2

The Direct Stiffness Method I

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§2.1. Foreword

This Chapter begins the exposition of the Direct Stiffness Method (DSM) of structural analysis. The DSM is by far the most common implementation of the Finite Element Method (FEM). In particular, all major commercial FEM codes are based on the DSM.

The exposition is done by following the DSM steps applied to a simple plane truss structure. The method has two major stages: breakdown, and assembly+solution. This Chapter covers primarily the breakdown stage.

§2.2. Why A Plane Truss?

The simplest structural finite element is the two-node bar (also called linear spring) element, which is illustrated in Figure 2.1(a). A six-node triangle that models thin plates, shown in Figure 2.1(b) displays intermediate complexity. Perhaps the most geometrically complex finite element (at least as regards number of degrees of freedom) is the curved, three-dimensional, 64-node "brick" element depicted in Figure 2.1(c).

Yet the remarkable fact is that, in the DSM, all elements, regardless of complexity, are treated alike! To illustrate the basic steps of this democratic method, it makes educational sense to keep it simple and use a structure composed of bar elements.

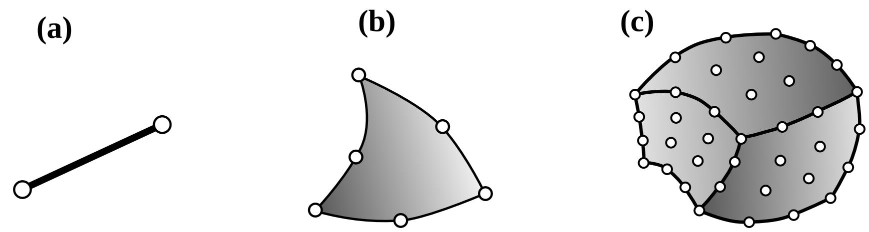

FIGURE 2.1. From the simplest through progressively more complex structural finite elements: (a) two-node bar element for trusses, (b) six-node triangle for thin plates, (b) 64-node tricubic, "brick" element for three-dimensional solid analysis.

A simple yet nontrivial structure is the *pin-jointed plane truss*, whose members may be modeled as two-node bars.[1] Using a plane truss to teach the stiffness method offers two additional advantages:

(a) Computations can be entirely done by hand as long as the structure contains just a few elements. This allows various steps of the solution procedure to be carefully examined and understood (learning by doing) before passing to the computer implementation. Doing hand computations on more complex finite element systems rapidly becomes impossible.

(b) The computer implementation on any programming language is relatively simple and can be assigned as preparatory computer homework before reaching Part III.

[1] A one dimensional bar assembly would be even simpler. That kind of structure would not adequately illustrate some of the DSM steps, however, notably the back-and-forth transformations from global to local coordinates.

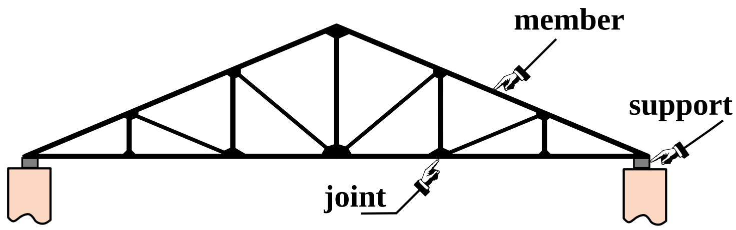

FIGURE 2.2. An actual plane truss structure. That shown is typical of a roof truss used in building construction for rather wide spans, say, over 10 meters. For shorter spans, as in residential buildings, trusses are simpler, with fewer bays.

§2.3. Truss Structures

Plane trusses, such as the one depicted in Figure 2.2, are often used in construction, particularly for roofing of residential and commercial buildings, and in short-span bridges. Trusses, whether two or three dimensional, belong to the class of *skeletal structures*. These structures consist of elongated structural components called *members*, connected at *joints*. Another important subclass of skeletal structures are frame structures or *frameworks*, which are common in reinforced concrete construction of buildings and bridges.

Skeletal structures can be analyzed by a variety of hand-oriented methods of structural analysis taught in beginning Mechanics of Materials courses: the Displacement and Force methods. They can also be analyzed by the computer-oriented FEM. That versatility makes those structures a good choice to illustrate the transition from the hand-calculation methods taught in undergraduate courses, to the fully automated finite element analysis procedures available in commercial programs.

§2.4. Idealization

The first analysis step carried out by a structural engineer is to replace the actual physical structure by a *mathematical model*. This model represents an *idealization* of the actual structure. For truss structures, by far the most common idealization is the *pin-jointed truss*, which directly maps to a FEM model. See Figure 2.3.

The replacement of true by idealized is at the core of the *physical interpretation* of the finite element method discussed in Chapter 1. The axially-carrying-load members and frictionless pins of the pin-jointed truss are only an approximation of the physical one. For example, building and bridge trusses usually have members joined to each other through the use of gusset plates, which are attached by nails, bolts, rivets or welds. Consequently members will carry some bending as well as direct axial loading.

Experience has shown, however, that stresses and deformations calculated using the simple idealized model will often be satisfactory for preliminary design purposes; for example to select the cross section of the members. Hence the engineer turns to the pin-jointed assemblage of axial-force-carrying elements and uses it to perform the structural analysis.

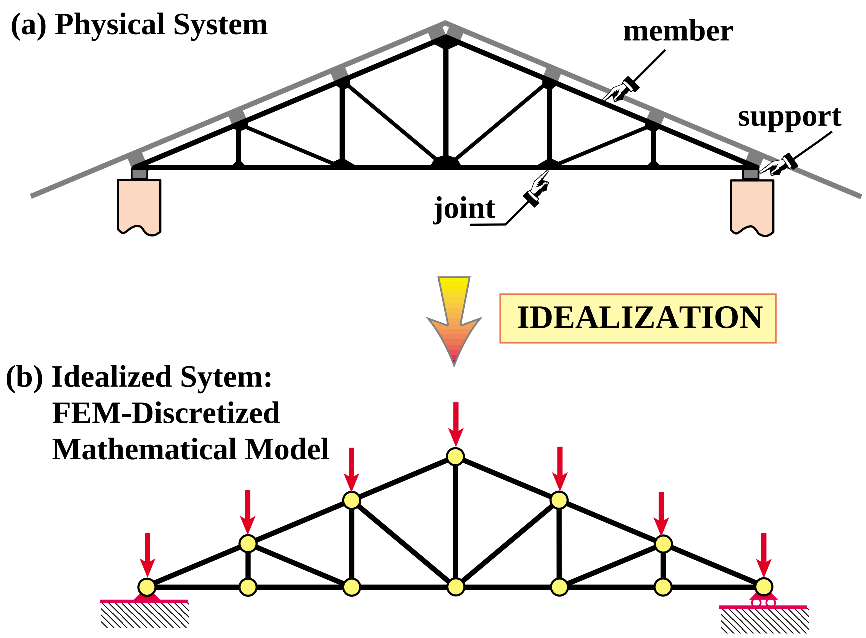

FIGURE 2.3. Idealization of roof truss: (a) physical system, (b) idealization as FEM discretized mathematical model.

In this and following Chapter we will go over the basic steps of the DSM in a "hand-computer" calculation mode. This means that although the steps are done by hand, whenever there is a procedural choice we shall either adopt the way that is better suited towards the computer implementation, or explain the difference between hand and computer computations. The actual computer implementation using a high-level programming language is presented in Chapter 4.

§2.5. The Example Truss

To keep hand computations manageable we will use just about the simplest structure that can be called a plane truss, namely the three-member truss illustrated in Figure 2.4(a). The *idealized* model of this physical truss as a pin-jointed assemblage of bars as shown in Figure 2.4(b), In this idealization truss members carry only axial loads, have no bending resistance, and are connected by frictionless pins.

Geometric, material and fabrication properties of the idealized truss are given in Figure 2.4(c),[2] while idealized loads and support conditions are provided in Figure 2.4(d).

It should be noted that as a practical structure the example truss is not particularly useful — that shown in Figure 2.2 is more common in construction. But with the example truss we can go over the basic DSM steps without getting mired into too many members, joints and degrees of freedom.

[2] Member mass densities are given in Figure 2.4(c) so that this truss can be used later for examples in vibration and dyamics analysis.

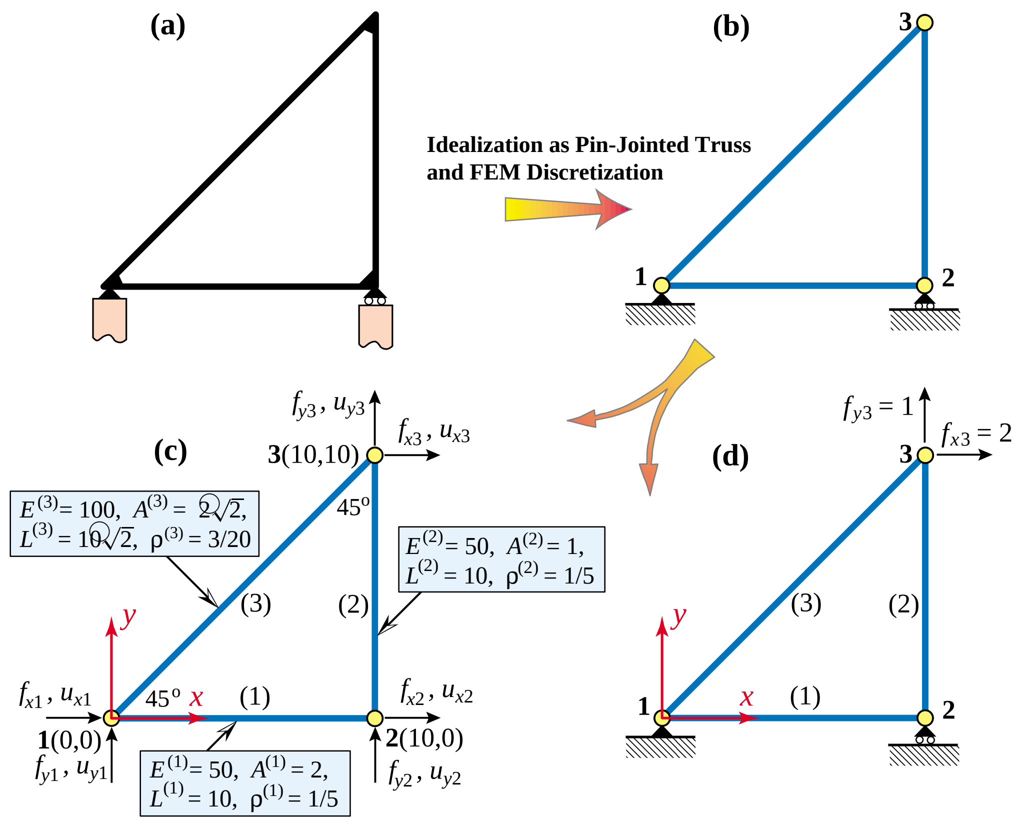

FIGURE 2.4. The three-member example truss: (a) physical structure; (b) idealization as a pin-jointed bar assemblage; (c) geometric, material and fabrication properties; (d) support conditions and applied loads.

§2.6. Members, Joints, Forces and Displacements

The pin-jointed idealization of the example truss, pictured in Figure 2.4(b,c,d), has three *joints*, which are labeled 1, 2 and 3, and three *members*, which are labeled (1), (2) and (3). Those members connect joints 1–2, 2–3, and 1–3, respectively. The member lengths are denoted by $L^{(1)}$, $L^{(2)}$ and $L^{(3)}$, their elastic moduli by $E^{(1)}$, $E^{(2)}$ and $E^{(3)}$, and their cross-sectional areas by $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$. Note that an element number superscript is enclosed in parenthesis to avoid confusion with exponents. Both E and A are assumed to be constant along each member.

Members are generically identified by index e (because of their close relation to finite elements, as explained below). This index is placed as superscript of member properties. For example, the cross-section area of a generic member is A^e. The member superscript is *not* enclosed in parentheses in this case because no confusion with exponents can arise. But the area of member 3 is written $A^{(3)}$ and not A^3.

Joints are generically identified by indices such as i, j or n. In the general FEM, the names “joint” and “member” are replaced by *node* and *element*, respectively. This dual nomenclature is used in the initial Chapters to stress the physical interpretation of the FEM.

The geometry of the structure is referred to a common Cartesian coordinate system $\{x, y\}$, which will be called the *global coordinate system*. Other names for it in the literature are *structure coordinate system* and *overall coordinate system*. For the example truss its origin is at joint 1.

The key ingredients of the stiffness method of analysis are the *forces* and *displacements* at the joints. In a idealized pin-jointed truss, externally applied forces as well as reactions *can act only at the joints*. All member axial forces can be characterized by the x and y components of these forces, denoted by f_x and f_y, respectively. The components at joint i will be identified as f_{xi} and f_{yi}, respectively. The set of all joint forces can be arranged as a 6-component column vector called $\mathbf{f}$.

The other key ingredient is the displacement field. Classical structural mechanics tells us that the displacements of the truss *are completely defined by the displacements of the joints*. This statement is a particular case of the more general finite element theory. The x and y displacement components will be denoted by u_x and u_y, respectively. The *values* of u_x and u_y at joint i will be called u_{xi} and u_{yi}. Like joint forces, they are arranged into a 6-component vector called $\mathbf{u}$. Here are the two vectors of nodal forces and nodal displacements, shown side by side:

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}. \tag{2.1}$$

In the DSM these six displacements are the primary unknowns. They are also called the *degrees of freedom* or *state variables* of the system.[3]

How about the displacement boundary conditions, popularly called support conditions? This data will tell us which components of $\mathbf{f}$ and $\mathbf{u}$ are actual unknowns and which ones are known *a priori*. In pre-computer structural analysis such information was used *immediately* by the analyst to discard unnecessary variables and thus reduce the amount of hand-carried bookkeeping.

The computer oriented philosophy is radically different: *boundary conditions can wait until the last moment*. This may seem strange, but on the computer the sheer volume of data may not be so important as the efficiency with which the data is organized, accessed and processed. The strategy "save the boundary conditions for last" will be followed here also for the hand computations.

Remark 2.1. Often column vectors such as (2.1) will be displayed in row form to save space, with a transpose symbol at the end. For example, $\mathbf{f} = [\, f_{x1} \;\; f_{y1} \;\; f_{x2} \;\; f_{y2} \;\; f_{x3} \;\; f_{y3} \,]^T$ and $\mathbf{u} = [\, u_{x1} \;\; u_{y1} \;\; u_{x2} \;\; u_{y2} \;\; u_{x3} \;\; u_{y3} \,]^T$.

§2.7. The Master Stiffness Equations

The *master stiffness equations* relate the joint forces $\mathbf{f}$ of the complete structure to the joint displacements $\mathbf{u}$ of the complete structure *before* specification of support conditions.

Because the assumed behavior of the truss is linear, these equations must be linear relations that connect the components of the two vectors. Furthermore it will be assumed that if all displacements vanish, so do the forces.[4] If both assumptions hold the relation must be homogeneous and

[3] *Primary unknowns* is the correct mathematical term whereas *degrees of freedom* has a mechanics flavor: "any of a limited number of ways in which a body may move or in which a dynamic system may change" (Merrian-Webster). The term *state variables* is used more often in nonlinear analysis, material sciences and statistics.

[4] This assumption implies that the so-called *initial strain* effects, also known as *prestress* or *initial stress* effects, are neglected. Such effects are produced by actions such as temperature changes or lack-of-fit fabrication, and are studied in Chapter 29.

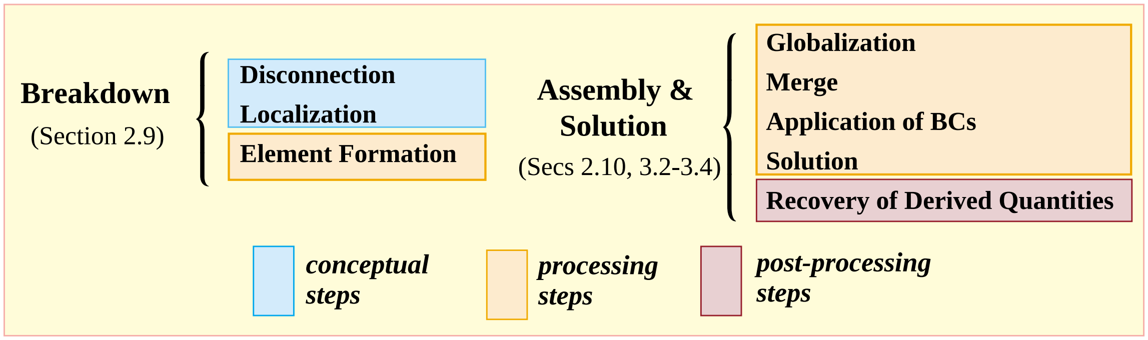

FIGURE 2.5. The Direct Stiffness Method steps.

expressable in component form as

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} \\ K_{y1x1} & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} \\ K_{x2x1} & K_{x2y1} & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} \\ K_{y2x1} & K_{y2y1} & K_{y2x2} & K_{y2y2} & K_{y2x3} & K_{y2y3} \\ K_{x3x1} & K_{x3y1} & K_{x3x2} & K_{x3y2} & K_{x3x3} & K_{x3y3} \\ K_{y3x1} & K_{y3y1} & K_{y3x2} & K_{y3y2} & K_{y3x3} & K_{y3y3} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}. \tag{2.2}$$

In matrix notation:

$$\boxed{\mathbf{f} = \mathbf{K}\,\mathbf{u}.} \tag{2.3}$$

Here $\mathbf{K}$ is the *master stiffness matrix*, also called *global stiffness matrix*, *assembled stiffness matrix*, or *overall stiffness matrix*. It is a 6×6 square matrix that happens to be symmetric, although this attribute has not been emphasized in the written-out form (2.2). The entries of the stiffness matrix are often called *stiffness coefficients* and have a physical interpretation discussed below.

The qualifiers ("master", "global", "assembled" and "overall") convey the impression that there is another level of stiffness equations lurking underneath. And indeed there is a *member level* or *element level*, into which we plunge in the **Breakdown** section.

Remark 2.2. *Interpretation of Stiffness Coefficients*. The following interpretation of the entries of $\mathbf{K}$ is valuable for visualization and checking. Choose a displacement vector $\mathbf{u}$ such that all components are zero except the i^{th} one, which is one. Then $\mathbf{f}$ is simply the i^{th} column of $\mathbf{K}$. For instance if in (2.3) we choose u_{x2} as unit displacement,

$$\mathbf{u} = [\,0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0\,]^T, \qquad \mathbf{f} = [\,K_{x1x2} \;\; K_{y1x2} \;\; K_{x2x2} \;\; K_{y2x2} \;\; K_{x3x2} \;\; K_{y3x2}\,]^T. \tag{2.4}$$

Thus K_{y1x2}, say, represents the y-force at joint 1 that would arise on prescribing a unit x-displacement at joint 2, while all other displacements vanish. In structural mechanics this property is called *interpretation of stiffness coefficients* as *displacement influence coefficients*. It extends unchanged to the general finite element method.

§2.8. The DSM Steps

The DSM steps, major and minor, are summarized in Figure 2.5 for the convenience of the reader. The two major stages are **Breakdown**, followed by **Assembly & Solution**. A postprocessing step may follow, although this is not part of the DSM proper.

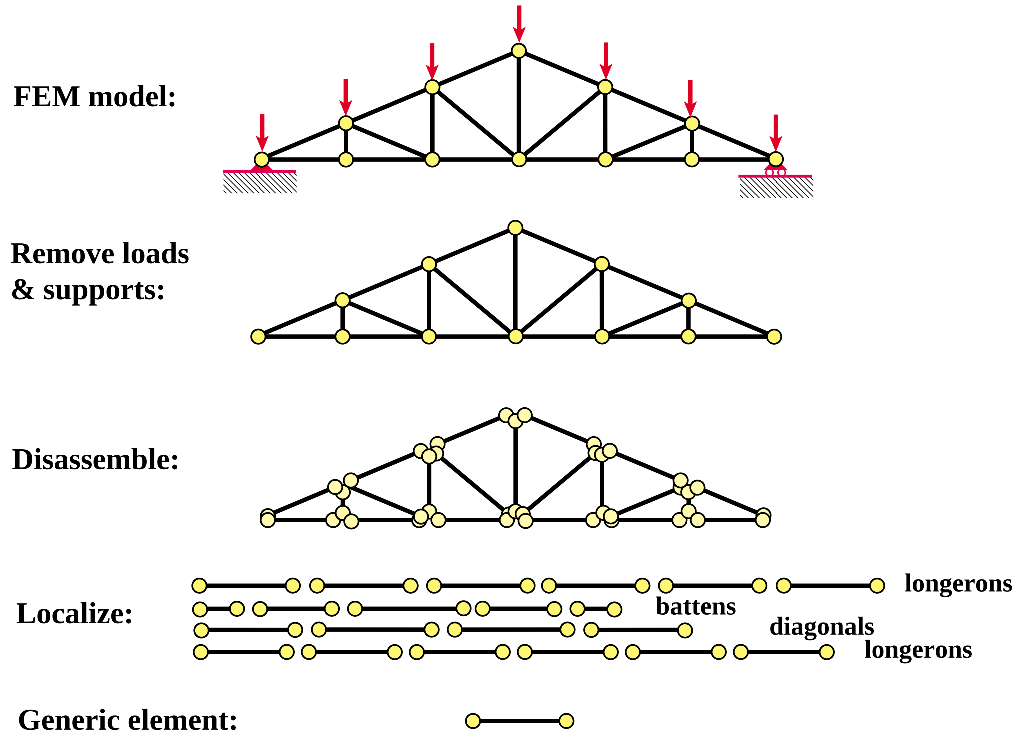

FIGURE 2.6. Visualization of the DSM breakdown stage.

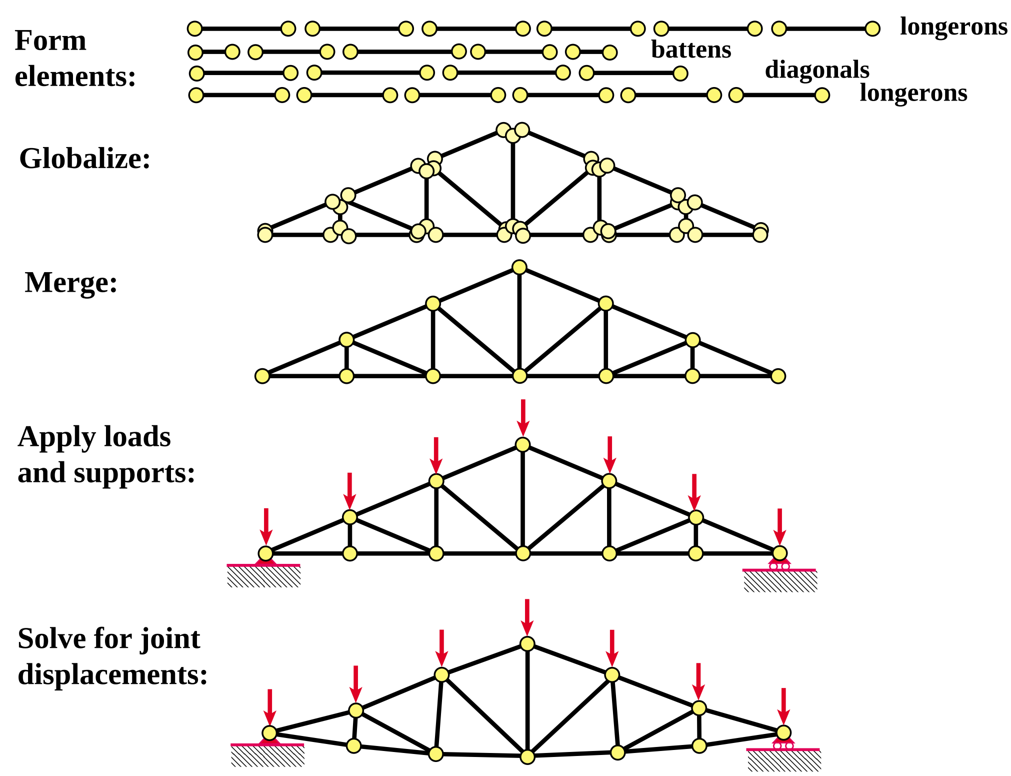

FIGURE 2.7. Visualization of the DSM assembly-and-solution stage.

The first three DSM steps are: (1) disconnection, (2) localization, and (3) computation of member stiffness equations. Collectively these form the *breakdown* stage. The first two are flagged as *conceptual* in Figure 2.5 because they are not actually programmed as such: they are implicitly carried out either through the user-provided problem definition, or produced by separate preprocessing programs such as CAD front ends. DSM processing actually begins at the element-stiffness-equation forming step.

Before starting with the detailed, step by step description of the DSM, it is convenient to exhibit the whole process through a graphic sequence. This is done in Figures 2.6 and 2.7, which are shown to students as animated slides. The sequence starts with the FEM model of the typical roof truss displayed in Figure 2.3(b). Thus the idealization step pictured there is assumed to have been carried out.[5]

§2.9. Breakdown Stage

The three breadown steps: disconnection, localization and formation of the element stiffness equations, are covered next.

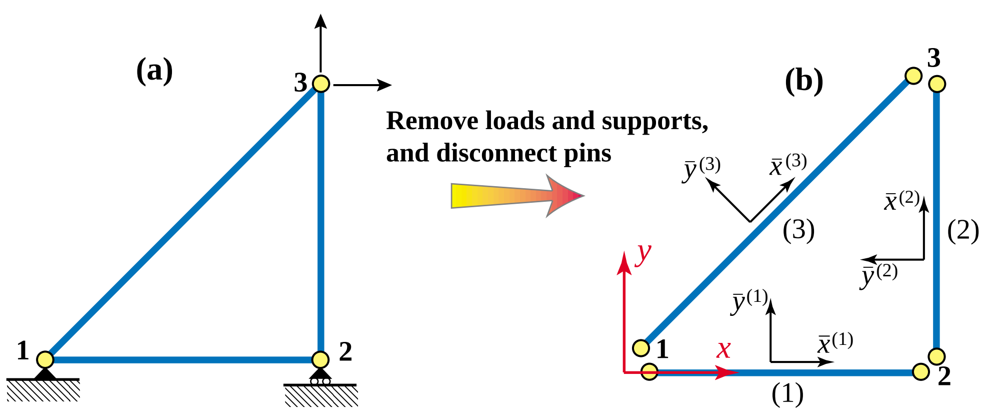

FIGURE 2.8. Disconnection step: (a) idealized example truss; (b) removal of loads and support, disconnection into members (1), (2) and (3), and selection of local coordinate systems. The latter are drawn offset from member axes for visualization convenience.

§2.9.1. Disconnection

To carry out the first breakdown step we begin by discarding all loads and supports (the so-called boundary conditions). Next we *disconnect* or *disassemble* the structure into its components. For a pin-jointed truss disconnection can be visualized as removing the pin connectors. This is illustrated for the example truss in Figure 2.8. To each member $e = 1, 2, 3$ assign a Cartesian system $\{\bar{x}^e, \bar{y}^e\}$. Axis $\bar{x}^e$ is aligned along the axis of the e^{th} member. Actually $\bar{x}^e$ runs along the member longitudinal axis; it is drawn offset in Figure 2.8 (b) for clarity.

By convention the positive direction of $\bar{x}^e$ runs from joint i to joint j, where $i < j$. The angle formed by $\bar{x}^e$ and x is the *orientation angle* φ^e, positive CCW. The axes origin is arbitrary and may be placed at the member midpoint or at one of the end joints for convenience.

[5] The sequence depicted in Figures 2.6 and 2.7 uses the typical roof truss of Figures 2.2 and 2.3, rather than the 3-member example truss. Reason: those pictures are more representative of actual truss structures, as seen often by students.

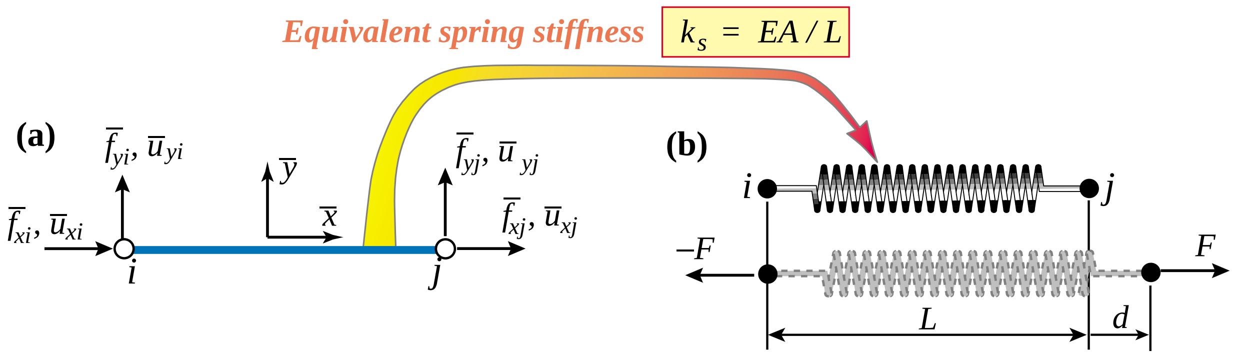

FIGURE 2.9. Generic truss member referred to its local coordinate system $\{\bar{x}, \bar{y}\}$: (a) idealization as 2-node bar element, (b) interpretation as equivalent spring. Element identification number e dropped to reduce clutter.

Systems $\{\bar{x}^e, \bar{y}^e\}$ are called *local coordinate systems* or *member-attached coordinate systems*. In the general finite element method they also receive the name *element coordinate systems*.

§2.9.2. Localization

To reduce clutter we drop the member identifier e so we are effectively dealing with a *generic* truss member, as illustrated in Figure 2.9(a). The local coordinate system is $\{\bar{x}, \bar{y}\}$. The two end joints are labelled i and j. As shown in that figure, a generic plane truss member has four joint force components and four joint displacement components (the member degrees of freedom). The member properties are length L, elastic modulus E and cross-section area A.

§2.9.3. Member Stiffness Equations

The force and displacement components of the generic truss member shown in Figure 2.9(a) are linked by the *member stiffness relations*

$$\bar{\mathbf{f}} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}, \tag{2.5}$$

which written out in full become

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{K}_{xixi} & \bar{K}_{xiyi} & \bar{K}_{xixj} & \bar{K}_{xiyj} \\ \bar{K}_{yixi} & \bar{K}_{yiyi} & \bar{K}_{yixj} & \bar{K}_{yiyj} \\ \bar{K}_{xjxi} & \bar{K}_{xjyi} & \bar{K}_{xjxj} & \bar{K}_{xjyj} \\ \bar{K}_{yjxi} & \bar{K}_{yjyi} & \bar{K}_{yjxj} & \bar{K}_{yjyj} \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}. \tag{2.6}$$

Vectors $\bar{\mathbf{f}}$ and $\bar{\mathbf{u}}$ are called the *member joint forces* and *member joint displacements*, respectively, whereas $\bar{\mathbf{K}}$ is the *member stiffness matrix* or *local stiffness matrix*. When these relations are interpreted from the standpoint of the general FEM, "member" is replaced by "element" and "joint" by "node."

There are several ways to construct the stiffness matrix $\bar{\mathbf{K}}$ in terms of L, E and A. The most straightforward technique relies on the Mechanics of Materials approach covered in undergraduate courses. Think of the truss member in Figure 2.9(a) as a linear spring of equivalent stiffness k_s, an interpretation illustrated in Figure 2.9(b). If the member properties are *uniform* along its length, Mechanics of Materials bar theory tells us that[6]

$$k_s = \frac{EA}{L}, \tag{2.7}$$

Consequently the force-displacement equation is

$$F = k_s d = \frac{EA}{L} d, \tag{2.8}$$

where F is the internal axial force and d the relative axial displacement, which physically is the bar elongation. The axial force and elongation can be immediately expressed in terms of the joint forces and displacements as

$$F = \bar{f}_{xj} = -\bar{f}_{xi}, \qquad d = \bar{u}_{xj} - \bar{u}_{xi}, \tag{2.9}$$

which express force equilibrium[7] and kinematic compatibility, respectively. Combining (2.8) and (2.9) we obtain the matrix relation[8]

$$\bar{\mathbf{f}} = \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}, \tag{2.10}$$

Hence

$$\boxed{\bar{\mathbf{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.} \tag{2.11}$$

This is the truss stiffness matrix in local coordinates.

Two other methods for deriving the local force-displacement relation (2.8) are covered in Exercises 2.6 and 2.7.

§2.10. Assembly and Solution Stage: Globalization

The first step in the assembly & solution stage, as shown in Figure 2.5, is *globalization*. This operation is done member by member. It refers the member stiffness equations to the global system $\{x, y\}$ so it can be merged into the master stiffness. Before entering into details we must establish relations that connect joint displacements and forces in the global and local coordinate systems. These are given in terms of *transformation matrices*.

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6 See for example, Chapter 2 of [67].

7 Equations $F = \bar{f}_{xj} = -\bar{f}_{xi}$ follow by considering the free body diagram (FBD) of each joint. For example, take joint i as a FBD. Equilibrium along x requires $-F - \bar{f}_{xi} = 0$ whence $F = -\bar{f}_{xi}$. Doing the same on joint j yields $F = \bar{f}_{xj}$.

8 The matrix derivation of (2.10) is the subject of Exercise 2.3.

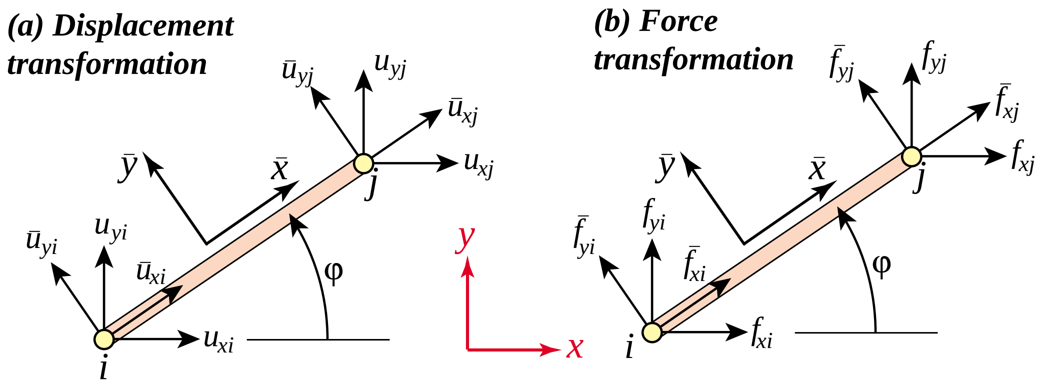

FIGURE 2.10. The transformation of node displacement and force components from the local system $\{\bar{x}, \bar{y}\}$ to the global system $\{x, y\}$.

§2.10.1. Displacement and Force Transformations

The necessary transformations are easily obtained by inspection of Figure 2.10. For the displacements

$$\begin{aligned} \bar{u}_{xi} &= u_{xi} c + u_{yi} s, \qquad & \bar{u}_{yi} &= -u_{xi} s + u_{yi} c, \\ \bar{u}_{xj} &= u_{xj} c + u_{yj} s, \qquad & \bar{u}_{yj} &= -u_{xj} s + u_{yj} c, \end{aligned} \tag{2.12}$$

in which $c = \cos\varphi$, $s = \sin\varphi$ and φ is the angle formed by $\bar{x}$ and x, measured positive CCW from x. The matrix form that collects these relations is

$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}. \tag{2.13}$$

The 4×4 matrix that appears above is called a *displacement transformation matrix* and is denoted[9] by $\mathbf{T}$. The node forces transform as $f_{xi} = \bar{f}_{xi}\, c - \bar{f}_{yi}\, s$, etc., which in matrix form become

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}. \tag{2.14}$$

The 4×4 matrix that appears above is called a *force transformation matrix*. A comparison of (2.13) and (2.14) reveals that the force transformation matrix is the *transpose* $\mathbf{T}^T$ of the displacement transformation matrix $\mathbf{T}$. This relation is not accidental and can be proved to hold generally.[10]

[9] This matrix will be called $\mathbf{T}_d$ when its association with displacements is to be emphasized, as in Exercise 2.5.

[10] A simple proof that relies on the invariance of external work is given in Exercise 2.5. However this invariance was only checked by explicit computation for a truss member in Exercise 2.4. The general proof relies on the Principle of Virtual Work, which is discussed later.

Remark 2.3. Note that in (2.13) the local system (barred) quantities appear on the left-hand side, whereas in (2.14) they show up on the right-hand side. The expressions (2.13) and and (2.14) are discrete counterparts of what are called covariant and contravariant transformations, respectively, in continuum mechanics. The continuum counterpart of the transposition relation is called *adjointness*. Colectively these relations, whether discrete or continuous, pertain to the subject of *duality*.

Remark 2.4. For this particular structural element $\mathbf{T}$ is square and orthogonal, that is, $\mathbf{T}^T = \mathbf{T}^{-1}$. But this property does not extend to more general elements. Furthermore in the general case $\mathbf{T}$ is not even a square matrix, and consequently does not possess an ordinary inverse. However the congruent transformation relations (2.15)–(2.17) given below do hold generally.

§2.10.2. Global Member Stiffness Equations

From now on we reintroduce the member (element) index, e. The member stiffness equations in global coordinates will be written

$$\mathbf{f}^e = \mathbf{K}^e \mathbf{u}^e. \tag{2.15}$$

The compact form of (2.13) and (2.14) for the e^{th} member is

$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e, \qquad \mathbf{f}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{f}}^e. \tag{2.16}$$

Inserting these matrix expressions into $\bar{\mathbf{f}}^e = \overline{\mathbf{K}}^e \, \bar{\mathbf{u}}^e$ and comparing with (2.15) we find that the member stiffness in the global system $\{x, y\}$ can be computed from the member stiffness $\bar{\mathbf{K}}^e$ in the local system $\{\bar{x}, \bar{y}\}$ through the congruent transformation[11]

$$\boxed{\mathbf{K}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{K}}^e \mathbf{T}^e.} \tag{2.17}$$

Carrying out the matrix multiplications in closed form (Exercise 2.8) we get

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}, \tag{2.18}$$

in which $c = \cos\varphi^e$, $s = \sin\varphi^e$, with e superscripts of c and s suppressed to reduce clutter. If the orientation angle φ^e is zero we recover (2.10), as may be expected. $\mathbf{K}^e$ is called a *member stiffness matrix in global coordinates*. The proof of (2.17) and verification of (2.18) is left as Exercise 2.8.

The globalized member stiffness equations for the example truss can now be easily obtained by inserting appropriate values provided in Figure 2.4(c). into (2.18).

For member (1), with end joints 1–2, $E^{(1)}A^{(1)} = 100$, $L^{(1)} = 10$, and $\varphi^{(1)} = 0^\circ$:

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}. \tag{2.19}$$

[11] Also known as *congruential transformation* and *congruence transformation* in linear algebra books.

For member (2), with end joints 2–3, $E^{(2)}A^{(2)} = 50$, $L^{(2)} = 10$, and $\varphi^{(2)} = 90^\circ$:

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}. \tag{2.20}$$

For member (3), with end joints 1–3, $E^{(3)}A^{(3)} = 200\sqrt{2}$, $L^{(3)} = 10\sqrt{2}$, and $\varphi^{(3)} = 45^\circ$:

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = 20 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}. \tag{2.21}$$

In the following Chapter we complete the DSM steps by putting the truss back together through the merge step, and solving for the unknown forces and displacements.

Notes and Bibliography

The Direct Stiffness Method has been the dominant FEM version since the mid-1960s, and is the procedure followed by all major commercial codes in current use. The general DSM was developed at Boeing in the mid and late 1950s, through the leadership of Jon Turner [759,761], and had defeated its main competitor, the Force Method, by 1970 [238].

All applications-oriented FEM books cover the DSM, although the procedural steps are sometimes not clearly delineated. In particular, the textbooks recommended in §1.9 offer adequate expositions.

Trusses, also called bar assemblies, are usually the first structures treated in Mechanics of Materials books written for undergraduate courses in Aerospace, Civil and Mechanical Engineering. Two widely used textbooks at this level are [67] and [588].

Steps in the derivation of stiffness matrices for truss elements are well covered in a number of early treatment of finite element books, of which Chapter 5 of Przemieniecki [596] is a good example.

Force and displacement transformation matrices for structural analysis were introduced by G. Kron [423].

References

Referenced items have been moved to Appendix R.

Homework Exercises for Chapter 2
The Direct Stiffness Method I

EXERCISE 2.1 [D:10] Explain why *arbitrarily oriented* mechanical loads on an *idealized* pin-jointed truss structure must be applied at the joints. [Hint: idealized truss members have no bending resistance.] How about actual trusses: can they take loads applied between joints?

EXERCISE 2.2 [A:15] Show that the sum of the entries of each row of the master stiffness matrix $\mathbf{K}$ of any plane truss, before application of any support conditions, must be zero. [Hint: apply translational rigid body motions at nodes.] Does the property hold also for the columns of that matrix?

EXERCISE 2.3 [A:15] Using matrix algebra derive (2.10) from (2.8) and (2.9). Note: Place *all equations in matrix form first* and eliminate d and F by matrix multiplication. Deriving the final form with scalar algebra and rewriting it in matrix form gets no credit.

EXERCISE 2.4 [A:15] By direct multiplication verify that for the truss member of Figure 2.9(a), $\bar{\mathbf{f}}^T \bar{\mathbf{u}} = F\, d$. Intepret this result physically. (Hint: what is a force times displacement in the direction of the force?)

EXERCISE 2.5 [A:20] The transformation equations between the 1-DOF spring and the 4-DOF generic truss member may be written in compact matrix form as

$$d = \mathbf{T}_d\, \bar{\mathbf{u}}, \qquad \bar{\mathbf{f}} = F\, \mathbf{T}_f, \tag{E2.1}$$

where $\mathbf{T}_d$ is 1×4 and $\mathbf{T}_f$ is 4×1. Starting from the identity $\bar{\mathbf{f}}^T \bar{\mathbf{u}} = F\, d$ proven in the previous exercise, and using *compact matrix notation*, show that $\mathbf{T}_f = \mathbf{T}_d^T$. Or in words: *the displacement transformation matrix and the force transformation matrix are the transpose of each other*. (This can be extended to general systems)

EXERCISE 2.6 [A:20] Derive the equivalent spring formula $F = (EA/L)\, d$ of (2.8) by the Theory of Elasticity relations $e = d\bar{u}(\bar{x})/d\bar{x}$ (strain-displacement equation), $\sigma = Ee$ (Hooke's law) and $F = A\sigma$ (axial force definition). Here e is the axial strain (independent of $\bar{x}$) and σ the axial stress (also independent of $\bar{x}$). Finally, $\bar{u}(\bar{x})$ denotes the axial displacement of the cross section at a distance $\bar{x}$ from node i, which is linearly interpolated as

$$\bar{u}(\bar{x}) = \bar{u}_{xi}\left(1 - \frac{\bar{x}}{L}\right) + \bar{u}_{xj}\,\frac{\bar{x}}{L} \tag{E2.2}$$

Justify that (E2.2) is correct since the bar differential equilibrium equation: $d[A(d\sigma/d\bar{x})]/d\bar{x} = 0$, is verified for all $\bar{x}$ if A is constant along the bar.

EXERCISE 2.7 [A:20] Derive the equivalent spring formula $F = (EA/L)\, d$ of (2.8) by the principle of Minimum Potential Energy (MPE). In Mechanics of Materials it is shown that the total potential energy of the axially loaded bar is

$$\Pi = \tfrac{1}{2}\int_0^L A\,\sigma\, e\, d\bar{x} - Fd, \tag{E2.3}$$

where symbols have the same meaning as the previous Exercise. Use the displacement interpolation (E2.2), the strain-displacement equation $e = d\bar{u}/d\bar{x}$ and Hooke's law $\sigma = Ee$ to express Π as a function $\Pi(d)$ of the relative displacement d only. Then apply MPE by requiring that $\partial\Pi/\partial d = 0$.

EXERCISE 2.8 [A:20] Derive (2.17) from $\bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{f}}^e$, (2.15) and (2.17). (*Hint*: premultiply both sides of $\bar{\mathbf{K}}^e \bar{\mathbf{u}}^e = \bar{\mathbf{f}}^e$ by an appropriate matrix). Then check by hand that using that formula you get (2.18). Falk's scheme is recommended for the multiplications.[12]

[12] This scheme is useful to do matrix multiplication by hand. It is explained in §B.3.2 of Appendix B.

EXERCISE 2.9 [D:5] Why are disconnection and localization labeled as "conceptual steps" in Figure 2.5?

EXERCISE 2.10 [C:20] (Requires thinking) Notice that the expression (2.18) of the globalized bar stiffness matrix may be factored as

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} = \begin{bmatrix} -c \\ -s \\ c \\ s \end{bmatrix} \frac{E^e A^e}{L^e} [\, -c \quad -s \quad c \quad s \,] \tag{E2.4}$$

Interpret this relation physically as a chain of global-to-local-to-global matrix operations: global displacements $\rightarrow$ axial strain, axial strain $\rightarrow$ axial force, and axial force $\rightarrow$ global node forces.