up to degree 3. The above equations can be explicitly written as

$$2 = w_1 + w_2 \qquad (9\text{-}14)$$

$$0 = w_1\zeta_1 + w_2\zeta_2 \qquad (9\text{-}15)$$

$$\frac{2}{3} = w_1\zeta_1^2 + w_2\zeta_2^2 \qquad (9\text{-}16)$$

$$0 = w_1\zeta_1^3 + w_2\zeta_2^3 \qquad (9\text{-}17)$$

We can now solve for these four unknowns. First, we use 9-15 and 9-17. Arranging in matrix form:

$$\begin{bmatrix} \zeta_1 & \zeta_2 \\ \zeta_1^3 & \zeta_2^3 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (9\text{-}18)$$

For this to be true, since the weights cannot be zero at the same time, the determinant must equal zero. That is,

$$\zeta_1\zeta_2{}^3 - \zeta_1{}^3\zeta_2 = \zeta_1\zeta_2\left(\zeta_2^2 - \zeta_1^2\right) = 0 \qquad (9\text{-}19)$$

for $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$:

$$\zeta_2^2 - \zeta_1^2 = 0 \tag{9-20}$$

For two distinct points:

$$\zeta_2 = -\zeta_1 \tag{9-21}$$

Using this information in 9-15 gives:

$$w_1 = w_2 \tag{9-22}$$

Then, using 9-14 gives:

$$w_1 = w_2 = 1 \tag{9-23}$$

and using 9-16 gives:

$$\zeta_1 = -\zeta_2 = \frac{1}{\sqrt{3}} \tag{9-24}$$

Finally, we get:

$$\int_{-1}^{1} f(\zeta)\, d\zeta = 1 f\left(\frac{1}{\sqrt{3}}\right) + 1 f\left(\frac{1}{\sqrt{3}}\right) \tag{9-25}$$

That is, we can obtain the exact integral for a polynomial of up to order $N = 3$ using two Gauss points with:

$$w_1 = w_2 = 1$$

and

$$\zeta_1 = \frac{1}{\sqrt{3}}, \quad \zeta_2 = -\frac{1}{\sqrt{3}}$$

For polynomials of order greater than three, the numerical integral with 2 Gauss points is only approximate.

Question 1: Why is Gaussian integration efficient?

9.4 Examples

To illustrate, let us integrate the following function:

$$f(\zeta) = 100 + 50\zeta + 75\zeta^2 \tag{9-26}$$

The exact result is:

$$\int_{-1}^{1} f(\zeta)d\zeta = 250 \tag{9-27}$$

Using our derived Gauss method:

$$\int_{-1}^{1} f(\zeta)d\zeta = (1)f\left(-\frac{1}{\sqrt{3}}\right) + (1)f\left(\frac{1}{\sqrt{3}}\right) \tag{9-28}$$

Evaluating gives:

$$\int_{-1}^{1} f(\zeta)d\zeta \cong (1)\left[100 + 50\left(-\frac{1}{\sqrt{3}}\right) + 75\left(-\frac{1}{\sqrt{3}}\right)^2\right] + (1)\left[100 + 50\left(\frac{1}{\sqrt{3}}\right) + 75\left(\frac{1}{\sqrt{3}}\right)^2\right] \tag{9-29}$$

or

$$\int_{-1}^{1} f(\zeta)d\zeta = 250 \tag{9-30}$$

This is the exact result, as expected, since the order of the function being integrated was 2.

As stated, if the function is of order $N > 3$, the 2-point Gauss numerical integral will be approximate. For example, if we the function is:

$$f(\zeta) = \cos(\zeta) \tag{9-31}$$

The exact integral from -1 to 1 is:

$$\int_{-1}^{1} \cos(\zeta)d\zeta = 1.682942 \tag{9-32}$$

While the approximate integral using two Gauss points is:

$$\int_{-1}^{1} \cos(\zeta)d\zeta = 1.675823 \tag{9-33}$$

Thus, the numerical integral is approximate. Using more Gauss points would lead to a more accurate numerical integral.

Question 2: For one-dimensional, 3-noded quadratic elements, to obtain accurate element stiffness matrix, how many integration points are needed?

9.5 Higher Order Gauss-Legendre Integration

Table for higher order integration using more Gauss points are given in many books. Table 9-1 gives values for up to three Gauss points. A polynomial of order $N \le 2n-1$, where

n is the number of Gauss points, will be integrated exactly.

n	ζ_i	w_i
1	0.0	2.0
2	±0.577350269189626 (or $1/\sqrt{3}$)	1.0
3	±0.774596669241483 (or $\sqrt{0.6}$) 0.0	0.555555555555555 (or 5/9) 0.888888888888889 (or 8/9)

Table 9-1: Gauss points and weights for Gauss-Legendre integration

9.6 Application of Numerical Integration to Evaluation of Element Matrices

We will now demonstrate how numerical integration can be used to calculate an element stiffness matrix. We previously changed variables for the element stiffness matrix in (9-2):

$$\left[\boldsymbol{K^e}\right] = AE\int_0^{L_e} \lfloor \boldsymbol{B} \rfloor^{\boldsymbol{T}} \lfloor \boldsymbol{B} \rfloor dx = AE\int_{-1}^{1} \lfloor \boldsymbol{B}(\zeta) \rfloor^{\boldsymbol{T}} \lfloor \boldsymbol{B}(\zeta) \rfloor \left(\frac{dx}{d\zeta}\right) d\zeta \tag{9-34}$$

We have also shown how we can implement a numerical integration scheme where we evaluate the function at a given number of points and multiply the values by a weight and then sum the results:

$$\int_{-1}^{1} f(\zeta)\, d\zeta = \sum_{i=1}^{n} w_i f(\zeta_i) \tag{9-35}$$

Applying (9-35) to (9-34) gives:

$$\left[\boldsymbol{K^e}\right] \cong \sum_{i=1}^{n} w_i \left[AE \lfloor \boldsymbol{B}(\zeta_i) \rfloor^{\boldsymbol{T}} \lfloor \boldsymbol{B}(\zeta_i) \rfloor \left(\frac{dx(\zeta_i)}{d\zeta}\right) \right] \tag{9-36}$$

This maybe looks a bit complicated, but all it means is that we will evaluate the term in square brackets at however many Gauss points we are using, multiply by the weights, and then summing the result.

Recall, for a linear element:

$$\lfloor \boldsymbol{B}(\zeta) \rfloor = \left\lfloor \left(\frac{dN_I(\zeta)}{dx}\right) \quad \left(\frac{dN_J(\zeta)}{dx}\right) \right\rfloor \tag{9-37}$$

and we have already shown in (9-8) that:

$$\frac{dN_I(\zeta)}{dx} = \left(\frac{-1}{L^e}\right) \tag{9-38}$$

Similarly:

$$\frac{dN_J(\zeta)}{dx} = \left(\frac{1}{L^e}\right) \tag{9-39}$$

Also, in (9-5) we derived that (for a linear element):

$$\frac{dx}{d\zeta} = \frac{L^e}{2} \tag{9-40}$$

Substitutin (9-37) through (9-40) into (9-36), we obtain:

$$\left[\boldsymbol{K^e}\right] \cong \sum_{i=1}^{n} w_i \left[AE \begin{Bmatrix} \left(\frac{-1}{L^e}\right) \\ \left(\frac{1}{L^e}\right) \end{Bmatrix} \left\lfloor \left(\frac{-1}{L^e}\right) \quad \left(\frac{1}{L^e}\right) \right\rfloor \left(\frac{L^e}{2}\right) \right] \tag{9-41}$$

or:

$$\left[\boldsymbol{K^e}\right] \cong \sum_{i=1}^{n} w_i \left[\frac{AE}{2L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right] \tag{9-42}$$

Obviously, this is a simple integral. The function we are integrating is a constant, so we only need to use one Gauss point to integrate it exactly. In that case, $n = 1$, $w_i = 2$, and $\zeta_i = 0$ (not used for constant function). Implementing the Gauss-Legendre integration gives:

$$\left[\boldsymbol{K^e}\right] \cong 2 \left[\frac{AE}{2L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right] \tag{9-43}$$

and we finally obtain our old friend:

$$\left[\boldsymbol{K^e}\right] \cong \frac{AE}{L^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9-44}$$

In this case, the integration is exact, but in general, the numerical integration will be approximate.

9.7 Numerical Integration in 2D

In Lecture 8 we derived our finite element statement:

$$\sum_{e=1}^{N_{elm}} \int_{V_e} \boldsymbol{B^T D B} dV \boldsymbol{u_a} = \sum_{e=1}^{N_{elm}} \int_{V_e} \boldsymbol{N^T b} dV + \sum_{e=1}^{N_{elm}} \int_{S_{et}} \boldsymbol{N^T t} dS - \sum_{e=1}^{N_{elm}} \int_{S_{et}} \boldsymbol{B^T D B_0} dV \boldsymbol{u_0} \tag{9-45}$$

and in later lectures we will present the details of the $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{N}$ matrices. The integration for each element will be performed numerically. As for one-dimensional numerical integration, we first transform from real coordinates to natural coordinates (note that in two dimensions, integration over the volume corresponds performing area integration multiplied by the thickness). In two dimensions, we correspondingly need two natural coordinates (ξ,η). The change of variables for integration is:

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dxdy = \int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} f\left(g_1(\xi,\eta), g_2(\xi,\eta)\right)\left|\boldsymbol{J}\right| d\xi d\eta \tag{9-46}$$

where:

$$x = g_1(\xi,\eta), \quad y = g_2(\xi,\eta)$$

$$\left|\boldsymbol{J}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial g_1}{\partial \xi} & \dfrac{\partial g_2}{\partial \xi} \\ \dfrac{\partial g_1}{\partial \eta} & \dfrac{\partial g_2}{\partial \eta} \end{vmatrix} = \text{Determinant of } \textit{Jacobian matrix.}$$

Question 3: What is the physical meaning of the determinant of Jacobian matrix in (9-46)?

In our case, the integration of the element stiffness matrix is transformed as:

$$\int_{V_e} \boldsymbol{B^T D B} dV = t\int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B^T D B}\left|\boldsymbol{J}\right| d\xi d\eta \tag{9-47}$$

where t is the element thickness and the $\boldsymbol{B}$, $\boldsymbol{D}$, and $\boldsymbol{J}$ matrices will be given in Lecture 10, where they are written in terms of ξ and η.

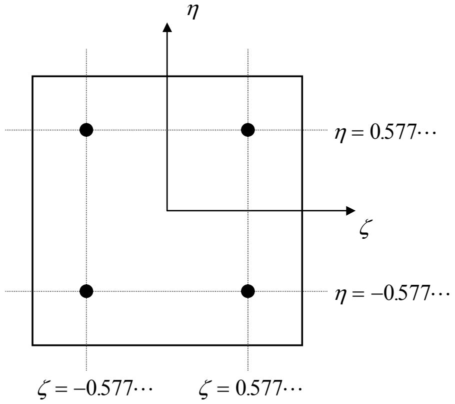

Figure 9-3: Location of 2×2 Gauss points

Similar to the one-dimensional case, in 2-D we integrate numerically by evaluating the function at selected points in each dimension, multiplying the value by a weight, and summing over all the integration points. In general this leads:

$$t\int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B^T D B}\left|\boldsymbol{J}\right| d\xi d\eta = t\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\boldsymbol{B}\left(\xi_i,\eta_j\right)^{\boldsymbol{T}} \boldsymbol{DB}\left(\xi_i,\eta_j\right)\left|\boldsymbol{J}\left(\xi_i,\eta_j\right)\right|\right] w_i w_j \tag{9-48}$$

The Gauss points defined in two dimensions are found to be at the same locations and weights for each natural coordinate as the one-dimensional case previously derived.

As before, a 2×2 Gauss point integration is exact for cubic (of each natural coordinate) polynomials and 3×3 integration is precise for 5th order polynomials. In our work, we will use 3×3 integration. This will be exact integration for parallelepiped elements with the side nodes at the midpoints (the students should verify this).

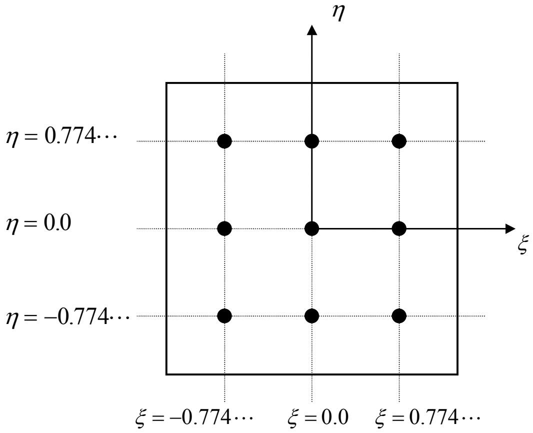

Figure 9-4: Location of 3×3 Gauss points

9.8 Integration of Body Forces and Surface Tractions

See next lecture.

9.9 Implementation

Numerical integration is typically implemented using nested loops for each direction of integration. For instance, 9-48 would be implemented as nested loops corresponding to the ξ and η natural coordinates. Inside the inner most loop, all matrices are evaluated, multiplied, and the sum is accumulated to obtain the integral. This is discussed in more detail in later lectures.

9.10 Hints to Questions

Q1: Because both the locations and weights are optimized.
Q2: 2 integration points.
Q3: The change of the integration area from one coordinate system to another.

9.11 References

Hughes, T. J. R., 1987, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall, Inc.