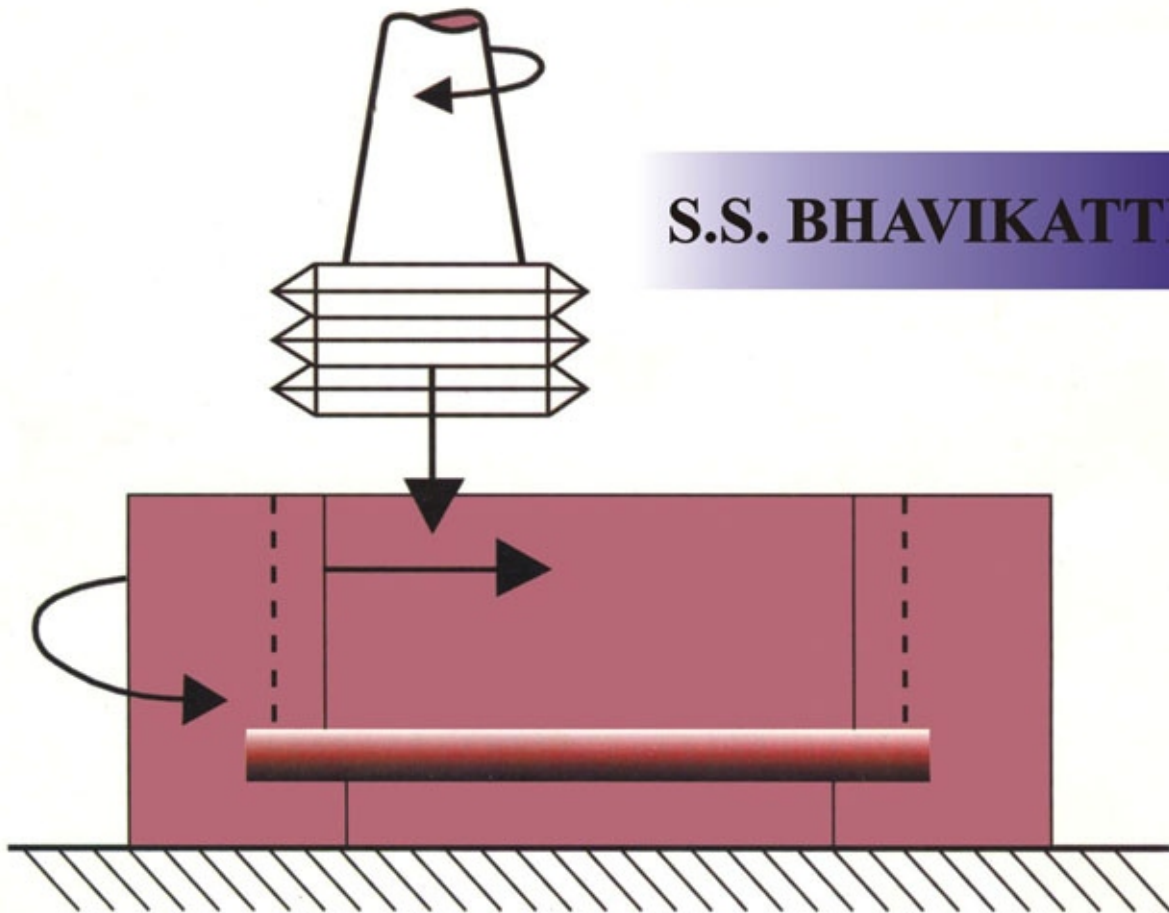


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# FINITE ELEMENT ANALYSIS

S.S. BHAVIKATTI



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ANALYSIS

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# FINITE ELEMENT ANALYSIS

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# Preface

Finite Element Analysis was developed as a numerical method of stress analysis, but now it has been extended as a general method of solution to many complex engineering and physical science problems. As it involves lot of calculations, its growth is closely linked with the developments in computer technology. Now-a-days a number of finite element analysis packages are available commercially and number of users is increasing. A user without a basic course on finite element analysis may produce dangerous results. Hence now-a-days in many M.Tech. programmes finite element analysis is a core subject and in undergraduate programmes many universities offer it as an elective subject. The experience of the author in teaching this course to M.Tech (Geotechnical Engineering) and M.Tech. (Industrial Structures) students at National Institute of Technology, Karnataka, Surathkal (formerly, K.R.E.C. Surathkal) and to undergraduate students at SDM College of Engineering and Technology, Dharwad inspired him to write this book. This is intended as a text book to students and as an introductory course to all users of finite element packages.

The author has developed the finite element concept, element properties and stiffness equations in first nine chapters. In chapter X the various points to be remembered in discretization for producing best results is presented. Isoparametric concept is developed and applications to simple structures like bars, trusses, beams and rigid frames is explained thoroughly taking small problems for hand calculations. Application of this method to complex problems like plates, shells and nonlinear analysis is introduced. Finally a list of commercially available packages is given and the desirable features of such packages is presented.

The author hopes that the students and teachers will find it as a useful text book. The suggestions for improvements are most welcome.

DR S.S. BHAVIKATTI

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# 1

## Introduction

### 1.1 GENERAL

The finite element analysis is a numerical technique. In this method all the complexities of the problems, like varying shape, boundary conditions and loads are maintained as they are but the solutions obtained are approximate. Because of its diversity and flexibility as an analysis tool, it is receiving much attention in engineering. The fast improvements in computer hardware technology and slashing of cost of computers have boosted this method, since the computer is the basic need for the application of this method. A number of popular brand of finite element analysis packages are now available commercially. Some of the popular packages are STAAD-PRO, GT-STRUDEL, NASTRAN, NISA and ANSYS. Using these packages one can analyse several complex structures.

The finite element analysis originated as a method of stress analysis in the design of aircrafts. It started as an extension of matrix method of structural analysis. Today this method is used not only for the analysis in solid mechanics, but even in the analysis of fluid flow, heat transfer, electric and magnetic fields and many others. Civil engineers use this method extensively for the analysis of beams, space frames, plates, shells, folded plates, foundations, rock mechanics problems and seepage analysis of fluid through porous media. Both static and dynamic problems can be handled by finite element analysis. This method is used extensively for the analysis and design of ships, aircrafts, space crafts, electric motors and heat engines.

### 1.2 GENERAL DESCRIPTION OF THE METHOD

In engineering problems there are some basic unknowns. If they are found, the behaviour of the entire structure can be predicted. The **basic unknowns** or the **Field variables** which are encountered in the engineering problems are displacements in solid mechanics, velocities in fluid mechanics, electric and magnetic potentials in electrical engineering and temperatures in heat flow problems.

In a continuum, these unknowns are infinite. The finite element procedure reduces such unknowns to a finite number by dividing the solution region into small parts called **elements** and by expressing the unknown field variables in terms of assumed **approximating functions** (Interpolating functions/Shape functions) within each element. The approximating functions are defined in terms of field variables of specified points called **nodes** or **nodal points**. Thus in the finite element analysis the unknowns are the field variables of the nodal points. Once these are found the field variables at any point can be found by using interpolation functions.

After selecting elements and nodal unknowns next step in finite element analysis is to assemble **element properties** for each element. For example, in solid mechanics, we have to find the force-displacement i.e. stiffness characteristics of each individual element. Mathematically this relationship is of the form

## 2 Finite Element Analysis

$$[k]_e \{\delta\}_e = \{F\}_e$$

where  $[k]_e$  is element stiffness matrix,  $\{\delta\}_e$  is nodal displacement vector of the element and  $\{F\}_e$  is nodal force vector. The element of stiffness matrix  $k_{ij}$  represent the force in coordinate direction 'i' due to a unit displacement in coordinate direction 'j'. Four methods are available for formulating these element properties viz. direct approach, variational approach, weighted residual approach and energy balance approach. Any one of these methods can be used for assembling element properties. In solid mechanics variational approach is commonly employed to assemble stiffness matrix and nodal force vector (consistant loads).

Element properties are used to assemble global properties/structure properties to get system equations  $[k] \{\delta\} = \{F\}$ . Then the boundary conditions are imposed. The solution of these simultaneous equations give the nodal unknowns. Using these nodal values additional calculations are made to get the required values e.g. stresses, strains, moments, etc. in solid mechanics problems.

Thus the various steps involved in the finite element analysis are:

- (i) Select suitable field variables and the elements.
- (ii) Discretise the continua.
- (iii) Select interpolation functions.
- (iv) Find the element properties.
- (v) Assemble element properties to get global properties.
- (vi) Impose the boundary conditions.
- (vii) Solve the system equations to get the nodal unknowns.
- (viii) Make the additional calculations to get the required values.

### 1.3 A BRIEF EXPLANATION OF FEA FOR A STRESS ANALYSIS PROBLEM

The steps involved in finite element analysis are clarified by taking the stress analysis of a tension strip with fillets (refer Fig.1.1). In this problem stress concentration is to be studied in the fillet zone. Since the problem is having symmetry about both x and y axes, only one quarter of the tension strip may be considered as shown in Fig.1.2. About the symmetric axes, transverse displacements of all nodes are to be made zero. The various steps involved in the finite element analysis of this problem are discussed below:

**Step 1:** Four noded isoparametric element (refer Fig 1.3) is selected for the analysis (However note that 8 noded isoparametric element is ideal for this analysis). The four noded isoparametric element can take quadrilateral shape also as required for elements 12, 15, 18, etc. As there is no bending of strip, only displacement continuity is to be ensured but not the slope continuity. Hence displacements of nodes in x and y directions are taken as basic unknowns in the problem.

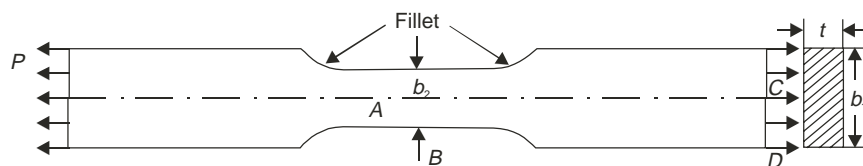


Fig. 1.1 Typical tension flat

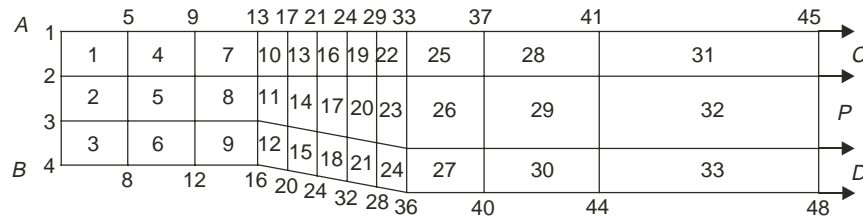


Fig. 1.2 Discretisation of quarter of tension flat

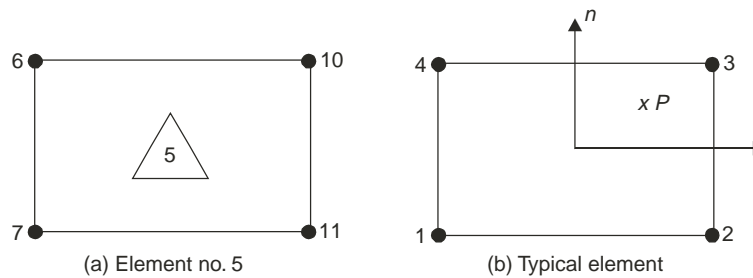


Fig. 1.3

**Step 2:** The portion to be analysed is to be discretised. Fig. 1.2 shows discretised portion. For this 33 elements have been used. There are 48 nodes. At each node unknowns are  $x$  and  $y$  components of displacements. Hence in this problem total unknowns (displacements) to be determined are  $48 \times 2 = 96$ .

**Step 3:** The displacement of any point inside the element is approximated by suitable functions in terms of the nodal displacements of the element. For the typical element (Fig. 1.3 b), displacements at  $P$  are

$$u = \sum N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

and 
$$v = \sum N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4 \quad \dots(1.2)$$

The approximating functions  $N_i$  are called shape functions or interpolation functions. Usually they are derived using polynomials. The methods of deriving these functions for various elements are discussed in this text in latter chapters.

**Step 4:** Now the stiffness characters and constant loads are to be found for each element. There are four nodes and at each node degree of freedom is 2. Hence degree of freedom in each element is  $4 \times 2 = 8$ . The relationship between the nodal displacements and nodal forces is called element stiffness characteristics. It is of the form

$$[k]_e \{\delta\}_e = \{F\}_e, \text{ as explained earlier.}$$

For the element under consideration,  $k_e$  is  $8 \times 8$  matrix and  $\delta_e$  and  $F_e$  are vectors of 8 values. In solid mechanics element stiffness matrix is assembled using variational approach i.e. by minimizing potential energy. If the load is acting in the body of element or on the surface of element, its equivalent at nodal points are to be found using variational approach, so that right hand side of the above expression is assembled. This process is called finding constant loads.



#### 4 Finite Element Analysis

**Step 5:** The structure is having  $48 \times 2 = 96$  displacement and load vector components to be determined. Hence global stiffness equation is of the form

$$\begin{array}{ccc} [k] & \{\delta\} & = \{F\} \\ 96 \times 96 & 96 \times 1 & 96 \times 1 \end{array}$$

Each element stiffness matrix is to be placed in the global stiffness matrix appropriately. This process is called assembling global stiffness matrix. In this problem force vector  $F$  is zero at all nodes except at nodes 45, 46, 47 and 48 in  $x$  direction. For the given loading nodal equivalent forces are found and the force vector  $F$  is assembled.

**Step 6:** In this problem, due to symmetry transverse displacements along  $AB$  and  $BC$  are zero. The system equation  $[k] \{\delta\} = \{F\}$  is modified to see that the solution for  $\{\delta\}$  comes out with the above values. This modification of system equation is called imposing the boundary conditions.

**Step 7:** The above 96 simultaneous equations are solved using the standard numerical procedures like Gauss-elimination or Choleski's decomposition techniques to get the 96 nodal displacements.

**Step 8:** Now the interest of the analyst is to study the stresses at various points. In solid mechanics the relationship between the displacements and stresses are well established. The stresses at various points of interest may be found by using shape functions and the nodal displacements and then stresses calculated. The stress concentrations may be studied by comparing the values obtained at various points in the fillet zone with the values at uniform zone, far away from the fillet (which is equal to  $P/b_2t$ ).

### 1.4 FINITE ELEMENT METHOD VS CLASSICAL METHODS

1. In classical methods exact equations are formed and exact solutions are obtained where as in finite element analysis exact equations are formed but approximate solutions are obtained.
2. Solutions have been obtained for few standard cases by classical methods, where as solutions can be obtained for all problems by finite element analysis.
3. Whenever the following complexities are faced, classical method makes the drastic assumptions' and looks for the solutions:
  - (a) Shape
  - (b) Boundary conditions
  - (c) Loading

Fig. 1.4 shows such cases in the analysis of slabs (plates).

To get the solution in the above cases, rectangular shapes, same boundary condition along a side and regular equivalent loads are to be assumed. In FEM no such assumptions are made. The problem is treated as it is.

4. When material property is not isotropic, solutions for the problems become very difficult in classical method. Only few simple cases have been tried successfully by researchers. FEM can handle structures with anisotropic properties also without any difficulty.
5. If structure consists of more than one material, it is difficult to use classical method, but finite element can be used without any difficulty.
6. Problems with material and geometric non-linearities can not be handled by classical methods. There is no difficulty in FEM.

Hence FEM is superior to the classical methods only for the problems involving a number of complexities which cannot be handled by classical methods without making drastic assumptions. For all regular problems, the solutions by classical methods are the best solutions. Infact, to check the validity of the FEM programs developed, the FEM solutions are compared with the solutions by classical methods for standard problems.

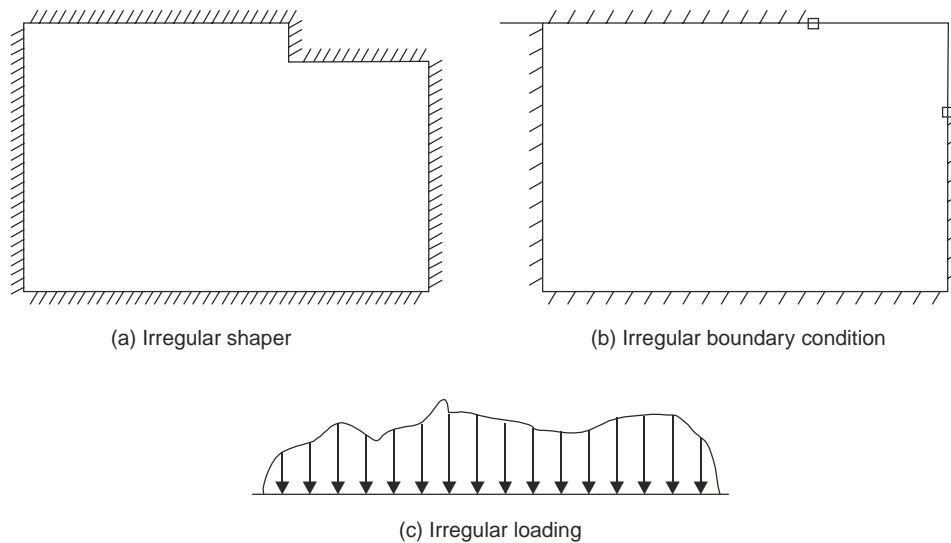


Fig. 1.4

## 1.5 FEM VS FINITE DIFFERENCE METHOD (FDM)

1. FDM makes **pointwise approximation** to the governing equations i.e. it ensures continuity only at the node points. Continuity along the sides of grid lines are not ensured.

FEM make piecewise approximation i.e. it ensures the continuity at node points as well as along the sides of the element.

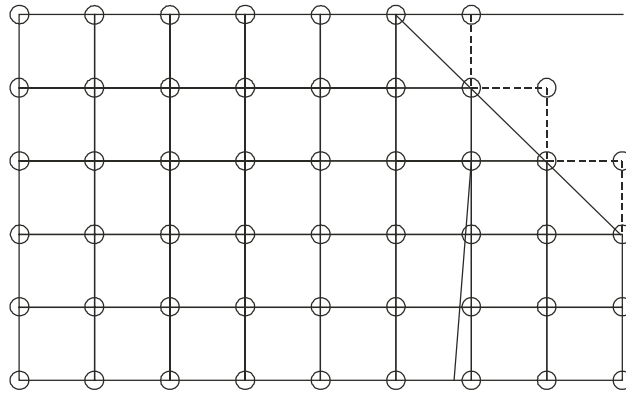
2. FDM do not give the values at any point except at node points. It do not give any approximating function to evaluate the basic values (deflections, in case of solid mechanics) using the nodal values.

FEM can give the values at any point. However the values obtained at points other than nodes are by using suitable interpolation formulae.

3. FDM makes stair type approximation to sloping and curved boundaries as shown in Fig. 1.5.

FEM can consider the sloping boundaries exactly. If curved elements are used, even the curved boundaries can be handled exactly.

4. FDM needs larger number of nodes to get good results while FEM needs fewer nodes.
5. With FDM fairly complicated problems can be handled where as FEM can handle all complicated problems.



**Fig. 1.5** *FDM approximation of shape*

## 1.6 A BRIEF HISTORY OF FEM

Engineers, physicists and mathematicians have developed finite element method independently. In 1943 Courant [1] made an effort to use piecewise continuous functions defined over triangular domain.

After that it took nearly a decade to use this distribution idea. In fifties renewed interest in this field was shown by Polya [2], Hersh [3] and Weinberger [4]. Argyris and Kelsey [5] introduced the concept of applying energy principles to the formation of structural analysis problems in 1960. In the same year Clough [6] introduced the word '**Finite Element Method**'.

In sixties convergence aspect of the finite element method was pursued more rigorously. One such study by Melesh [7] led to the formulation of the finite element method based on the principles of minimum potential energy. Soon after that de Veubeke [8] introduced equilibrium elements based on the principles of minimum potential energy, Pion [9] introduced the concept of hybrid element using the dual principle of minimum potential energy and minimum complementary energy.

In Late 1960's and 1970's, considerable progress was made in the field of finite element analysis. The improvements in the speed and memory capacity of computers largely contributed to the progress and success of this method. In the field of solid mechanics from the initial attention focused on the elastic analysis of plane stress and plane strain problems, the method has been successfully extended to the cases of the analysis of three dimensional problems, stability and vibration problems, non-linear analysis. A number of books [10 – 20] have appeared and made this field interesting.

## 1.7 NEED FOR STUDYING FEM

Now, a number of users friendly packages are available in the market. Hence one may ask the question 'What is the need to study FEA?'.

The above argument is not sound. The finite element knowledge makes a good engineer better while just user without the knowledge of FEA may produce more dangerous results. To use the FEA packages properly, the user must know the following points clearly:

1. Which elements are to be used for solving the problem in hand.
2. How to discretise to get good results.
3. How to introduce boundary conditions properly.

4. How the element properties are developed and what are their limitations.
5. How the displays are developed in pre and post processor to understand their limitations.
6. To understand the difficulties involved in the development of FEA programs and hence the need for checking the commercially available packages with the results of standard cases.

Unless user has the background of FEA, he may produce worst results and may go with overconfidence. Hence it is necessary that the users of FEA package should have sound knowledge of FEA.

## 1.8 WARNING TO FEA PACKAGE USERS

When hand calculations are made, the designer always gets the feel of the structure and get rough idea about the expected results. This aspect cannot be ignored by any designer, whatever be the reliability of the program, a complex problem may be simplified with drastic assumptions and FEA results obtained. Check whether expected trend of the result is obtained. Then avoid drastic assumptions and get more refined results with FEA package. User must remember that structural behaviour is not dictated by the computer programs. Hence the designer should develop feel of the structure and make use of the programs to get numerical results which are close to structural behaviour.

### QUESTIONS

1. Explain the concept of FEM briefly and outline the procedure.
2. Discuss the advantages and disadvantages of FEM over
  - (i) Classical method
  - (ii) Finite difference method.
3. Clearly point out the situations in which FEM is preferred over other methods.
4. When there are several FEM packages are available is there need to study this method? Discuss.

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# 2

## Basic Equations in Elasticity

### 2.1 INTRODUCTION

This chapter summarizes the results from theory of elasticity which are useful in solving the problems in structural and continuum mechanics by the finite element method.

### 2.2 STRESSES IN A TYPICAL ELEMENT

In theory of elasticity, usually right hand rule is used for selecting the coordinate system. Fig. 2.1 shows various orientations of right hand rule of the coordinate systems. Equations derived for any one such orientation hold good for all other orientations of

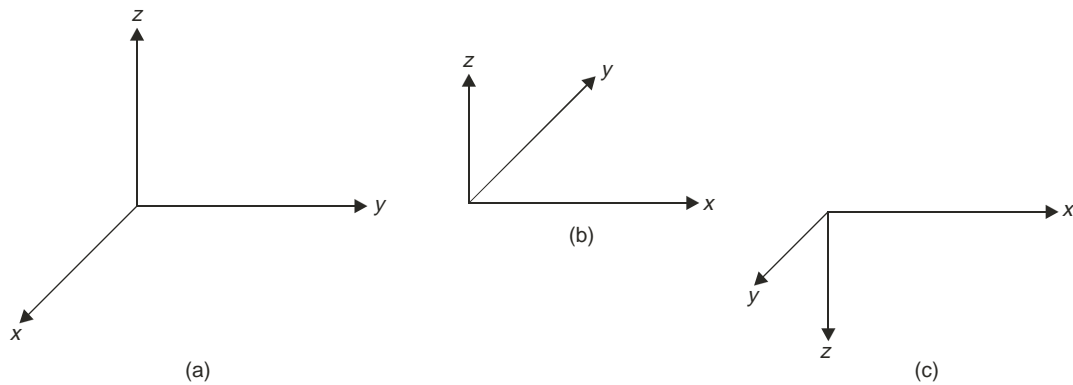


Fig. 2.1

coordinate system with right hand rule. In this Chapter orientation shown in Fig. 2.1(a) is used for the explanation. Fig. 2.2 shows a typical three dimensional element of size  $dx \times dy \times dz$ . Face  $abcd$  may be called as negative face of  $x$  and the face  $efgh$  as the positive face of  $x$  since the  $x$  value for face  $abcd$  is less than that for the face  $efgh$ . Similarly the face  $aehd$  is negative face of  $y$  and  $bfgc$  is positive face of  $y$ . Negative and positive faces of  $z$  are  $dhgc$  and  $aefb$ .

The direct stresses  $\sigma$  and shearing stresses  $\tau$  acting on the negative faces are shown in the Fig. 2.3 with suitable subscript. It may be noted that the first subscript of shearing stress is the plane and the second subscript is the direction. Thus the  $\tau_{xy}$  means shearing stress on the plane where  $x$  value is constant and  $y$  is the direction.

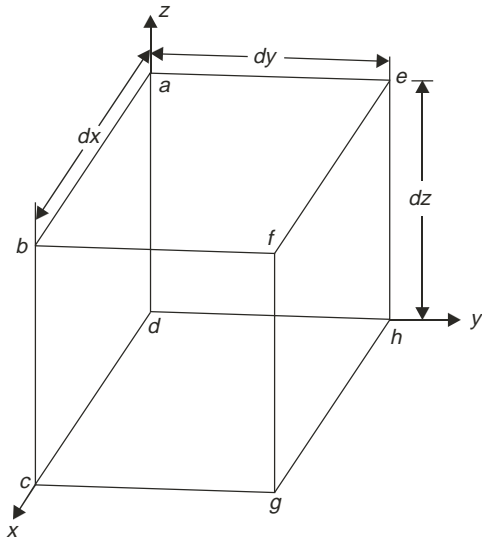


Fig. 2.2

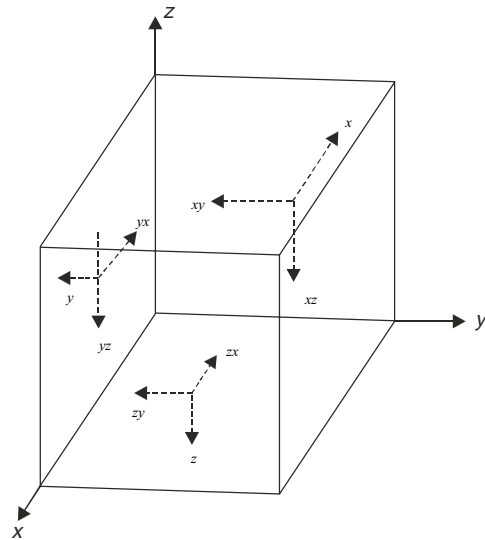


Fig. 2.3

In a stressed body, the values of stresses change from face to face of an element. Hence on positive face the various stresses acting are shown in Fig. 2.4 with superscript '+'.

All these forces are listed in table 2.1

**Note the sign convention:** A stress is positive when it is on positive face in positive direction or on negative face in negative direction. In other words the stress is + ve when it is as shown in Figs 2.3 and 2.4.

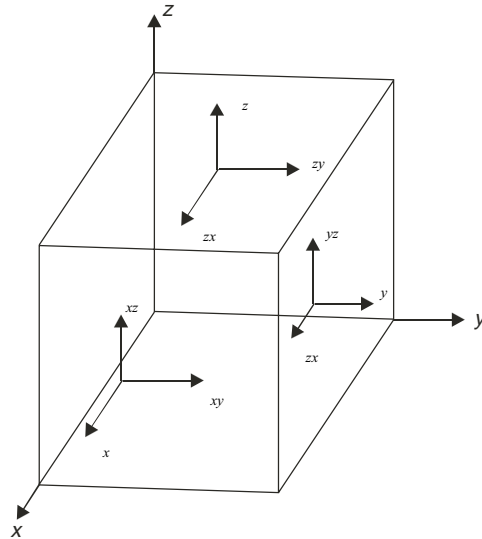
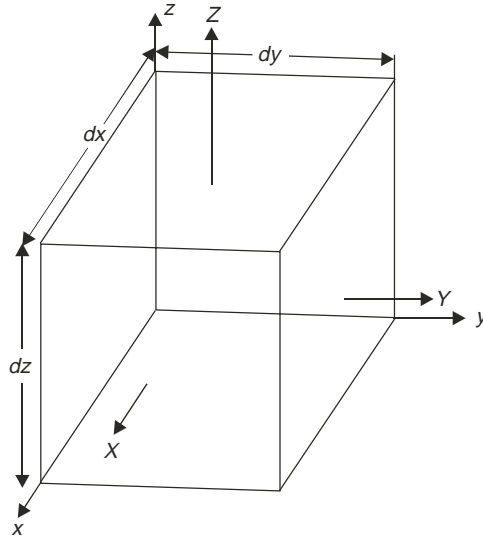


Fig. 2.4

**Table 2.1** Stresses on a typical element

Face	Stress on -ve Face	Stresses on +ve Face
$x$	$\sigma_x$ $\tau_{xy}$ $\tau_{xz}$	$\sigma_x^+ = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx$ $\tau_{xy}^+ = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx$ $\tau_{xz}^+ = \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} dx$
$y$	$\sigma_y$ $\tau_{yx}$ $\tau_{yz}$	$\sigma_y^+ = \sigma_y + \frac{\partial \sigma_y}{\partial y} dy$ $\tau_{yx}^+ = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy$ $\tau_{yz}^+ = \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy$
$z$	$\sigma_z$ $\tau_{zx}$ $\tau_{zy}$	$\sigma_z^+ = \sigma_z + \frac{\partial \sigma_z}{\partial z} dz$ $\tau_{zx}^+ = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz$ $\tau_{zy}^+ = \tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz$

Note that stress on positive face is equal to the stress on negative face plus rate of change of that stress multiplied by the distance between the faces.



**Fig. 2.5**



## 12 Finite Element Analysis

Let the intensity of body forces acting on the element in  $x$ ,  $y$ ,  $z$  directions be  $X$ ,  $Y$  and  $Z$  respectively as shown in Fig 2.5. The intensity of body forces are uniform over entire body. Hence the total body force in  $x$ ,  $y$ ,  $z$  direction on the element shown are given by

- (i)  $X dx dy dz$  in  $x$  – direction
- (ii)  $Y dx dy dz$  in  $y$  – direction and
- (iii)  $Z dx dy dz$  in  $z$  – direction

### 2.3 EQUATIONS OF EQUILIBRIUM

Considering all the forces acting, we can write equations of equilibrium for the element.

$$\sum F_x = 0$$

$$\sigma_x^+ dy dz - \sigma_x dy dz + \tau_{yx}^+ dx dz - \tau_{yx} dx dz + \tau_{zx}^+ dx dy - \tau_{zx} dx dy + X dx dy dz = 0$$

i.e.  $\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy dz - \sigma_x dy dz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx dz - \tau_{yx} dx dz$

$$+ \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) dx dy - \tau_{zx} dx dy + X dx dy dz = 0 \quad \dots(i)$$

Simplifying and then dividing throughout by  $dx dy dz$ , we get

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

Similarly  $\Sigma F_y = 0$  and  $\Sigma F_z = 0$  equilibrium conditions give,

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \quad \dots(ii)$$

and  $\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \quad \dots(iii)$

Now,  $\Sigma$  moment about x-axis = 0 through the centroid of the element gives

$$\tau_{yz}^+ dx dz \frac{dy}{2} + \tau_{yz} dx dz \frac{dy}{2} - \left[ \tau_{zy}^+ dx dz \frac{dy}{2} + \tau_{zy} dx dz \frac{dy}{2} \right] = 0$$

i.e.  $\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy\right) dx dy \frac{dz}{2} + \tau_{yz} dx dy \frac{dz}{2} - \left[ \left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z} dz\right) dx dy \frac{dz}{2} + \tau_{zy} dx dz \frac{dz}{2} \right] = 0$

Neglecting the small quantity of higher (4<sup>th</sup>) order and dividing throughout by  $dx dy dz$ , we get

$$\tau_{yz} = \tau_{zy} \quad \dots(iv)$$

Similarly the moment equilibrium conditions about y-axis and z-axis result into

$$\tau_{xz} = \tau_{zx} \quad \dots(v)$$

and 
$$\tau_{xy} = \tau_{yx} \quad \dots(\text{vi})$$

Thus the stress vector is

$$|\sigma|^T = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{xy} \ \tau_{yz} \ \tau_{xz}] \quad \dots(2.1)$$

and the equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0$$

and 
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \quad \dots(2.2)$$

and note that

$$\tau_{xy} = \tau_{yx}, \ \tau_{yz} = \tau_{zy} \ \text{and} \ \tau_{xz} = \tau_{zx} \quad \dots(2.3)$$

## 2.4 STRAINS

Corresponding to the six stress components given in equation 2.1, the state of strain at a point may be divided into six strain components as shown below:

$$\{\varepsilon\}^T = [\varepsilon_x \ \varepsilon_y \ \varepsilon_z \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{yx}] \quad \dots(2.4)$$

## 2.5 STRAIN DISPLACEMENT EQUATIONS

Taking displacement components in  $x, y, z$  directions as  $u, v$  and  $w$  respectively, the relations among components of strains and components of displacements are

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right]$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right]$$

$$\varepsilon_z = \frac{\partial w}{\partial z} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right]$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \cdot \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \cdot \frac{\partial w}{\partial z} \quad \dots(2.5)$$

and

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial z}$$

In equation 2.5, strains are expressed upto the accuracy of second order (quadratic) changes in displacements. These equations may be simplified to the first (linear) order accuracy only by dropping the second order changes terms. Then linear strain – displacement relation is given by:

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} & \gamma_{xy} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ \epsilon_y &= \frac{\partial v}{\partial y} & \gamma_{yz} &= \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\ \epsilon_z &= \frac{\partial w}{\partial z} & \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned} \quad \dots(2.6)$$

Equations 2.6 are used in small deflection theories and equations 2.5 in large deflection theories.

## 2.6 LINEAR CONSTITUTIVE EQUATIONS

The constitutive law expresses the relationship among stresses and strains. In theory of elasticity, usually it is considered as linear. In one dimensional stress analysis, the linear constitutive law is stress is proportional to strain and the constant of proportionality is called Young’s modulus. It is very well known as Hooke’s law. The similar relation is expressed among the six components of stresses and strains and is called ‘**Generalized Hookes Law**’. This may be stated as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{Bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{Bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad \dots(2.7)$$

or in matrix form

$$\{\sigma\} = [D] \{\epsilon\},$$

where  $D$  is  $6 \times 6$  matrix of constants of elasticity to be determined by experimental investigations for each material. As  $D$  is symmetric matrix [ $D_{ij} = D_{ji}$ ], there are 21 material properties for linear elastic **Anisotropic Materials**.

Certain materials exhibit symmetry with respect to planes within the body. Such materials are called **Ortho tropic materials**. Hence for orthotropic materials, the number of material constants reduce to 9 as shown below:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{Bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ & D_{22} & D_{23} & 0 & 0 & 0 \\ & & D_{33} & 0 & 0 & 0 \\ & Sym & & D_{44} & 0 & 0 \\ & & & & D_{55} & 0 \\ & & & & & D_{66} \end{Bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \quad \dots(2.8)$$

Using the Young's Moduli and Poisons ratio terms the above relation may be expressed as:

$$\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E_x} - \mu_{yx} \frac{\sigma_y}{E_y} - \mu_{zx} \frac{\sigma_z}{E_z} \\ \epsilon_y &= -\mu_{xy} \frac{\sigma_x}{E_x} + \frac{\sigma_y}{E_y} - \mu_{zy} \frac{\sigma_z}{E_z} \\ \epsilon_z &= -\mu_{xz} \frac{\sigma_x}{E_x} - \mu_{yz} \frac{\sigma_y}{E_y} + \frac{\sigma_z}{E_z} \\ \gamma_{xy} &= \frac{\tau_{xy}}{G_{xy}}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G_{yz}}, \quad \gamma_{zx} = \frac{\tau_{zx}}{G_{zx}} \end{aligned} \quad \dots(2.9)$$

Note that there are 12 material properties in equations 2.9. However only nine of these are independent because the following relations exist

$$\frac{E_x}{\mu_{xy}} = \frac{E_y}{\mu_{yx}}, \quad \frac{E_y}{\mu_{yz}} = \frac{E_z}{\mu_{zy}}, \quad \frac{E_z}{\mu_{zx}} = \frac{E_x}{\mu_{xz}} \quad \dots(2.10)$$

For **Isotropic Materials** the above set of equations are further simplified. An isotropic material is the one that has same material property in all directions. In other word for isotropic materials,

$$\begin{aligned} E_x &= E_y = E_z \text{ say } E \text{ and} \\ \mu_{xy} &= \mu_{yx} = \mu_{yz} = \mu_{zy} = \mu_{xz} = \mu_{zx} \text{ say } \mu \end{aligned} \quad \dots(2.11)$$

Hence for a three dimensional problem, the strain stress relation for isotropic material is,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{E} & -\frac{\mu}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\mu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{1-\mu}{2} & 0 & 0 \\ & & & & \frac{1-\mu}{2} & 0 \\ & & & & & \frac{1-\mu}{2} \end{Bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \quad \dots(2.12)$$

Since  $G = \frac{E}{2(1-\mu)}$  and stress – strain relation is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{Bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ & 1-\mu & \mu & 0 & 0 & 0 \\ & & 1-\mu & 0 & 0 & 0 \\ & & & \frac{1-2\mu}{2} & 0 & 0 \\ & & & & \frac{1-2\mu}{2} & 0 \\ & & & & & \frac{1-2\mu}{2} \end{Bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \dots(2.13)$$

In case of two dimensional elasticity, the above relations get further simplified. There are two types of two dimensional elastic problems, namely plane stress and plane strain problems.

### Plane Stress Problems

The thin plates subject to forces in their plane only, fall under this category of the problems. Fig. 2.6 shows a typical plane stress problem. In this, there is

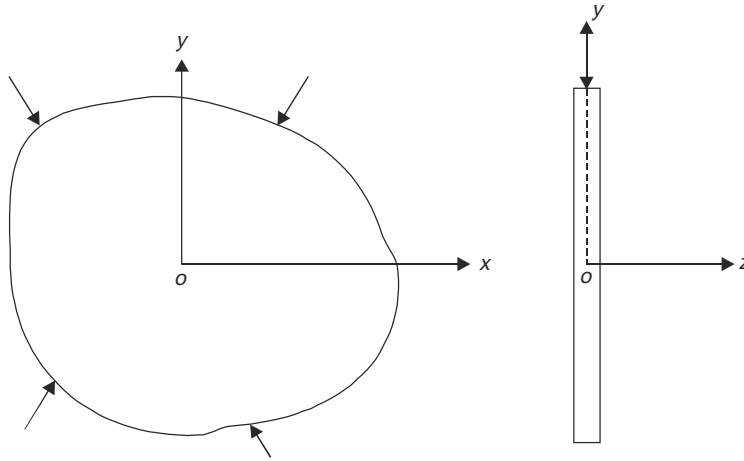


Fig. 2.6

no force in the z-direction and no variation of any forces in z-direction. Hence

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

The conditions  $\tau_{xz} = \tau_{yz} = 0$  give  $\gamma_{xz} = \gamma_{yz} = 0$  and the condition  $\sigma_z = 0$  gives,

$$\sigma_z = \mu\epsilon_x + \mu\epsilon_y + (1-\mu)\epsilon_z = 0$$

i.e. 
$$\epsilon_z = -\frac{\mu}{1-\mu} (\epsilon_x + \epsilon_y)$$

If this is substituted in equation 2.13 the constitutive law reduces to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \dots(2.14)$$

### Plane Strain Problems

A long body subject to significant lateral forces but very little longitudinal forces falls under this category of problems. Examples of such problems are pipes, long strip footings, retaining walls, gravity dams, tunnels, etc. (refer Fig. 2.7). In these problems, except for a small distance at the ends, state of stress is represented by any small longitudinal strip. The displacement in longitudinal direction (z-direction) is zero in typical strip. Hence the strain components,

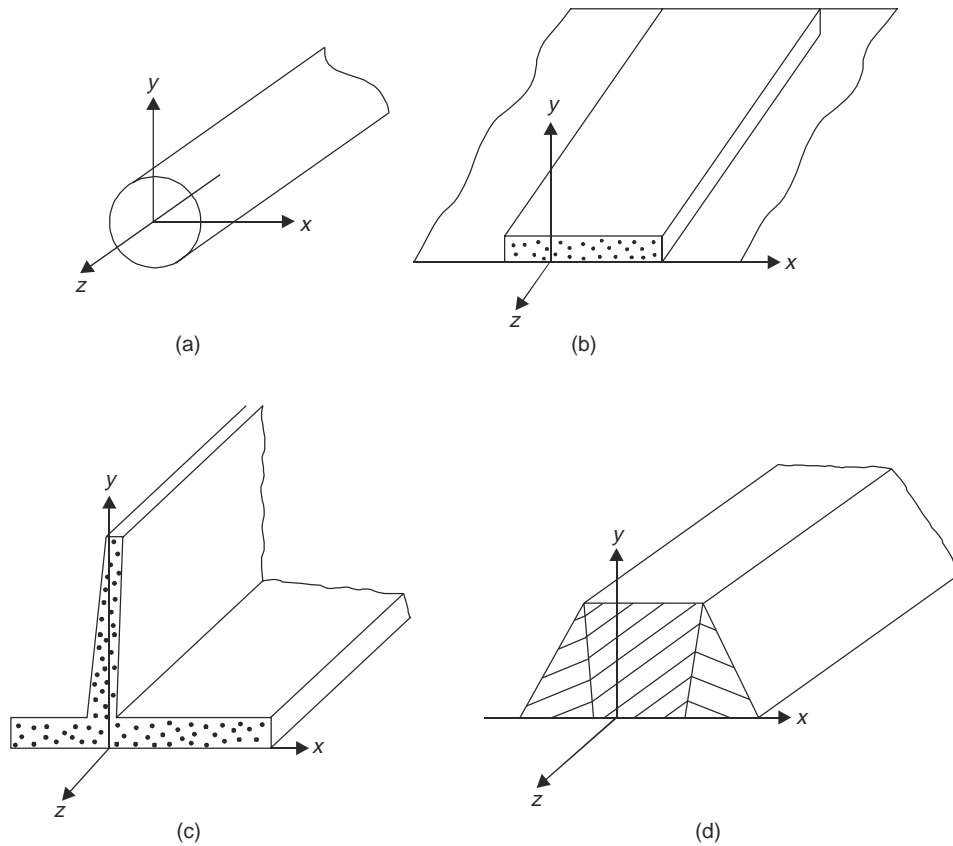


Fig. 2.7

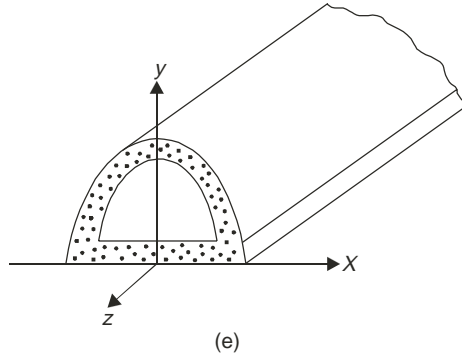


Fig. 2.7 (contd)

$$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$\gamma_{xz} = \gamma_{yz} = 0$  means  $\tau_{xz}$  and  $\tau_{yz}$  are zero.

$\epsilon_z = 0$  means

$$\epsilon_z = \frac{\sigma_z}{E} - \mu \frac{(\sigma_x + \sigma_y)}{E} = 0$$

i.e.

$$\sigma_z = \mu(\sigma_x + \sigma_y)$$

Hence equation 2.13 when applied to plane strains problems reduces to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1 + \mu)(1 - 2\mu)} \begin{pmatrix} 1 - \mu & \mu & 0 \\ \mu & 1 - \mu & 0 \\ 0 & 0 & \frac{1 - 2\mu}{2} \end{pmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \dots(2.15)$$

### Axi-Symmetric Problems

Axi-symmetric structures are those which can be generated by rotating a line or curve about an axis. Cylinders (refer Fig. 2.8) are the common examples of axisymmetric structures. If such structures are subjected to axisymmetric loadings like uniform internal or external pressures, uniform self weight or live load uniform over the surface,

there exist symmetry about any axis. The advantage of symmetry may be made use to simplify the analysis. In these problems cylindrical coordinates can be used advantageously. Because of symmetry, the stress components are independent of the angular ( $\theta$ ) coordinate. Hence all derivatives with respect to  $\theta$  vanish i.e. in these cases.

$$v = \gamma_{r\theta} = \gamma_{\theta z} = \tau_{r\theta} = \tau_{\theta z} = 0$$

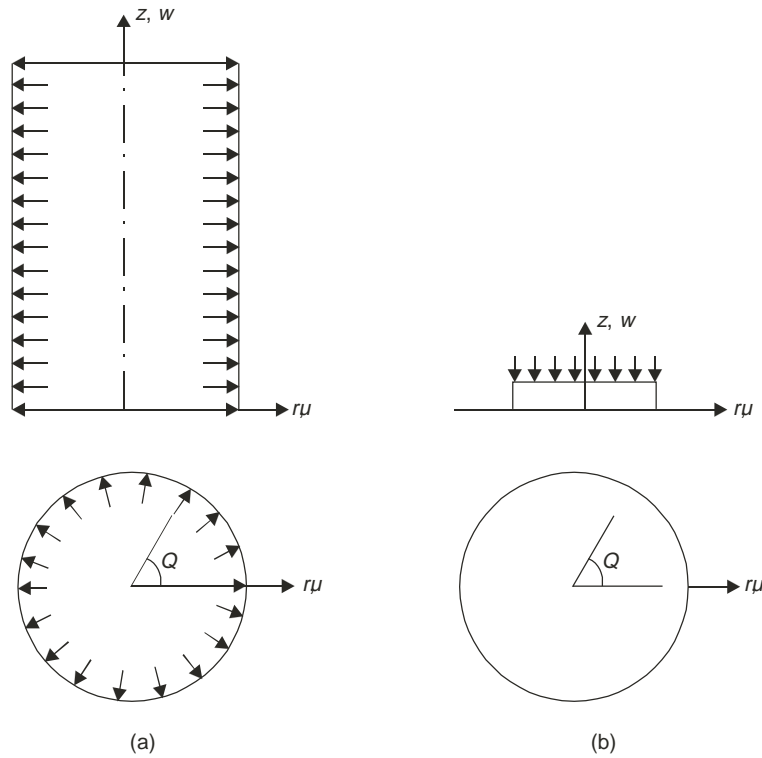


Fig. 2.8

Hence there are only four nonzero components. The strain displacement relations for these components are

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r}, & \epsilon_\theta &= \frac{u}{r}, & \epsilon_z &= \frac{\partial w}{\partial z} \text{ and} \\ \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{aligned} \quad \dots(2.16)$$

In these cases stress-strain relation is

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1 + \mu)(1 - 2\mu)} \begin{pmatrix} 1 - \mu & \mu & \mu & 0 \\ & 1 - \mu & \mu & 0 \\ & & 1 - \mu & 0 \\ & & & \frac{1 - 2\mu}{2} \end{pmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \quad \dots(2.17)$$

**QUESTIONS**

1. Draw a typical three dimensional element and indicate state of stress in their positive senses.
2. Derive the equations of equilibrium in case of a three dimensional stress system.



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3. State and explain generalized Hooke's law.
4. Give strain displacement relations in case of a three dimensional elasticity problem upto
  - (i) accuracy of linear terms only
  - (ii) accuracy of quadratic terms.
5. Explain the terms, 'Anisotropic', 'Orthotropic' and 'Isotropic' as applied to material properties.
6. Give constitutive laws for three dimensional problems of
  - (i) orthotropic materials
  - (ii) isotropic materials.
7. Explain the terms 'Plane stress' and 'Plane strain' problems. Give constitutive laws for these cases.
8. Explain the term 'Axi-symmetric problems' and give constitutive law for such problems.

# 3

## Matrix Displacement Formulation

### 3.1 INTRODUCTION

Though mathematicians, physicists and stress analysts worked independently in the field of FEM, it is the matrix displacement formulation of the stress analysts which lead to fast development of FEM. Infact till the word FEM became popular, stress analyst worked in this field in the name of matrix displacement method. In matrix displacement method stiffness matrix of an element is assembled by direct approach while in FEM though direct stiffness matrix may be treated as an approach for assembling element properties (stiffness matrix as far as stress analysis is concerned), it is the energy approach which has revolutionized entire FEM.

Hence in this chapter, a brief explanation of matrix displacement method is presented and solution techniques for simultaneous equations are discussed briefly.

### 3.2 MATRIX DISPLACEMENT EQUATIONS

The standard form of matrix displacement equation is,

$$[k] \{\delta\} = \{F\}$$

where  $[k]$  is stiffness matrix

$\{\delta\}$  is displacement vector and

$\{F\}$  is force vector in the coordinate directions

The element  $k_{ij}$  of stiffness matrix maybe defined as the force  $k$  at coordinate  $i$  due to unit displacement in coordinate direction  $j$ .

The direct method of assembling stiffness matrix for few standard cases is briefly given in this article.

#### 1. Bar Element

Common problems in this category are the bars and columns with varying cross section subjected to axial forces as shown in Fig. 3.1.

For such bar with cross section  $A$ , Young's Modulus  $E$  and length  $L$  (Fig. 3.2 (a)) extension/shortening  $\delta$  is given by

$$\delta = \frac{PL}{EA}$$

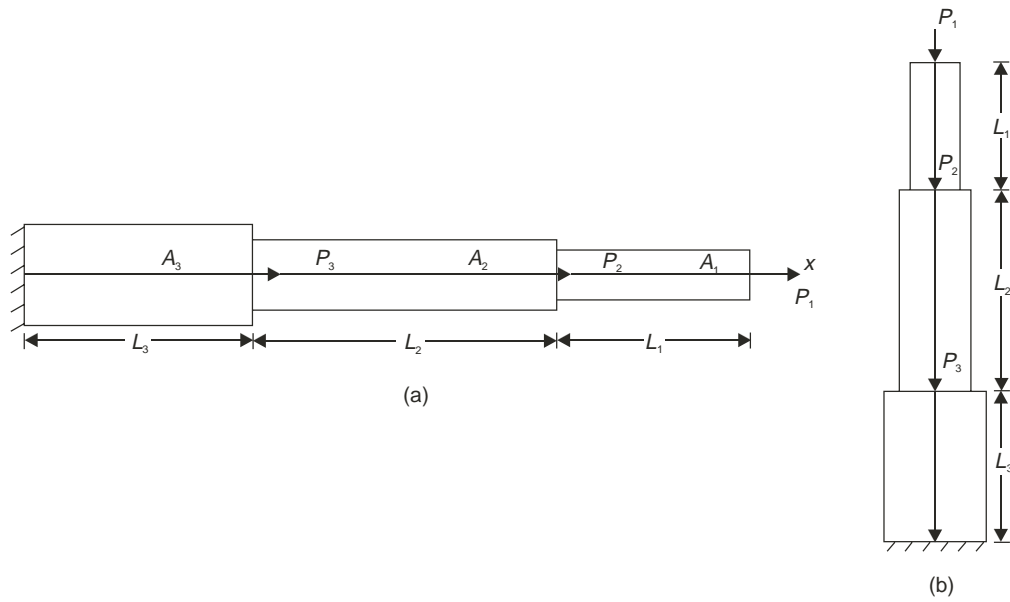


Fig. 3.1

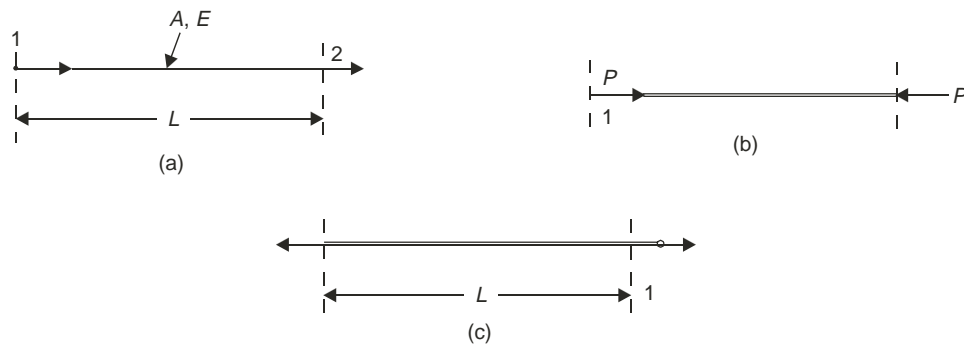


Fig. 3.2

$$\therefore P = \frac{EA}{L} \delta$$

$$\therefore \text{If } \delta = 1, P = \frac{EA}{L}$$

By giving unit displacement in coordinate direction 1, the forces development in the coordinate direction 1 and 2 can be found (Fig. 3.2 (b)). Hence from the definition of stiffness matrix,

$$k_{11} = \frac{EA}{L} \text{ and } k_{21} = -\frac{EA}{L}$$

Similarly giving unit displacement in coordinate direction 2 (refer Fig. 3.2 (c)), we get

$$k_{12} = -\frac{EA}{L} \text{ and } k_{22} = \frac{EA}{L}$$

Thus,

$$[k] = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \dots(3.5)$$

## 2. Truss Element

Members of the trusses are subjected to axial forces only, but their orientation in the plane may be at any angle to the coordinate directions selected. Figure 3.3 shows a typical case in a plane truss. Figure 3.4 (a) shows a typical member of the truss with Young's Modulus  $E$ , cross sectional area  $A$ , length  $L$  and at angle  $\theta$  to x-axis

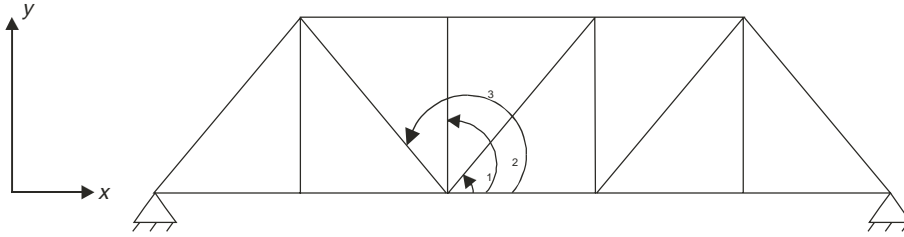


Fig. 3.3

- (i) Unit displacement of end 1 in x-direction.

Due to this, displacement along the axis is  $1 \times \cos \theta$  as shown in Fig. 3.4 (b). Hence forces development at the ends are as shown in figure.

$$P = \frac{EA}{L} \cos \theta$$

From the definition of elements of stiffness matrix, we get

$$k_{11} = P \cos \theta = \frac{EA}{L} \cos^2 \theta$$

$$k_{21} = P \sin \theta = \frac{EA}{L} \cos \theta \sin \theta$$

$$k_{31} = -P \cos \theta = -\frac{EA}{L} \cos^2 \theta$$

$$k_{41} = -P \sin \theta = -\frac{EA}{L} \cos \theta \sin \theta$$

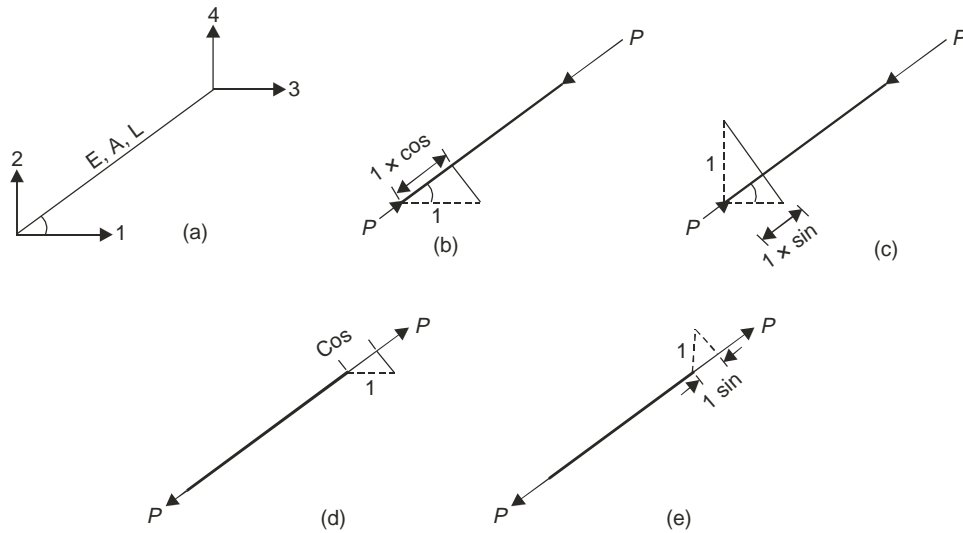


Fig. 3.4

(ii) Unit displacement in coordinate direction 2;

This case is shown in Fig. 3.4 (c). In this case axial deformation is  $1 \times \sin \theta$  and the forces developed at each end are as shown in the figure.

$$\therefore P = \frac{EA}{L} \sin \theta$$

$$k_{12} = P \cos \theta = \frac{EA}{L} \sin \theta \cos \theta$$

$$k_{22} = P \sin \theta = \frac{EA}{L} \sin^2 \theta$$

$$k_{32} = -P \cos \theta = -\frac{EA}{L} \sin \theta \cos \theta$$

$$k_{42} = -P \sin \theta = -\frac{EA}{L} \sin^2 \theta$$

(iii) Unit displacement in coordinate direction 3,

Extension along the axis is  $1 \times \sin \theta$  and hence the forces developed are as shown in the Fig. 3.4 (d)

$$\therefore P = \frac{EA}{L} \cos \theta$$

$$k_{13} = -P \cos \theta = -\frac{EA}{L} \cos^2 \theta$$

$$k_{23} = -P \sin \theta = -\frac{EA}{L} \cos \theta \sin \theta$$

$$k_{33} = P \cos \theta = \frac{EA}{L} \cos^2 \theta$$

$$k_{43} = P \sin \theta = \frac{EA}{L} \cos \theta \sin \theta$$

(vi) Due to unit displacement in coordinate direction 4,

Extension of the bar is equal to  $1 \times \sin \theta$ , and hence the forces developed are as shown in Fig. 3.4 (e).

$$\therefore P = \frac{EA}{L} \sin \theta$$

$$k_{14} = -P \cos \theta = -\frac{EA}{L} \sin \theta \cos \theta$$

$$k_{24} = -P \sin \theta = -\frac{EA}{L} \sin^2 \theta$$

$$k_{34} = P \cos \theta = \frac{EA}{L} \sin \theta \cos \theta$$

$$k_{44} = P \sin \theta = \frac{EA}{L} \sin^2 \theta$$

$\therefore$  The stiffness matrix is

$$[k] = \frac{EA}{L} \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta & -\cos^2 \theta & -\cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta & -\cos \theta \sin \theta & -\sin^2 \theta \\ -\cos^2 \theta & -\cos \theta \sin \theta & \cos^2 \theta & \cos \theta \sin \theta \\ -\cos \theta \sin \theta & -\sin^2 \theta & \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$$

$$= \frac{EA}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \quad \dots(3.6)$$

Where  $l$  and  $m$  are the direction cosines of the member i.e.  $l = \cos \theta$  and  $m = \cos (90 - \theta) = \sin \theta$ .

(v) Beam Element

In the analysis of continuous beams normally axial deformation is negligible (small deflection theory) and hence only two unknowns may be taken at each end of a element (Fig. 3.5). Typical element and the coordinates of displacements selected are shown in Fig. 3.5 (b). The end forces

developed due to unit displacement in all the four coordinate directions are shown in Fig. 3.6 (a, b, c, d).

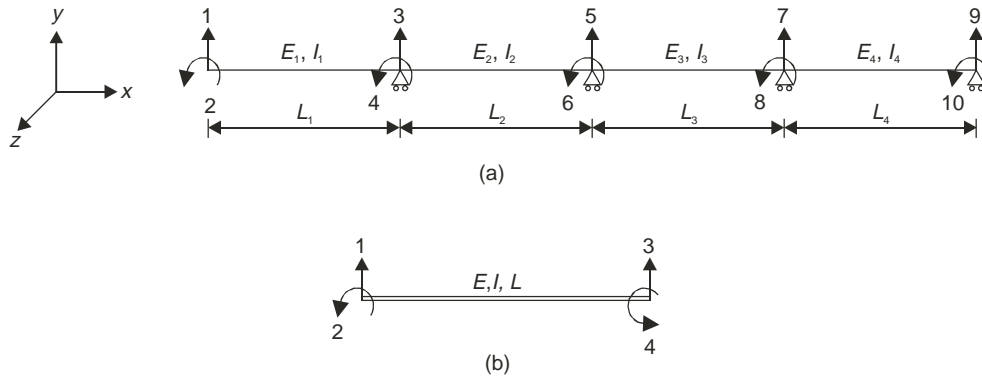


Fig. 3.5

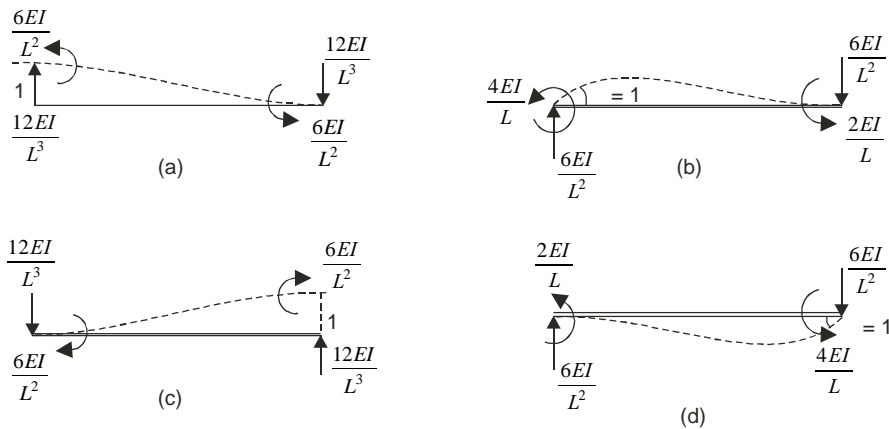


Fig. 3.6

From the definition of stiffness matrix and looking at positive senses indicated, we can write

(a) Due to unit displacement in coordinate direction 1,

$$k_{11} = \frac{12EI}{L^3} \quad k_{21} = \frac{6EI}{L^2} \quad k_{31} = -\frac{12EI}{L^3} \quad k_{41} = \frac{6EI}{L^2}$$

(b) Due to unit displacement in coordinate direction 2,

$$k_{12} = \frac{6EI}{L^2} \quad k_{22} = \frac{4EI}{L} \quad k_{32} = -\frac{6EI}{L^2} \quad k_{42} = \frac{2EI}{L}$$

(c) Due to unit displacement in coordinate direction 3,

$$k_{13} = -\frac{12EI}{L^3} \quad k_{23} = -\frac{6EI}{L^2} \quad k_{33} = \frac{12EI}{L^3} \quad k_{43} = -\frac{6EI}{L^2}$$

(d) Due to unit displacement in coordinate direction 4,

$$k_{14} = \frac{6EI}{L^2} \quad k_{24} = \frac{2EI}{L} \quad k_{34} = -\frac{6EI}{L^2} \quad k_{44} = \frac{4EI}{L}$$

$$\therefore [k] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad \dots(3.7)$$

If axial deformations in the beam elements are to be considered as in case of columns of frames, etc. (Fig. 3.7), it may be observed that axial force do not affect values of bending moment and shear force and vice versa is also true. Hence stiffness matrix for the element shown in Fig. 3.8 is obtained by combining the stiffness matrices of bar element and beam element and arranging in proper locations. For this case

$$[k] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad \dots(3.8)$$

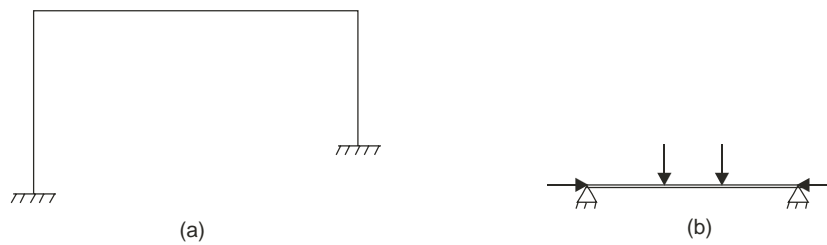


Fig. 3.7

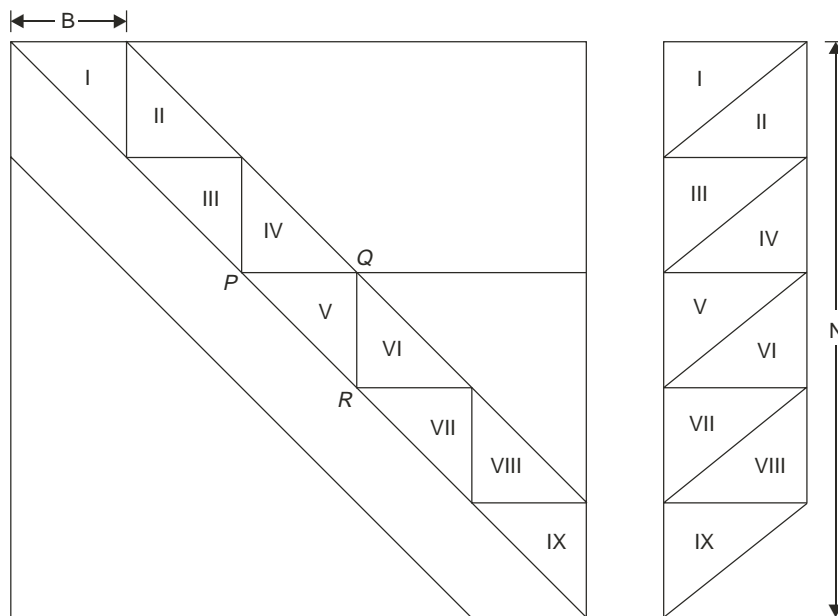


Fig. 3.8



The following special features of matrix displacement equations are worth noting:

- (i) The matrix is having diagonal dominance and is positive definite. Hence in the solution process there is no need to rearrange the equations to get diagonal dominance.
- (ii) The matrix is symmetric. It is obvious from Maxwell's reciprocal theorem. Hence only upper or lower triangular elements may be formed and others obtained using symmetry.
- (iii) The matrix is having banded nature i.e. the nonzero elements of stiffness matrix are concentrated near the diagonal of the matrix. The elements away from the diagonal are zero. Considerable saving is effected in storage requirement of stiffness matrix in the memory of computers by avoiding storage of zero values of stiffness matrices. The banded nature of matrix is shown in Fig. 3.9.



**Fig. 3.9**

In this case instead of storing  $N \times N$  size matrix only  $N \times B$  size matrix can be stored.

### 3.3 SOLUTION OF MATRIX DISPLACEMENT EQUATIONS

The matrix displacement equations are linear simultaneous equations. These equations can be solved using Gaussian elimination method. Let the equations to be solved be

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2k} & \dots & a_{2n} \\ \vdots & & & & & & \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{kk} & \dots & a_{kn} \\ \vdots & & & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nk} & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \\ \vdots \\ b_n \end{Bmatrix} \quad \dots(3.2)$$

i.e.  $[A] \{x\} = \{b\}$

The Gauss elimination method consists in reducing A matrix to upper triangular matrix and then finding the variables  $x_n, x_{n-1}, \dots, x_k, \dots, x_2, x_1$  by back substitution

**Step I:** To eliminate  $x_1$  in the lower equations:

- (i) First equation is maintained as it is
- (ii) For equations below 1,

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{ij}$$

$$\text{and } b_i^{(1)} = b_i - \frac{a_{i1}}{a_{11}} b_1$$

At the end of this, the equations will be

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2k}^{(1)} & \dots & a_{2n}^{(1)} \\ \vdots & & & & & & \\ 0 & a_{k2}^{(1)} & a_{k3}^{(1)} & \dots & a_{kk}^{(1)} & \dots & a_{kn}^{(1)} \\ \vdots & & & & & & \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \dots & a_{nk}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2^{(1)} \\ \vdots \\ b_k^{(1)} \\ \vdots \\ b_n^{(1)} \end{Bmatrix}$$

The above process is called pivotal operation on  $a_{11}$ . For pivotal operation on  $a_{kk}$ , no changes are made in  $k^{\text{th}}$  row but for the rows below  $k^{\text{th}}$ ,

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)} \text{ for } i, j = k + 1, \dots, n.$$

$$\text{and } b_i^{(k)} = \frac{b_i^{(k-1)}}{a_{kk}^{(k-1)}} b_k^{(k-1)} \text{ for } i = k + 1, \dots, n.$$

After  $n - 1$  pivotal operations, matrix equation is of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2k}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3k}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & a_{kk}^{(k-1)} & \dots & a_{kn}^{(k-1)} \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 0 & \dots & a_{nn}^{(n-1)} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_k^{(k-1)} \\ \vdots \\ b_n^{(k-1)} \end{Bmatrix} \quad \dots(3.3)$$

From the last equation,

$$x_n = \frac{b_n}{a_{nn}}$$

and then,

$$x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij} x_j}{a_{ij}}, \quad i = n - 1, n - 2 \dots 1 \quad \dots(3.4)$$

Thus the required solution is obtained.

### 3.4 TECHNIQUES OF SAVING COMPUTER MEMORY REQUIREMENTS

In FEM size of stiffness matrix of size  $1000 \times 1000$  or even more is not uncommon. Hence memory requirement for storing stiffness matrix is very high. If user tries to implement Gaussian elimination straight way as described above, ends up with the problem of shortage of memory. The following techniques are used to reduce memory requirement for storing the stiffness matrices:

- (i) Use of symmetry and banded nature
- (ii) Partitioning of matrix (Frontal solution).
- (iii) Skyline storage.

#### (i) Use of Symmetry and Banded Nature

Since the stiffness matrix is always symmetric and banded in nature, techniques have been developed to store only semiband width of non-zero elements and get the solution. If  $B$  is the semiband width of  $N \times N$  matrix we need to store only  $N \times B$  elements as indicated in Fig. 3.9(b). The diagonal of the given matrix is stored as the first column of the modified matrix. The computer coding is modified to use modified matrix for the solution of the given problem. The modification required is,

$$a_{ij} = a_i, (j - i + 1)$$

#### (ii) Partitioning of the Matrix

For larger systems, even this method of storage may be inadequate. In such cases the partitioning of the matrix is made as shown in Fig. 3.10.

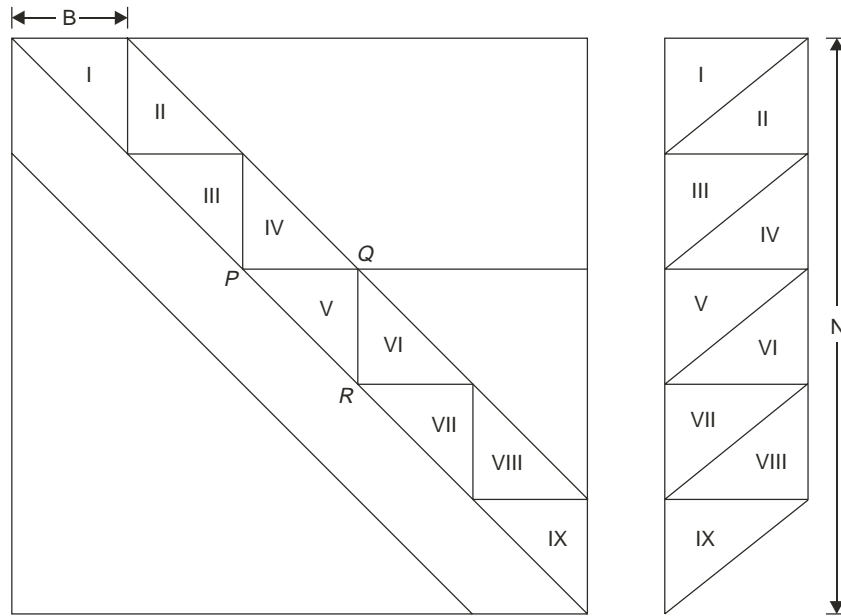


Fig. 3.10

Then only few of the triangular sub-matrices need to be stored in the computer core at a given time, while the remaining portions are kept in peripheral storage like hard disk. It may be noted that the elimination performed using one row affects only the triangle of element within the band below that row. For example, in Fig. 3.10 reduction involving row  $PQ$  modifies only the triangle  $PQR$ . This permits us to carry out the elimination with only few of the sub-matrices of Fig. 3.10 in core. Frontal Solution Technique is developed on this scheme.

### (iii) Skyline Storage

Further saving in memory requirement is by making use of skyline storage technique. In this system of storage, if there are zeros at the top of a column, only the elements starting from non-zero value need to be stored. The line separating the top zeros from the first non-zero element is called the skyline. For the matrix given below the skyline is indicated.

$$\begin{bmatrix}
 a_{11} & a_{12} & 0 & a_{14} & 0 & 0 & 0 & 0 \\
 & a_{22} & a_{23} & 0 & a_{25} & 0 & 0 & 0 \\
 & & a_{33} & a_{34} & 0 & a_{36} & 0 & 0 \\
 & & & a_{44} & 0 & a_{46} & 0 & a_{48} \\
 & & & & a_{55} & a_{56} & a_{57} & 0 \\
 & & & & & a_{66} & a_{67} & a_{68} \\
 & & & & & & a_{77} & a_{78} \\
 & & & & & & & a_{88}
 \end{bmatrix}$$

## QUESTIONS

1. Define stiffness matrix and explain its special features.
2. By direct stiffness matrix approach, determine stiffness matrix for
  - (a) Bar Element
  - (b) Truss Element
  - (c) Beam element neglecting axial deformation
  - (d) Beam element (Frame Element), considering axial deformation also.
3. Briefly explain various attempts made to reduce memory requirement in storing stiffness matrix.
4. Explain the term “Skyline Storage Technique”.

# 4

## Element Shapes, Nodes, Nodal Unknowns and Coordinate Systems

### 4.1 INTRODUCTION

In this chapter, element shapes, types of nodes, order of the element, types of nodal unknowns, are discussed. Global – Local coordinate systems and natural coordinate systems are explained. Before taking up mathematical aspect of finite element analysis, these preliminaries are to be understood.

### 4.2 ELEMENT SHAPES

Based on the shapes elements can be classified as

- (i) One dimensional elements
- (ii) Two dimensional elements
- (iii) Axi-symmetric elements and
- (iv) Three dimensional elements.

#### One Dimensional Elements

These elements are suitable for the analysis of one dimensional problem and may be called as line elements also. Figure 4.1 shows different types of one dimensional elements.

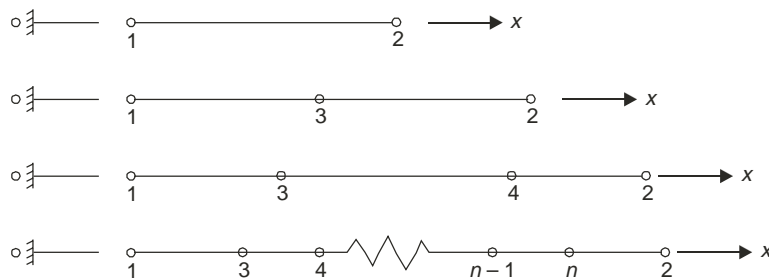


Fig. 4.1 One dimensional elements

## Two Dimensional Elements

We need two dimensional elements to solve two dimensional problems. Common two dimensional problems in stress analysis are plane stress, plane strain and plate problems. Two dimensional elements often used is three noded triangular element shown in Fig. 4.2. It has the distinction of being the first and most used element. These elements are known as **Constant Strain Triangles (CST)** or **Linear Displacement Triangles**.

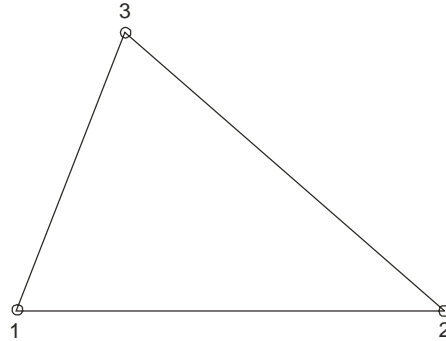


Fig. 4.2 *Constant strain triangle*

Six noded and ten noded triangular elements (Fig. 4.3) are also used by the analysts. Six noded triangular element is known as **Linear Strain Triangle (LST)** or as **Quadratic Displacement Triangle**. Ten noded

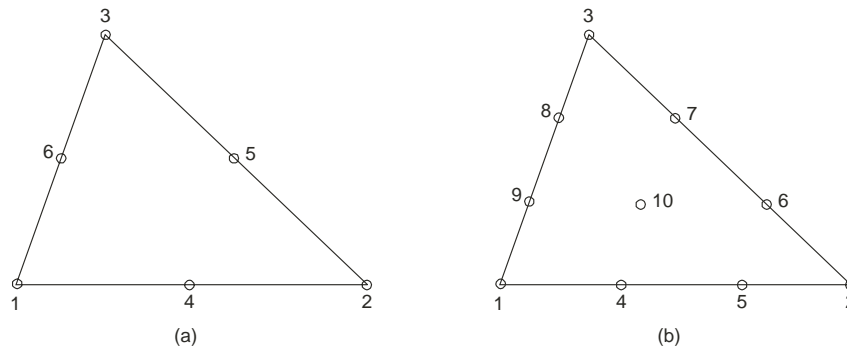


Fig. 4.3 (a) *Linear strain triangle* (b) *Quadratic strain triangle*

triangular elements are known as **Quadratic Strain Triangles (QST)** or Cubic Displacement Triangles. One can think of trying the use of still higher order triangular elements like Cubic Strain Triangles and Quartic Strain Triangles.

A simple but less used two dimensional element is the four noded rectangular element whose sides are parallel to the global coordinate systems (Fig. 4.5). This systems is easy to construct automatically but it is not well suited to approximate inclined boundaries.

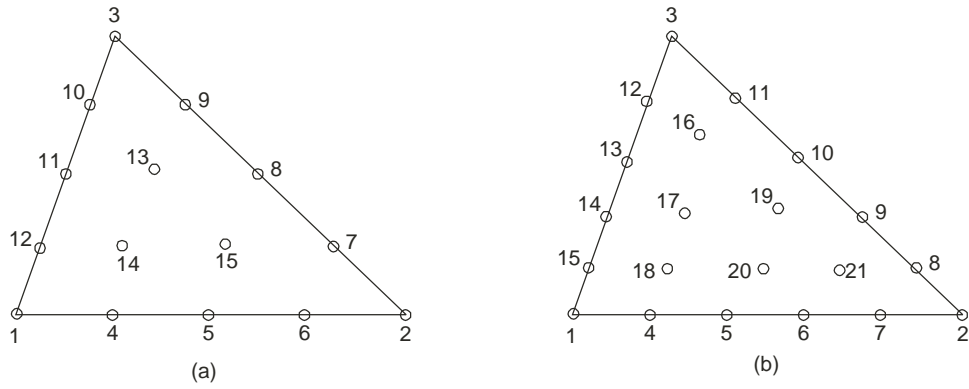


Fig. 4.4 (a) Cubic strain triangle (15 noded) (b) Quartic strain triangle (21 noded)

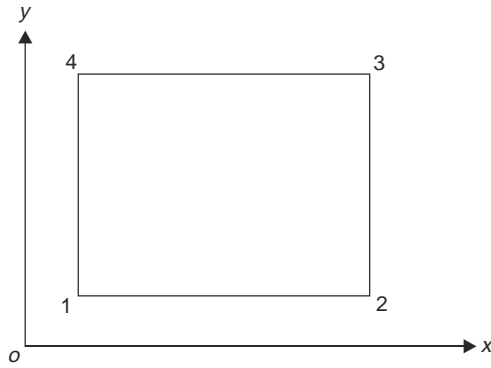


Fig. 4.5 4 noded rectangular element

Rectangular elements of higher order also can be used. Figure 4.6 shows a family of **Lagrange rectangle** in which nodes are in the form of grid points. Figure 4.7 shows the family of **Serendipity rectangles** which are having nodes only along the external boundaries.

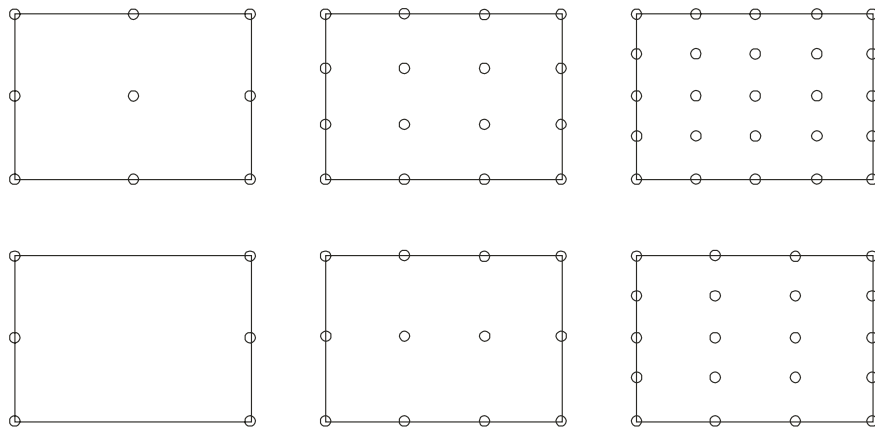
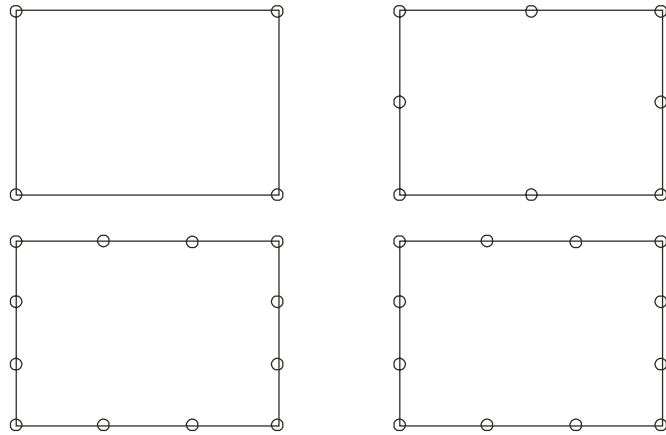


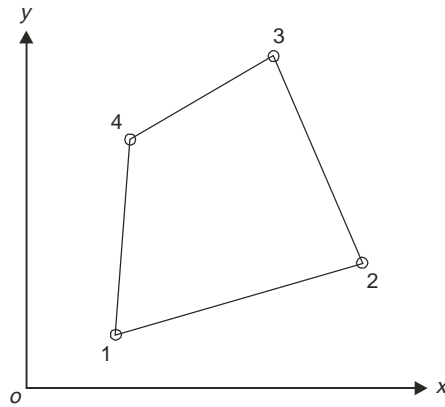
Fig. 4.6 Lagrange family rectangular elements



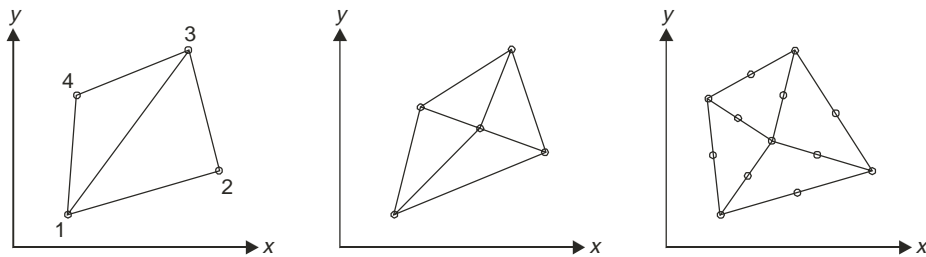


**Fig. 4.7** *Serendipity family rectangular elements*

**Quadrilateral Elements** are also used in finite element analysis (Fig. 4.8). Initially quadrilateral elements were developed by combining triangular elements (Fig. 4.9). But it has taken back stage after isoparametric concept was developed. Isoparametric concept is based on using same functions for defining geometries and nodal unknowns. Even higher order triangular elements may be used to generate quadrilateral elements.



**Fig. 4.8** *Quadrilateral element*



**Fig. 4.9** *Quadrilateral elements generated using triangular elements*

Using isoparametric concept even curved elements are developed to take care of boundaries with curved shapes (Fig. 4.10).

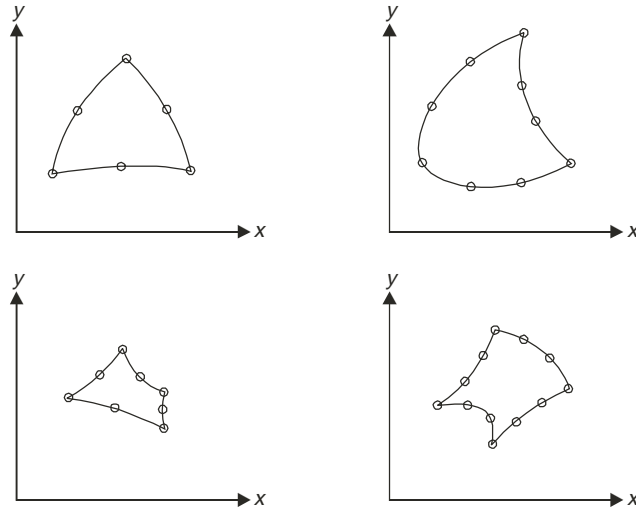


Fig. 4.10 Curved two dimensional elements

### Axi-symmetric Elements

These are also known as ring type elements. These elements are useful for the analysis of axi-symmetric problems such as analysis of cylindrical storage tanks, shafts, rocket nozzles. Axi-symmetric elements can be constructed from one or two dimensional elements. One dimensional axi-symmetric element is a conical frustum and a two dimensional axi-symmetric element is a ring with a triangular or quadrilateral cross section. Two such elements are shown in Fig. 4.11.

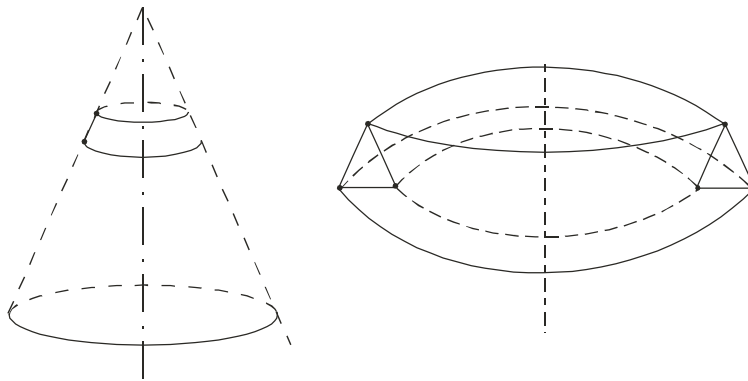
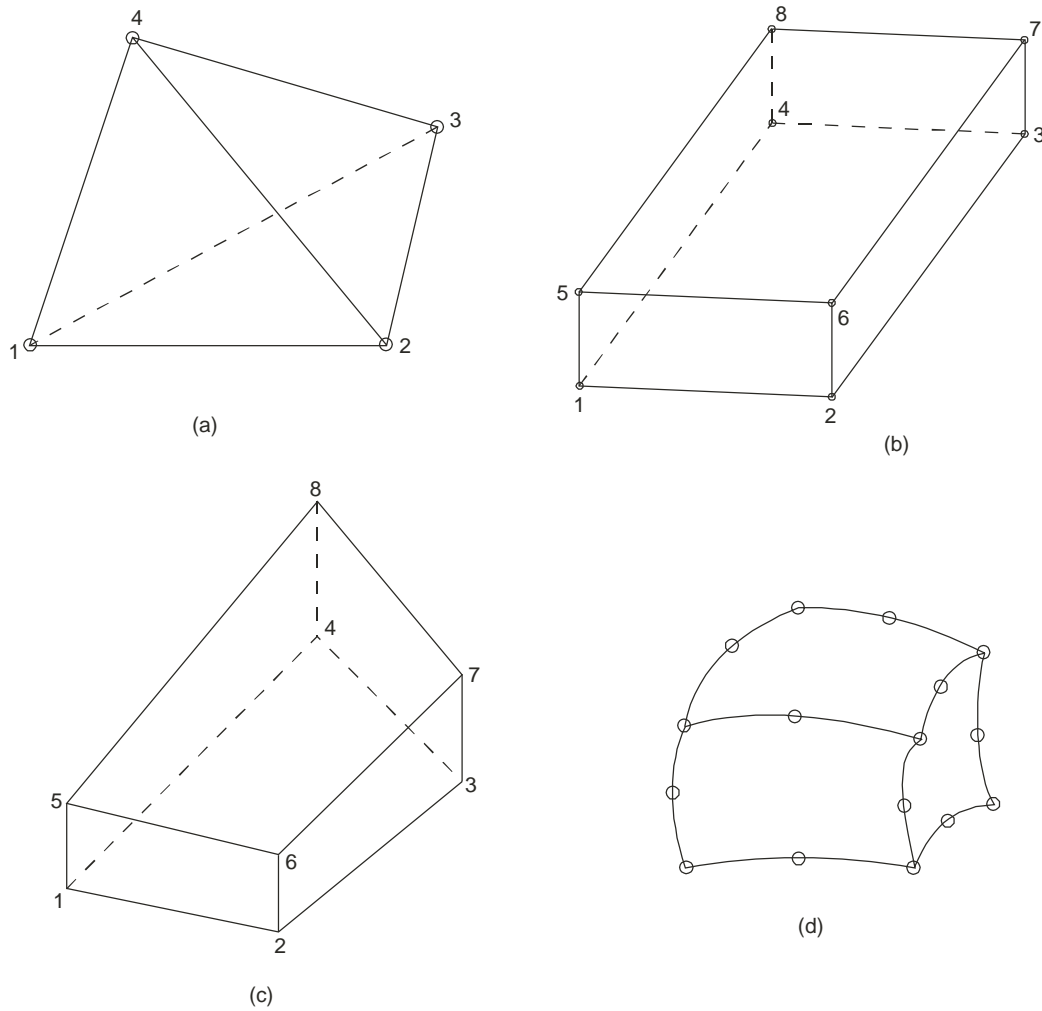


Fig. 4.11 Axis-symmetric elements

### Three Dimensional Elements

Similar to the triangle for two dimensional problems tetrahedron is the basic element for three dimensional problems (Fig. 4.12). Tetrahedron is having four nodes, one at each corner. Three dimensional elements with eight nodes are either in the form of a general hexahedron or a rectangular prism, which is a particular case of a hexahedron. The rectangular prism element is many times called as a **brick element** also. In these elements also one can think of using higher order elements. (Fig. 4.12).



**Fig. 4.12** (a) Tetrahedron element (b) Rectangular prism (brick) element (c) Arbitrary hexahedron element (d) Three dimensional quadratic element

### 4.3 NODES

Nodes are the selected finite points at which basic unknowns (displacements in elasticity problems) are to be determined in the finite element analysis. The basic unknowns at any point inside the element are determined

by using approximating/interpolation/shape functions in terms of the nodal values of the element. There are two types of nodes viz. external nodes and internal nodes. External nodes are those which occur on the edges/surface of the elements and they may be common to two or more elements. In Fig. 4.13, nodes, 1 and 2 in one dimensional element, nodes 1 to 9 in 10 noded triangular element and nodes 1 to 8 in 9 noded lagrangian element are external nodes. These nodes may be further classified as (i) Primary nodes and (ii) Secondary nodes.

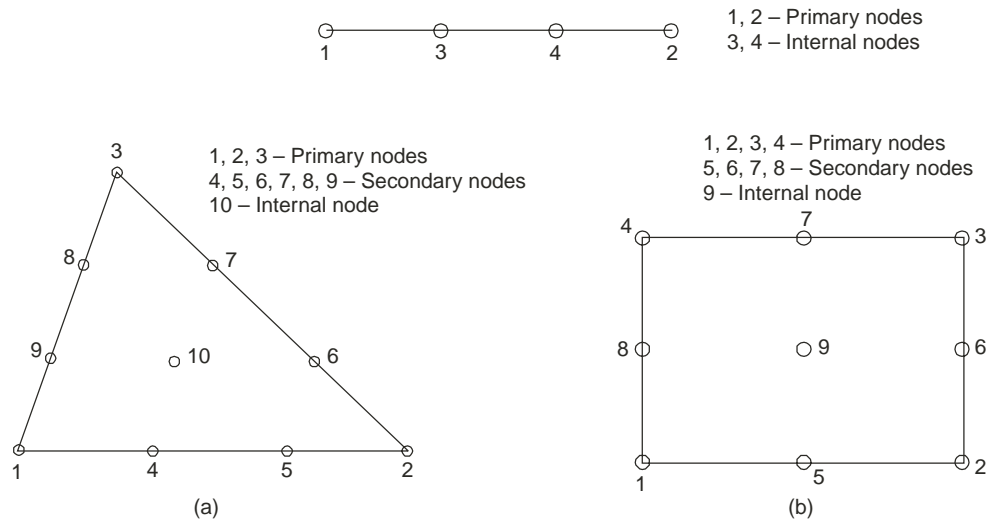


Fig. 4.13 (a) 10 noded triangular element (b) 9 noded Lagrange element

Primary nodes occur at the ends of one dimensional elements or at the corners in the two or three dimensional elements. Secondary nodes occur along the side of an element but not at corners. Figure 4.13 shows such nodes.

Internal nodes are the one which occur inside an element. They are specific to the element selected i.e. there will not be any other element connecting to this node. Such nodes are selected to satisfy the requirement of geometric isotropy while choosing interpolation functions. Figure 4.13 shows such nodes for few typical cases.

#### 4.4 NODAL UNKNOWNNS

Basic unknowns may be displacements for stress analysis, temperatures for heat flow problems and the potentials for fluid flow or in the magneticfield problems. In the problems like truss analysis, plane stress and plane strain, it is enough if the continuity of only displacements are satisfied, since there is no change in the slopes at any nodal point. Such problems are classified as ‘**zeroth**’ continuity problems and are indicated as **C<sup>0</sup>-continuity** problem. In case of beams and plates, not only the continuity of displacements, but the slope continuity also should be ensured. Since the slope is the first derivative of displacement, this type of problems are classified as ‘**First order continuity problems** and are denoted as **C<sup>1</sup> – continuity problems**. In exact

plate bending analysis even second order  $\left( \frac{\partial^2 w}{\partial x \partial y} \right)$  continuity should be ensured. Hence the actual nodal

unknowns in these problems are  $w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y}$  where  $w$  is displacement. Such problems are classified as

**C<sup>2</sup>- continuity problems.** In general **C<sup>r</sup> continuity problems** are those in which nodal unknowns are to be basic unknowns and up to  $r^{\text{th}}$  derivatives of the basic unknowns.

### 4.5 COORDINATE SYSTEMS

The following terms are commonly referred in FEM

- (i) Global coordinates
- (ii) Local coordinates and
- (iii) Natural coordinates.

However there is another term '**generalized coordinates**' used for defining a polynomial form of interpolation function. This has nothing to do with the 'coordinates' term used here to define the location of points in the element.

#### Global Coordinates

The coordinate system used to define the points in the entire structure is called global coordinate system. Figure 4.14 shows the cartesian global coordinate system used for some of the typical cases.

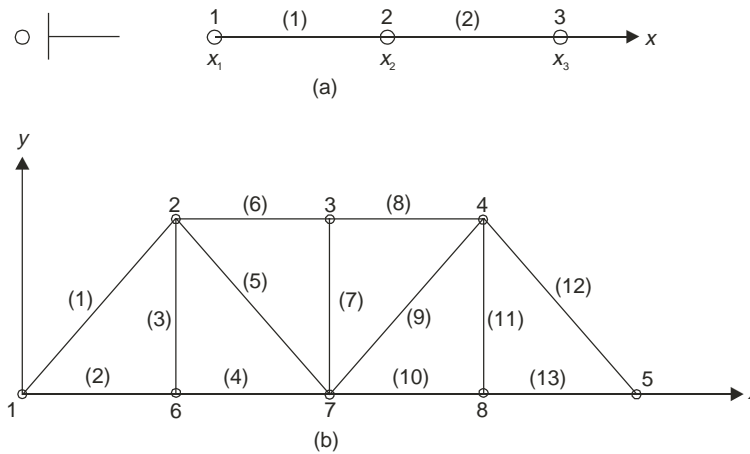


Fig. 4.14 Global coordinate system

#### Local Coordinates

For the convenience of deriving element properties, in FEM many times for each element a separate coordinate system is used. For example, for typical elements shown in Fig. 4.14, the local coordinates may be as shown in Fig. 4.15. However the final equations are to be formed in the common coordinate system i.e. global coordinate system only.

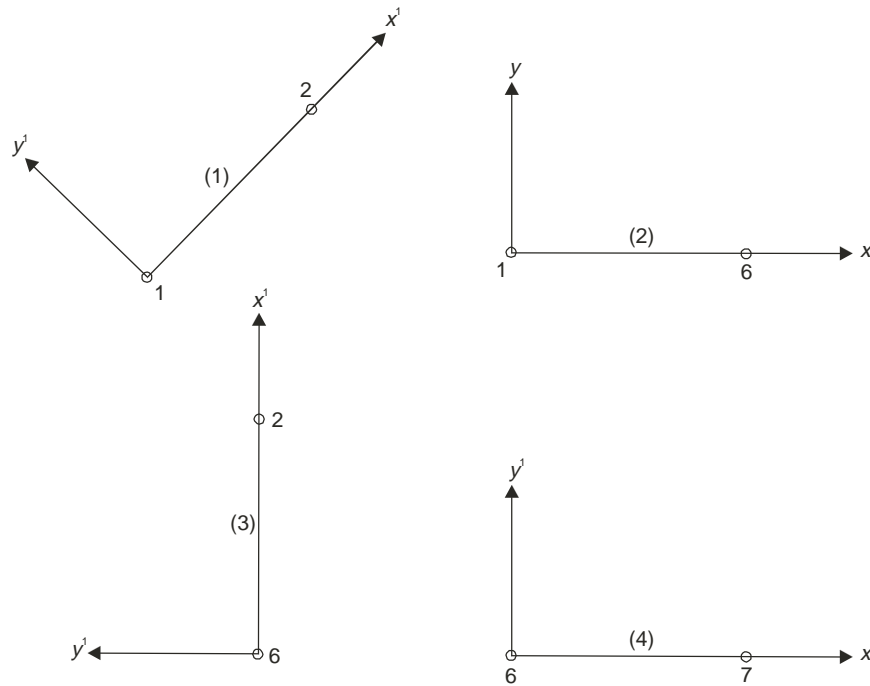


Fig. 4.15 Local coordinate system

### Natural Coordinates

A natural coordinate system is a coordinate system which permits the specification of a point within the element by a set of dimensionless numbers, whose magnitude never exceeds unity. It is obtained by assigning weightages to the nodal coordinates in defining the coordinate of any point inside the element. Hence such system has the property that  $i$ th coordinate has unit value at node  $i$  of the element and zero value at all other nodes.

The use of natural coordinate system is advantages in assembling element properties (stiffness matrices), since closed form integrations formulae are available when the expressions are in natural coordinate systems.

Natural coordinate systems for one dimensional, two dimensional and three dimensional elements are discussed below:

### Natural Coordinates in One Dimension

Consider the two noded line element shown in Fig. 4.16. let the natural coordinate of point  $P$  be  $(L_1, L_2)$  and the Cartesian coordinate be  $x$ . Node 1 and node 2 have the Cartesian coordinates  $x_1$  and  $x_2$ .

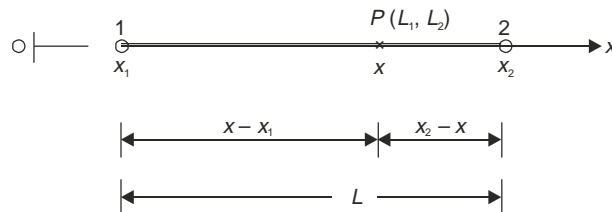


Fig. 4.16

Since natural coordinates are nothing but weightage to the nodal coordinates, total weightage at any point is unity i.e.,

$$L_1 + L_2 = 1 \quad \dots(4.1)$$

$$\text{and also } L_1 x_1 + L_2 x_2 = x \quad \dots(4.2)$$

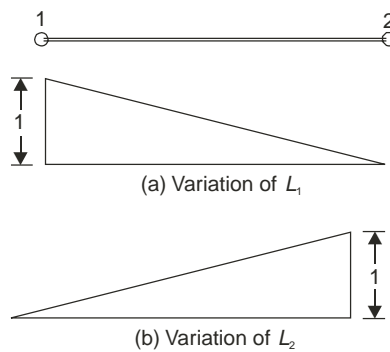
i.e. In matrix form

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} &= \begin{Bmatrix} 1 \\ x \end{Bmatrix} \\ \therefore \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} &= \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \\ \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} &= \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix}^T \begin{Bmatrix} 1 \\ x \end{Bmatrix} \\ &= \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ -x_1 + x \end{Bmatrix} \end{aligned}$$

Noting that  $x_2 - x_1$  is the length of the element say,  $l$ , we can write

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \frac{x_2 - x}{l} \\ \frac{x - x_1}{l} \end{Bmatrix} \quad \dots(4.3)$$

The variation of  $L_1$  and  $L_2$  is shown in Fig. 4.17.  $L_1$  is 1 at node 1 and is zero at node 2 where as  $L_2$  is zero when referred to node 1 and is one when referred to node 2. The variation is linear.



**Fig. 4.17** Variation of natural coordinates  $L_1, L_2$

The standard closed form integration over entire length is

$$\int_{x_1}^{x_2} L_1^p L_2^q dx = \frac{p! q!}{(p + q + 1)!} l \quad \dots(4.4)$$

**Example. 4.1:** Integrate the following over the entire length  $l$  of the element:

(i)  $\int_0^l L_1^2 dx$       (ii)  $\int_0^l L_1^3 L_2 dx$

**Solution:** (i)  $\int_0^l L_1^2 dx$

Using the standard formula,  $\int_0^l L_1^p L_2^q dx = \frac{p! q!}{(p + q + 1)!}$

We note,  $p = 2, q = 0$ .

Hence 
$$\int_0^l L_1^2 dx = \frac{2! 0!}{(2 + 0 + 1)!} l$$

$$= \frac{2}{2 \times 3} l = \frac{1}{3} \text{ Answer}$$

(ii) 
$$\int_0^l L_1^3 L_2 dx = \frac{3! 1!}{(3 + 1 + 1)!} l$$

$$= \frac{3 \times 2 \times 1}{5 \times 4 \times 3 \times 2} l = \frac{l}{20} \text{ Answer}$$

### Natural Coordinate $\xi$

In one dimensional problem, the following type of natural coordinate is also used. The natural coordinator  $\xi$

for any point in the element shown in Fig. 4.18 is defined as  $\xi = \frac{PC}{\left(\frac{x_2 - x_1}{2}\right)}$  where  $P$  is the point referred and

$C$  is the centre point of nodes 1 and 2.

The relationship between natural coordinate  $\xi$  and Cartesian coordinate  $x$  can be easily expressed as shown next page:



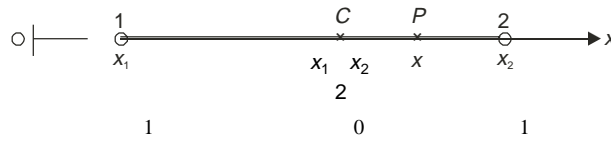


Fig. 4.18 Natural coordinate  $\xi$

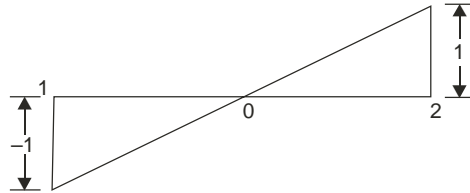


Fig. 4.19 Variation of  $\xi$

$$\begin{aligned} \xi &= \frac{PC}{\left(\frac{l}{2}\right)} = \frac{x - x_c}{\frac{l}{2}}, \text{ where } l \text{ is length of the element} = x_2 - x_1 \\ &= \frac{2}{l} \left( x - \frac{x_2 + x_1}{2} \right) \\ &= \frac{2}{l} \left( x - \frac{x_2 - x_1 + 2x_1}{2} \right) \\ &= \frac{2}{l} \left( x - \frac{l + 2x_1}{2} \right) \\ &= \frac{2}{l} \left[ x - x_1 - \frac{l}{2} \right] \end{aligned}$$

$$\frac{l}{2} \xi = x - x_1 - \frac{l}{2}$$

or  $\frac{l}{2}(1 + \xi) = x - x_1$  ...(4.5)

It may be noted that, at node 1 where  $x = x_1$ ,

$$\frac{l}{2}(1 + \xi) = x - x_1 = 0$$

$$\therefore \xi = -1$$

and at node 2, where  $x = x_2$ , we get

$$\frac{l}{2}(1 + \xi) = x_2 - x_1 = l$$

or

$$\xi = 1$$

The variation of local coordinate  $\xi$  is as shown in Fig 4.19.

The integration formula for integration over entire length is

$$\int_{x_1}^{x_2} \xi^p dx = \int_{-1}^1 \xi^p \frac{l}{2} d\xi \quad \text{since } dx = \frac{l}{2} d\xi \quad (\text{refer equation 4.5})$$

$$= \frac{l}{2} \frac{1}{p+1} [\xi^{p+1}]_{-1}^1$$

$$= 0, \text{ if } p \text{ is odd} \quad \dots(4.6a)$$

$$= \frac{l}{2} \frac{1}{p+1} (2) = \frac{l}{p+1}, \text{ if } p \text{ is even} \quad \dots(4.6b)$$

### Natural Coordinates in Two Dimensions

Natural coordinates for triangular and rectangular elements are discussed below:

**1. Natural Coordinates for Triangular Elements:** Consider the typical 3 noded triangular element shown in Fig. 4.20. Since there are three nodes, for any point there are three coordinates, say  $L_1, L_2$  and  $L_3$ . From the definition of natural coordinates, we have

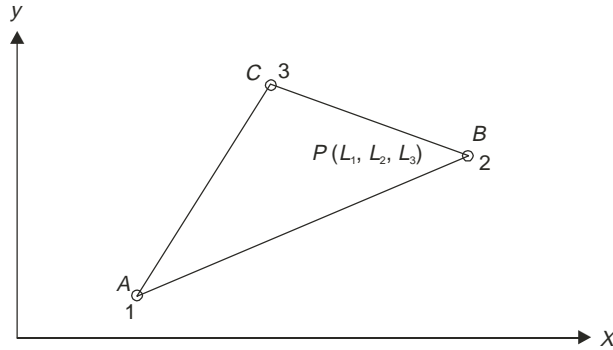


Fig. 4.20 Typical 3 noded triangular element

$$L_1 + L_2 + L_3 = 1 \quad \dots(4.7a)$$

$$L_1 x_1 + L_2 x_2 + L_3 x_3 = x \quad \dots(4.7b)$$

$$L_1 y_1 + L_2 y_2 + L_3 y_3 = y \quad \dots(4.7c)$$

Expressing the above equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

It can be shown that the determinant,

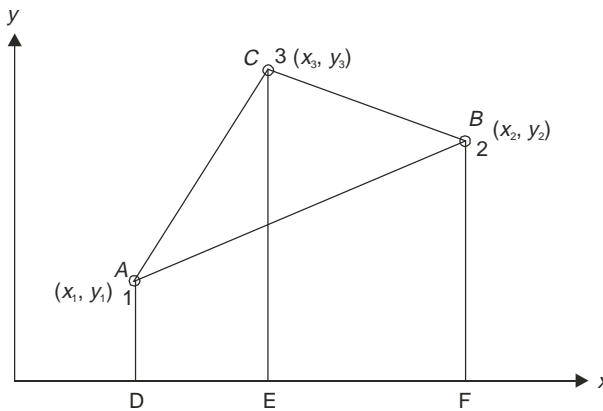
$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

is equal to twice the area of triangle with corners  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$

**Proof:** Now,

$$\text{Det} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)$$

Consider the triangle  $ABC$  shown in Fig. 4.21. Drop perpendiculars  $AD$ ,  $BE$  and  $CF$  on to x-axis.



**Fig. 4.21**

Now, Area of triangle  $ABC$

$$= \text{Area } ADEC + \text{Area } CEFB - \text{Area } ADFB$$

$$= \frac{1}{2} (AD + CE) DE + \frac{1}{2} (CE + BF) EF - \frac{1}{2} (AD + BF) DF$$

$$\begin{aligned}
 &= \frac{1}{2} (y_1 + y_3) (x_3 - x_1) + \frac{1}{2} (y_3 + y_2) (x_2 - x_3) - \frac{1}{2} (y_1 + y_2) (x_2 - x_1) \\
 &= \frac{1}{2} [y_1x_3 - y_1x_1 + y_3x_3 - y_3x_1 + y_3x_2 - y_3x_3 + y_2x_2 - y_2x_3 - y_1x_2 + y_1x_1 - y_2x_2 + y_2x_1] \\
 &= \frac{1}{2} [y_1x_3 - y_3x_1 + y_3x_2 - y_2x_3 - y_1x_2 + y_2x_1] \\
 &= \frac{1}{2} [(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)] \\
 &= \frac{1}{2} \text{Det}
 \end{aligned}$$

$$\therefore \text{Det} = 2 \text{ Area of triangle } ABC = 2A \quad \dots(4.8)$$

$$\begin{aligned}
 \therefore \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} &= \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}^T \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \\
 &= \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}
 \end{aligned}$$

where  $a_1 = x_2y_3 - x_3y_2$      $a_2 = x_3y_1 - x_1y_3$      $a_3 = x_1y_2 - x_2y_1$   
 $b_1 = y_2 - y_3$      $b_2 = y_3 - y_1$      $b_3 = y_1 - y_2$   
 $c_1 = x_3 - x_2$      $c_2 = x_1 - x_3$      $c_3 = x_2 - x_1$

[Note the cyclic order of subscript and absence of subscript of left hand term in right hand terms]

Thus

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \frac{a_1 + b_1x + c_1y}{2A} \\ \frac{a_2 + b_2x + c_2y}{2A} \\ \frac{a_3 + b_3x + c_3y}{2A} \end{Bmatrix} \quad \dots(4.9)$$

Referring to Fig. 4.22 and applying equation 4.8, we get Area of subtriangle *CPB*

$$= 2A_1 = \begin{bmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{bmatrix}$$

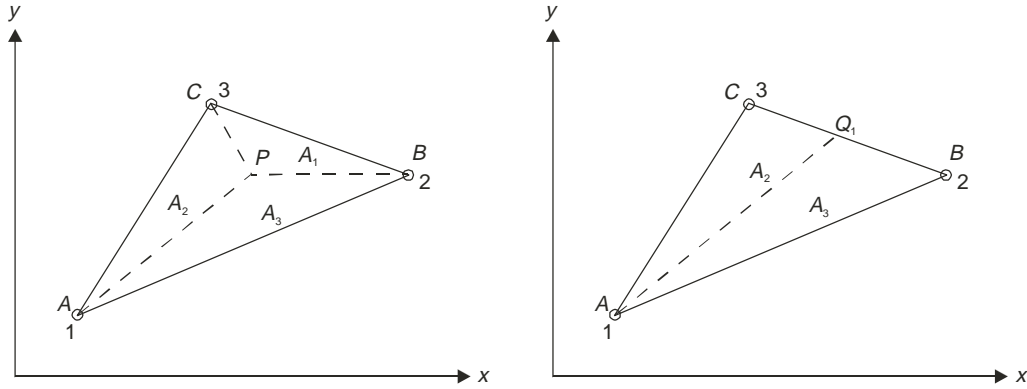


Fig. 4.22 Area coordinates for a triangle

i.e., 
$$2A_1 = x_2y_3 - x_3y_2 - (xy_3 - x_3y) + xy_2 - x_2y$$

$$= x_2y_3 - x_3y_2 + x(y_2 - y_3) + y(x_3 - x_2)$$

$$= a_1 + b_1x + c_1y$$

Thirdly  $2A_2 = a_2 + b_2x + c_2y$

and  $2A_3 = a_3 + b_3x + c_3y$

∴ Equation 4.9 reduces to

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \frac{2A_1}{2A} \\ \frac{2A_2}{2A} \\ \frac{2A_3}{2A} \end{Bmatrix} = \frac{1}{A} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \quad \dots(4.10)$$

where  $A_1, A_2$  and  $A_3$  are the areas of sub-triangles  $PCB, PAC$  and  $PAB$ , which are opposite to nodes 1, 2 and 3 respectively. Hence the natural coordinates in triangles are also known as **area coordinates**.

Note the following:

To get the natural coordinate of A, P is to be moved to A. Then we find area  $A_1 = A, A_2 = A_3 = 0$

∴  $A(1, 0, 0)$

Similarly natural coordinates of B and C are

$A(0, 1, 0), \quad C(0, 0, 1)$

For the point  $Q_1$ , natural coordinates are to be found by shifting P to Q, (refer Fig. 4.22(b)). In this case

$A_1 = 0;$

$A_2 = \frac{1}{2} CQ_1 h$

$A_3 = \frac{1}{2} BQ_1 h$

Where  $h$  is perpendicular distance of A from BC.

Then  $A = \frac{1}{2} BC h$

$$\therefore L_1 = \frac{A_1}{A} = 0$$

$$L_2 = \frac{A_2}{A} = \frac{\frac{1}{2} C Q_1 h}{\frac{1}{2} BC h} = \frac{C Q_1}{BC}$$

$$L_3 = \frac{A_3}{A} = \frac{\frac{1}{2} B Q_1 h}{\frac{1}{2} BC h} = \frac{B Q_1}{BC}$$

Thus natural coordinate representation of  $Q_1$  is

$$Q_1 = \left( 0, \frac{C Q_1}{BC}, \frac{B Q_1}{BC} \right)$$

Thirdly,

$$Q_2 = \left( \frac{C Q_2}{AC}, 0, \frac{A Q_2}{BC} \right)$$

and

$$Q_3 = \left( \frac{B Q_3}{AB}, \frac{A Q_3}{AB}, 0 \right)$$

The closed form integration for the function

$$\oint L_1^p L_2^q L_3^r dA = \frac{p! q! r!}{(p + q + r + 2)!} 2A \quad \dots(4.11)$$

**Example 4.2:** Determine the values of

(i)  $\oint_A L_1 L_2 L_3 dA$

(ii)  $\oint_A L_1^3 L_2 dA$

(iii)  $\oint_A L_1^2 L_2^2 L_3^3 dA$

**Solution:** (i)  $\oint_A L_1 L_2 L_3 dA$

Now,  $p = 1, q = 1, r = 1$

$$\therefore \oint_A L_1 L_2 L_3 dA = \frac{1! 1! 1!}{(1+1+1+2)!} 2A = \frac{1 \times 1 \times 1}{5 \times 4 \times 3 \times 2 \times 1} 2A = \frac{A}{60}$$

(ii)  $\oint_A L_1^3 L_2 dA$

In this case,  $p = 3, q = 1, r = 0$

$$\oint_A L_1^3 L_2 dA = \frac{3! 1! 0!}{(3+1+0+2)!} 2A = \frac{3 \times 2 \times 1 \times 1 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} 2A = \frac{A}{60}$$

(iii)  $\oint_A L_1^2 L_2^2 L_3^3 dA$

In this case  $p = 2, q = 2, r = 3$

$$\begin{aligned} \therefore \oint_A L_1^2 L_2^2 L_3^3 dA &= \frac{2! 2! 3!}{(2+2+3+2)!} 2A \\ &= \frac{2 \times 1 \times 2 \times 1 \times 3 \times 2 \times 1}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} 2A = \frac{A}{7960} \end{aligned}$$

### Natural Coordinates for Rectangular Elements

Natural coordinates for rectangular elements are as shown in Fig. 4.23. In these cases the centroid of the area is the origin. The relationships between the local coordinates and the Cartesian coordinates are based on isoparametric concept, which is taken up in the latter chapter. It may be noted here that the coordinates  $\xi$  and  $\eta$  vary from  $-1$  to  $1$ . The relationship between global coordinates and the natural coordinates are  $x = \sum L_i x_i$  and  $y = \sum L_i y_i$ . The derivation of  $L_i$  are discussed in the chapter 'isoparametric elements'. When the expressions are formed in these coordinate systems, instead of seeking integrations in the closed form expressions, numerical technique is usually employed.

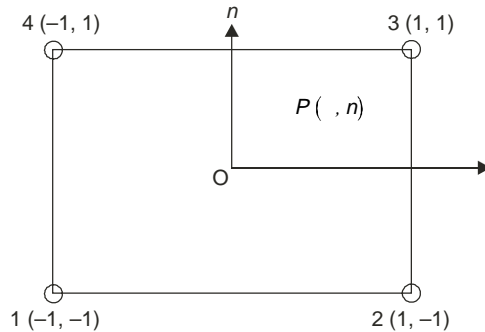


Fig. 4.23 Local coordinate system for a rectangular element

### Natural Coordinates in Three Dimensions

Natural coordinates for a 4 noded tetrahedron may be derived and it results into volume coordinates. Consider the typical tetrahedron shown in Fig. 4.24.

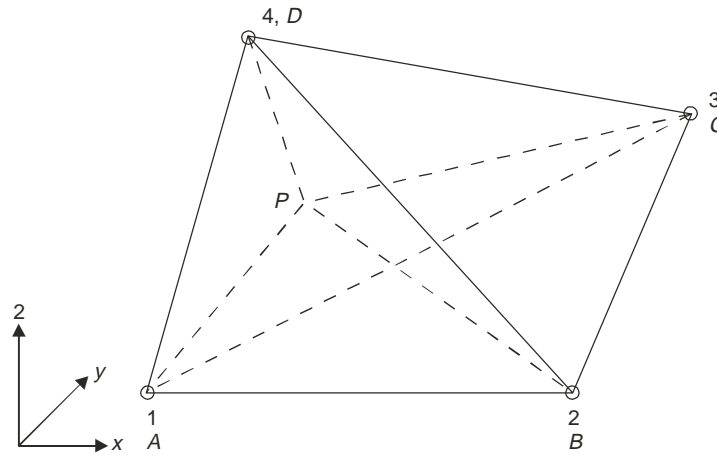


Fig. 4.24 Tetrahedron coordinates

The natural coordinates are related to the Cartesian coordinates as follows:

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{Bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \quad \dots(4.12)$$

The above equation may be solved by inverting the  $4 \times 4$  matrix. It gives

$$L_i = \frac{1}{6V} (a_i + b_i x + c_i y + d_i z), \text{ for } i = 1, 2, 3 \text{ and } 4 \quad \dots(4.13)$$

where  $6V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix} = 6 \times \text{volume of tetrahedron defined by nodes 1, 2, 3 and 4}$

and  $a_1 = \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}$   $b_1 = \begin{vmatrix} 1 & 1 & 1 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}$



$$c_1 = \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \end{vmatrix} \quad \text{and} \quad d_1 = \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{vmatrix}$$

The other constants are obtained by cyclic changes in the subscripts. It may be noted that the above equations are valid only when the nodes are numbered so that nodes 1, 2 and 3 are ordered counter clockwise when viewed from node 4. It is also necessary that for coordinates system of right hand rule is strictly adhered to.

If  $V_i$  is the volume of the smaller tetrahedron which has vertices  $P$  and the three nodes other than the node  $i$ , then the tetrahedron coordinates can be considered as volume coordinates, defined as

$$L_i = \frac{V_i}{V} \quad \text{for } i=1, 2, 3 \text{ and } 4 \quad \dots(4.14)$$

The closed form integration formula for the volume coordinates is

$$\oint_v L_1^p L_2^q L_3^r L_4^s dV = \frac{p! q! r! s!}{(p+q+r+s+3)!} 6V \quad \dots(4.15)$$

**Example 4.3:** Find the values of the following:

$$(i) \oint_v L_1 L_2 L_3 L_4 dV \quad (ii) \oint_v L_1^2 L_2 L_4 dV$$

**Solution:** (i)  $\oint_v L_1 L_2 L_3 L_4 dV$

In this case  $p = 1, q = 1, r = 1, s = 1$

Hence using equation

$$\oint_v L_1^p L_2^q L_3^r L_4^s dV = \frac{p! q! r! s!}{(p+q+r+s+3)!} \times 6V$$

we get,  $\oint_v L_1 L_2 L_3 L_4 dV = \frac{1! 1! 1! 1!}{(1+1+1+1+3)!} 6V = \frac{1 \times 1 \times 1 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 6V$

$$= \frac{V}{840} \quad \text{Answer}$$

(ii)  $\oint_v L_1^2 L_2 L_4 dV$

In this case  $p = 2, q = 1, r = 0$  and  $s = 1$

$$\therefore \oint_v L_1^2 L_2 L_4 dV = \frac{2! 1! 0! 1!}{(2+1+0+1+3)!} 6V$$

$$= \frac{2 \times 1 \times 1 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} 6V = \frac{V}{420} \text{ Answer}$$

### Natural Coordinates for Hexahedron

Figure 4.25 shows a typical hexahedron with natural coordinates. It has origin at centroid of the hexahedron. It may be noted that natural coordinates vary from  $-1$  to  $+1$ . The natural coordinates are related to global coordinates as

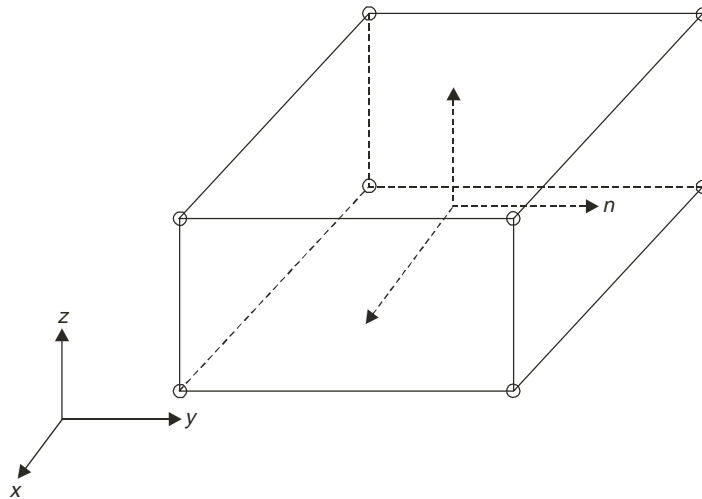


Fig. 4.25

$$x = \sum L_i x_i$$

$$y = \sum L_i y_i \text{ and } z = \sum L_i z_i \quad \dots(4.16)$$

The derivation of  $L_i$  is discussed under the chapter 'Isoparametric Elements'. Later for integration numerical technique is preferred. Hence no discussion is taken up here about the closed form integrations for such cases.

### QUESTIONS

1. Explain the following terms clearly
  - (i) Nodes, primary nodes, secondary nodes and internal nodes
  - (ii) Local coordinates, global coordinates, natural coordinates and area coordinates.
  - (iii) Higher order elements and lower order elements.
2. Explain the terms
  - (i) Constant strain triangle (CST)
  - (ii) Linear strain triangle(LST) and
  - (iii) Quadratic strain triangles (QST).
3. Explain the term  $C^r$ -continuity.

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4. Derive the expressions for natural coordinates for a two noded element
  - (i) In terms of  $L_1$  and  $L_2$ , when range is 0 to 1
  - (ii) In terms of  $\xi$ , when range is  $-1$  to  $1$ .
5. Derive expressions for natural coordinates in a CST element. Show that they are nothing but area coordinates.

# 5

## Shape Functions

### 5.1 INTRODUCTION

In the finite element analysis aim is to find the field variables at nodal points by rigorous analysis, assuming at any point inside the element basic variable is a function of values at nodal points of the element. This function which relates the field variable at any point within the element to the field variables of nodal points is called shape function. This is also called as interpolation function and approximating function. In two dimensional stress analysis in which basic field variable is displacement,

$$u = \sum N_i u_i, v = \sum N_i v_i \quad \dots(5.1)$$

where summation is over the number of nodes of the element. For example for three noded triangular element, displacement at  $P(x, y)$  is

$$u = \sum N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = \sum N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

i.e.,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

or

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

or

$$\{\delta\}_{2 \times 1} = [N]_{2 \times 6} \{\delta\}_e_{6 \times 1} \quad \dots(5.2a)$$

where  $q$  is displacement at any point in the element

$[N]$  shape function

$\{\delta\}_e$  is vector of nodal displacements

Similarly in case of 6 noded triangular element

$$\begin{matrix} \{\delta\} \\ 2 \times 1 \end{matrix} = \begin{matrix} [N] \\ 2 \times 12 \end{matrix} \begin{matrix} \{\delta\}_e \\ 12 \times 1 \end{matrix} \quad \dots(5.2b)$$

In case of 4 noded rectangular element

or 
$$\begin{matrix} \{\delta\} \\ 2 \times 1 \end{matrix} = \begin{matrix} [N] \\ 2 \times 8 \end{matrix} \begin{matrix} \{\delta\}_e \\ 8 \times 1 \end{matrix} \quad \dots(5.2c)$$

### 5.2 POLYNOMIAL SHAPE FUNCTIONS

Polynomials are commonly used as shape functions. There are two reasons for using them:

- (i) They are easy to handle mathematically i.e. differentiation and integration of polynomials is easy.
- (ii) Using polynomial any function can be approximated reasonably well. If a function is highly nonlinear we may have to approximate with higher order polynomial. Fig. 5.1 shows approximation of a nonlinear one dimensional function by polynomials of different order.

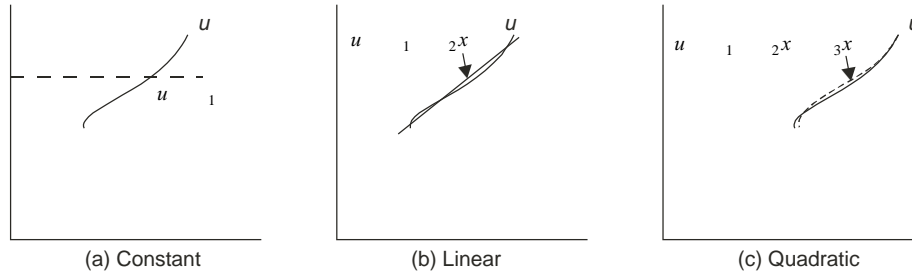


Fig. 5.1 Approximation with polynomials

#### One Dimensional Polynomial Shape Function

A general one dimensional polynomial shape function of  $n$ th Order is given by,

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots \alpha_{n+1} x^n \quad \dots(5.3)$$

In matrix form  $u = [G] \{\alpha\} \quad \dots(5.4)$

where

$$[G] = [1, x, x^2 \dots x^n]$$

and

$$\{\alpha\}^T = [\alpha_1 \alpha_2 \alpha_3 \dots \alpha_{n+1}]$$

Thus in one dimensional  $n^{\text{th}}$  order complete polynomial there are  $m = n + 1$  terms.  $\dots(5.5)$

### Two Dimensional Polynomial Shape Function

A general form of two dimensional polynomial model is

$$\begin{aligned}
 u(x, y) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 \dots + \alpha_m y^n \\
 v(x, y) &= \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \dots + \alpha_{2m} y^n
 \end{aligned}
 \tag{5.6}$$

or

$$\{\delta\} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [G]\{\alpha\} = \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix} \{\alpha\}
 \tag{5.7}$$

where

$$G_1 = [1 \ x \ y \ x^2 \ xy \ y^2 \ x^3 \dots y^n]$$

$$\{\alpha\}^T = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4 \dots \alpha_{2m}]$$

It may be observed that in two dimensional problem, total number of terms  $m$  in a complete  $n$ th degree polynomial is

$$m = \frac{(n + 1)(n + 2)}{2}
 \tag{5.8}$$

For first order complete polynomial  $n = 1$ ,

$$\therefore m = \frac{(1 + 1)(1 + 2)}{2} = 3$$

The first three terms are  $\alpha_1 + \alpha_2 x + \alpha_3 y$

Similarly for  $n = 2$ ,  $m = \frac{(2 + 1)(2 + 2)}{2} = 6$

and we know the first six terms are,

$$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

Another convenient way to remember complete two dimensional polynomial is in the form of Pascal Triangle shown in Fig. 5.2

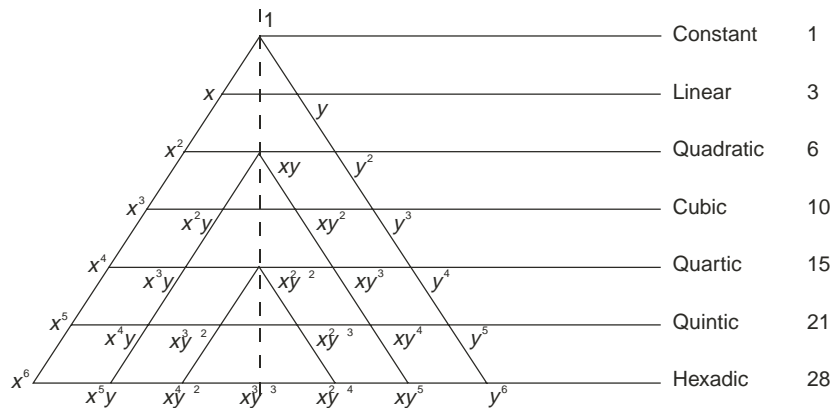


Fig. 5.2 Pascal triangle

### Three Dimensional Polynomial Shape Function

A general three dimensional shape function of  $n$ th order complete polynomial is given by

$$\begin{aligned} u(x, y, z) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \dots + \alpha_m x^{n-1} z \\ v(x, y, z) &= \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \alpha_{m+4} z + \alpha_{m+5} x^2 + \dots + \alpha_{2m} x^{n-1} z \\ w(x, y, z) &= \alpha_{2m+1} + \alpha_{2m+2} x + \alpha_{2m+3} y + \alpha_{2m+4} z + \dots + \alpha_{3m} x^{n-1} z \end{aligned} \quad \dots(5.9)$$

or

$$\delta(x, y, z) = \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix} = [G] \{\alpha\} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & G_1 \end{bmatrix} \{\alpha\} \quad \dots(5.10)$$

Where  $G_1 = [1 \ x \ y \ z \ x^2 \ xy \ y^2 \ yz \ z^2 \ zx \ \dots \ z^n \ z^{n-1}x \ \dots \ zx^{n-1}]$

and  $\{\alpha\}^T = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \dots \ \alpha_{3m}]$

It may be observed that a complete  $n$ th order polynomial in three dimensional case is having number of terms  $m$  given by the expression

$$m = \frac{(n+1)(n+2)(n+3)}{6}$$

Thus when  $n = 1$ ,  $m = \frac{(1+1)(1+2)(1+3)}{6} = 4$

i.e.  $\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$

For  $n = 2$ ,  $m = \frac{(2+1)(2+2)(2+3)}{6} = 10$

Thus second degree complete polynomial is

$$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \alpha_6 xy + \alpha_7 y^2 + \alpha_8 yz + \alpha_9 z^2 + \alpha_{10} zx$$

Complete polynomial in three dimensions may be expressed conveniently by a tetrahedron as shown in Fig. 5.3.

### 5.3 CONVERGENCE REQUIREMENTS OF SHAPE FUNCTIONS

Numerical solutions are approximate solutions. Stiffness coefficients for a displacements model have higher magnitudes compared to those for the exact solutions. In other words the displacements obtained by finite element analysis are lesser than the exact values. Thus the FEM gives lower bound values. Hence it is desirable that as the finite element analysis mesh is refined, the solution approaches the exact values. This requirement is shown graphically in Fig. 5.4. In order to ensure this convergence criteria, the shape functions should satisfy the following requirement:

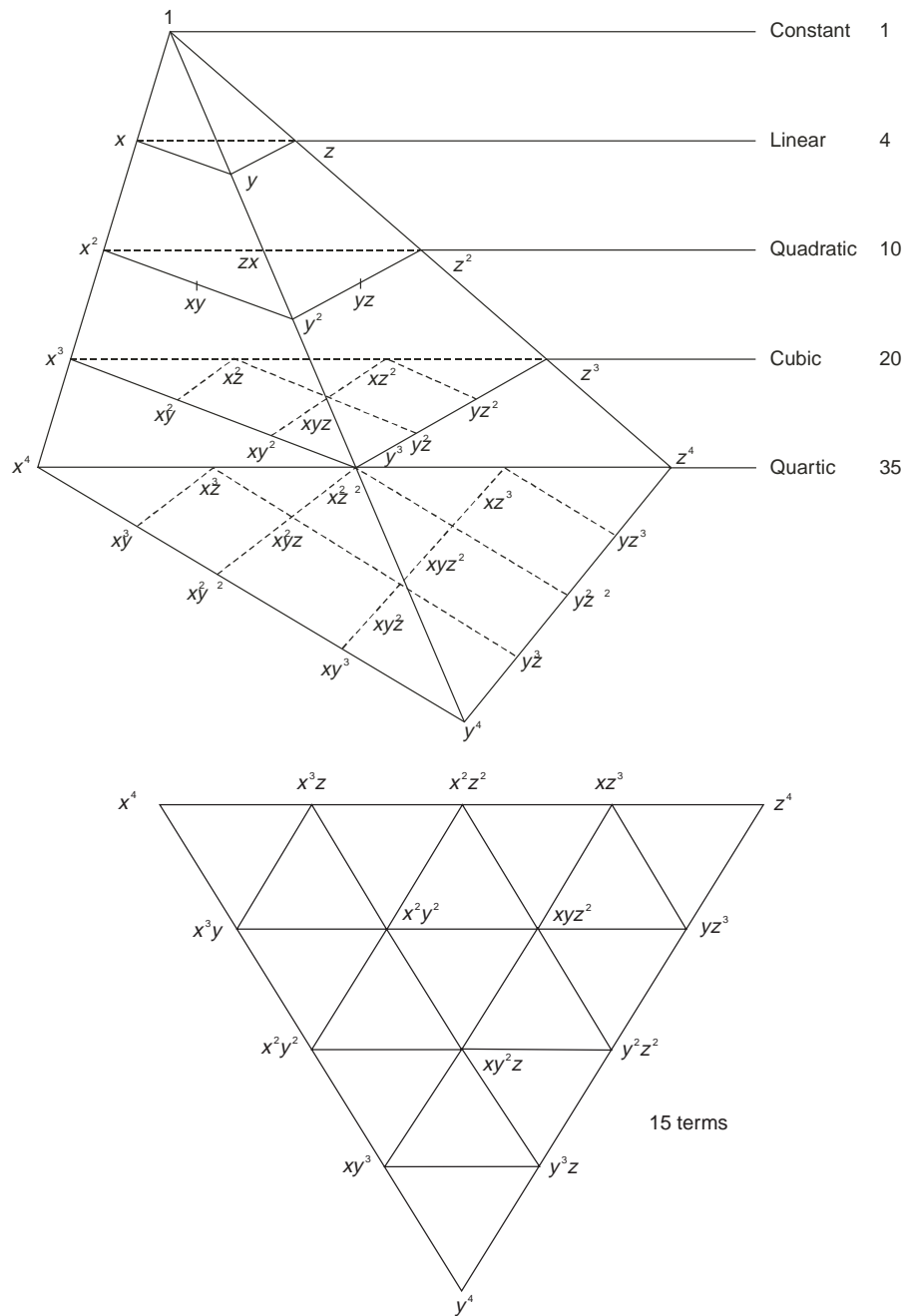


Fig. 5.3 Terms at base of quartic tetrahedron of polynomial

1. The displacement models must be continuous within the elements and the displacements must be compatible between the adjacent elements. The second part implies that the adjacent elements must deform without causing openings, overlaps or discontinuities between the elements. This requirement is called ‘**compatibility requirement**’ .



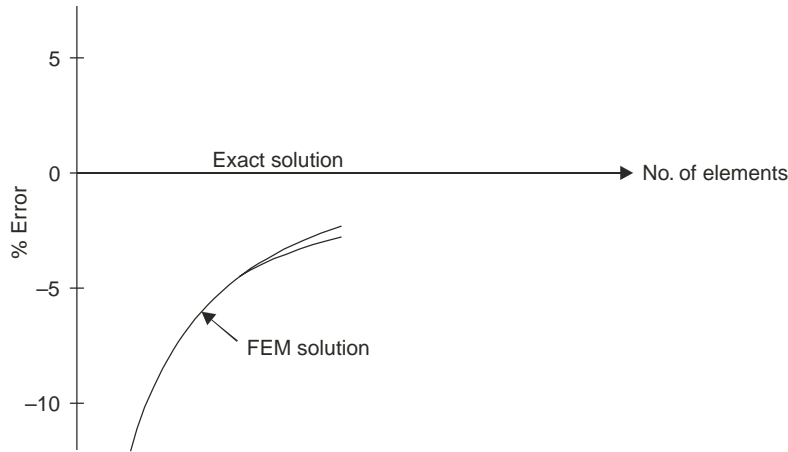


Fig. 5.4 Convergence of FEM solution

According to Felippa and Clough this requirement is satisfied, if the displacement and its partial derivatives upto one order less than the highest order derivative appearing in strain energy function is continuous. Hence in plane stress and plane strain problems, it is enough if continuity of displacement is satisfied, since strain energy function includes only first order derivatives of the displacement ( $SE = \frac{1}{2} \text{ stress} \times \text{strain}$ ). It implies, it is enough if  $C^0$  continuity is ensured in plane stress and plane strain problems. In case of flexure problems (beams, plates, shells) the strain

energy terms include second derivatives of displacements (like  $\frac{1}{2} \frac{M^2}{EI}$  where  $M = -EI \frac{d^2 w}{dx^2}$ ).

Hence to satisfy compatibility requirement, not only displacement continuity but slope continuity ( $C^1$ -continuity) should be satisfied. Hence in flexure problems displacements and their first derivatives are selected as nodal field variables.

2. The displacement model should include the **rigid body displacements** of the element. It means in displacement model there should be a term which permit all points on the element to experience the same displacement. It is obvious, if such term do not exists, shifting of the origin of the coordinate system will cause additional stresses and strains, which should not occur. In the displacement model,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

the term  $\alpha_1$  provides for the rigid body displacement. Hence to satisfy the requirement of rigid body displacement, there should be constant term in the shape function selected.

3. The displacement models must include the **constant strain state** of the element. This means, there should exist combination of values of polynomial terms that cause all points in the element to experience the same strain. One such combination should occur for each possible strain. The necessity of this requirement is understood physically, if we imagine the refinement of the mesh. As these elements approach infinitesimal size, the strains within the element approach constant values. Unless the shape function term includes these constant strain terms, we cannot hope to converge to a correct solution. In the displacement model,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \dots + \alpha_m y^n$$

$$v = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \alpha_{m+4} x^2 + \dots + \alpha_{2m} y^n$$

$\alpha_2$  and  $\alpha_{m+2}$  provide for uniform strain  $\epsilon_x$ ,

$\alpha_3$  and  $\alpha_{m+3}$  provide for uniform strain  $\epsilon_y$

**An additional consideration** in the selection of polynomial shape function for the displacement model is that the pattern should be independent of the orientation of the local coordinate system. This property is known as **Geometric Isotropy, Spatial Isotropy** or **Geometric Invariance**. There are two simple guidelines to construct polynomial series with the desired property of isotropy:

1. Polynomial of order  $n$  that are complete, have geometric isotropy.
2. Polynomial of order  $n$  that are not complete, yet contain appropriate terms to preserve 'symmetry' have geometric isotropy. The simple test for this property is to interchange  $x$  and  $y$  in two dimensional problems or  $x, y, z$  in cyclic order in three dimensional problems and see that the total expression do not change. However the arbitrary constants may change.

For example, we wish to construct a cubic polynomial expression for an element that has eight nodal values assigned to it. In this situation, we have to drop two terms from the complete cubic polynomial which contains 10 terms. To maintain geometric isotropy drop only terms that occur in symmetric pairs i.e.  $x^3, y^3$  or  $x^2y, xy^2$ . Thus the acceptable eight term cubic polynomials shape function exhibiting geometric isotropy are

$$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^2 y + \alpha_8 xy^2$$

and  $\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 y^3$

In finite element analysis, the safest approach to reach correct solution is to pick the shape functions that satisfy all the requirements. For some problems, however, choosing shape functions that meet all the requirements may be difficult and may involve excessive numerical computations. For this reason some investigators have ventured to formulate shape functions for the elements that do not meet compatibility requirements. In some cases acceptable convergence has been obtained. Such elements are called '**non-conforming elements**'. The main disadvantage of using non-conforming elements is that we no longer know in advance that correct solution is reached.

## 5.4 DERIVATION OF SHAPE FUNCTIONS USING POLYNOMIALS

Initially shape functions were derived in terms of Cartesian coordinates. Polynomial function were used for this. After natural coordinates were identified and its advantage was noticed researchers started deriving shape functions in terms of natural coordinates. By this approach more elements could be developed. Advantages of using Lagrangian and Hermetian functions were discovered latter. To handle few special cases, degeneration technique was also developed. In this artiae these methods are illustrated for various elements.

### Polynomial Functions In Terms of Cartesian Coordinates

In this approach polynomials with number of constants exactly equal to nodal degrees of freedom of the element are selected. Care is taken to see that geometric isotropy is not lost. Using nodal values number of equations equal to number of constants in the polynomials are formed and then the constants found. Then the shape functions are identified. This procedure may also be called as **generalized coordinate approach**, since

the constants in the polynomial are called as generalized coordinates. This procedure is illustrated with few cases below.

**Example 5.4:** Using generalized coordinate approach, find shape functions for two noded bar/truss element.

**Solution:** Figure 5.5 shows the typical truss element. In this case nodal unknowns are displacements  $u_1$  and  $u_2$  along x-axis. For this element we have to select polynomial with only two constants to represent displacement at any point in the elements. Hence we select

$$u = \alpha_1 + \alpha_2 x \quad \dots(5.11)$$

where  $\alpha_1$  and  $\alpha_2$  are generalized coordinates. This polynomial satisfies compatibility and completeness requirement. Writing equation 5.11 in the matrix form we have,

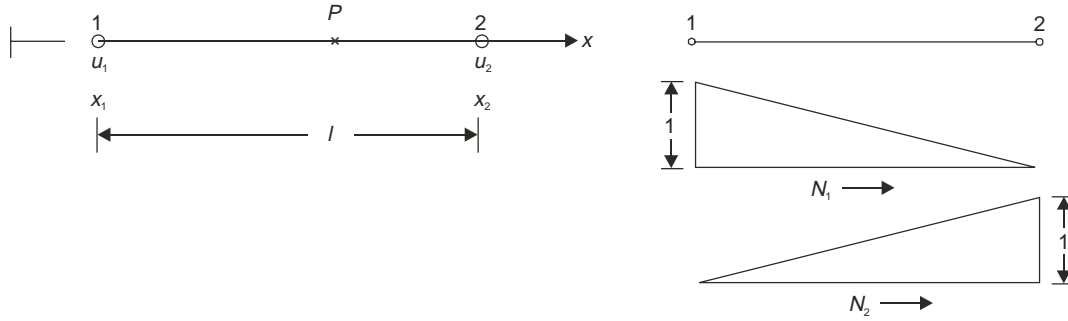


Fig. 5.5 Bar/Truss element with two nodes

$$u = [1 \ x] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

since  $u = u_1$  at node 1 and equal to  $u_2$  at node 2, we have

$$\{\delta\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{x_2 - x_1} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore u = [1 \ x] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = [1 \ x] \frac{1}{l} \begin{bmatrix} x_2 & -x_1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [x_2 - x \quad -x_1 + x] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \frac{x_2 - x}{l} & \frac{x - x_1}{l} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= [N_1 \ N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = N_1 u_1 + N_2 u_2$$

where

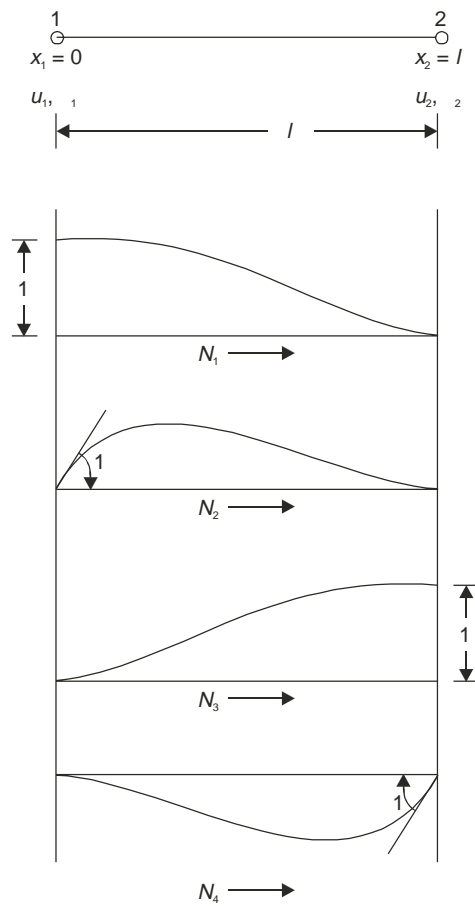
$$N_1 = \frac{x_2 - x}{l} \text{ and } N_2 = \frac{x - x_1}{l}$$

Thus the shape function  $[N]$  is

$$[N] = [N_1 \ N_2] = \left[ \frac{x_2 - x}{l} \quad \frac{x - x_1}{l} \right] \text{ Answer} \quad \dots 5.12.$$

Variation of shape function  $N_1$  and  $N_2$  is shown in Fig. 5.5. (b).

**Example 5.5:** Using polynomial functions (generalized coordinates) determine shape functions for a two noded beam element.



**Fig. 5.6** Variation of shape functions in a beam element

**Solution:** The typical beam element is shown in Fig. 5.6. (a). In this case  $C^1$  – continuity is to be satisfied, since strain energy expression involves second differentiation term  $\frac{d^2w}{dx^2}$ . Hence in this case at each node, unknowns are the displacement and slope. i.e.,

$$\{\delta\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

where

$$\theta_1 = \frac{\partial w_1}{\partial x}$$

and

$$\theta_2 = \frac{\partial w_2}{\partial x}$$

Since there are four nodal values, we select polynomial with four constants. Thus

$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \quad \dots(5.13)$$

Equation 5.13 satisfies compatibility and completeness requirement. Now,

$$\theta = \frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2$$

For convenience we select local coordinate system.

i.e.,

$$x_1 = 0$$

$$x_2 = l$$

$$\therefore w_1 = \alpha_1$$

$$\theta_1 = \alpha_2$$

$$w_2 = \alpha_1 + \alpha_2 l + \alpha_3 l^2 + \alpha_4 l^3$$

$$\theta_2 = \alpha_2 + 2\alpha_3 l + 3\alpha_4 l^2$$

i.e.,

$$\{\delta\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}^{-1} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \frac{1}{3l^4 - 2l^4} \begin{bmatrix} l^4 & 0 & -3l^2 & 2l \\ 0 & l^4 & -2l^3 & l^2 \\ 0 & 0 & 3l^2 & -2l \\ 0 & 0 & -l^3 & l^2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ \frac{2}{l^2} & \frac{1}{l} & \frac{-2}{l^2} & \frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$\therefore w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ \frac{2}{l^2} & \frac{1}{l} & \frac{-2}{l^2} & \frac{1}{l} \\ \frac{2}{l^3} & \frac{1}{l^2} & \frac{-2}{l^3} & \frac{1}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3} & x - \frac{2x^2}{l} + \frac{x^3}{l^2} & \frac{3x^2}{l^2} - \frac{2x^3}{l^3} & -\frac{x^2}{l} + \frac{x^3}{l^2} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

$$= [N_1 \ N_2 \ N_3 \ N_4] \{\delta\}_e = [N] \{\delta\}_e$$

where  $[N] = [N_1 \ N_2 \ N_3 \ N_4]$

and  $N_1 = 1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}$   $N_2 = x - \frac{2x^2}{l} + \frac{x^3}{l^2}$  ... (5.13)

$$N_3 = \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \quad N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

Variation of these function is shown in Fig. 5.6 (b) (Note that at node 1,  $N_1=1$ ,

$$N_2=N_3=N_4=0, \text{ and } \frac{\partial N_2}{\partial x} = 1, \frac{\partial N_1}{\partial x} = \frac{\partial N_3}{\partial x} = \frac{\partial N_4}{\partial x} = 0 \text{ similarly at node 2,}$$

$$N_1=N_2=N_4=0, N_3=1 \text{ and } \frac{\partial N_1}{\partial x} = \frac{\partial N_2}{\partial x} = \frac{\partial N_3}{\partial x} = 0 \text{ and } \frac{\partial N_4}{\partial x} = 1,$$

**Example 5.6:** Determine the shape functions for the Constant Strain Triangle (CST). Use polynomial functions.

**Solution:** Figure 5.7 shows a typical CST element. Let the nodal variables be  $u_1, u_2, u_3, v_1, v_2$  and  $v_3$  i.e.,

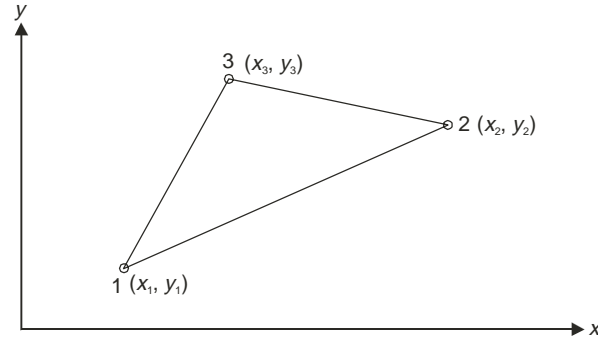


Fig. 5.7

$$\{\delta\}^T = [u_1 \ u_2 \ u_3 \ v_1 \ v_2 \ v_3]$$

From the consideration of compatibility and completeness the following displacement model is selected

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \quad \dots(5.14)$$

$$\therefore u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

i.e.,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Now

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 2A$$

Where  $A$  is the area of triangle with vertices at  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  i.e., the area of the element.

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^T \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

where  $a_1 = x_2 y_3 - x_3 y_2$        $a_2 = x_3 y_1 - x_1 y_3$        $a_3 = x_1 y_2 - x_2 y_1$   
 $b_1 = y_2 - y_3$        $b_2 = y_3 - y_1$        $b_3 = y_1 - y_2$   
 $c_1 = x_3 - x_2$        $c_2 = x_1 - x_3$        $c_3 = x_2 - x_1$ ,

same as used in deriving natural coordinates.

$$\therefore u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$= \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{a_1 + b_1 x + c_1 y}{2A} & \frac{a_2 + b_2 x + c_2 y}{2A} & \frac{a_3 + b_3 x + c_3 y}{2A} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} N \end{bmatrix} \{\delta\}_e \quad \dots(5.15)$$

where  $\begin{bmatrix} N \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}$

and  $N_1 = \frac{a_1 + b_1 x + c_1 y}{2A}$      $N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$     and  $N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$

Similarly  $v = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$

$$\therefore u(x, y) = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \{\delta\}_e \quad \dots(5.16)$$



### Shape Functions In Terms of Natural Coordinate Systems

Using polynomial functions and natural coordinate systems, shape functions can be derived easily. This approach makes it possible to find shape functions for more elements. This approach is illustrated with few standard cases below:

**Example 5.7:** For a two noded bar element, determine the shape functions. Use natural coordinate system.

**Solution:** As there are only two nodal values in this case, only linear function in natural coordinates are to be taken. Fig. 5.8 shown the typical element. Thus

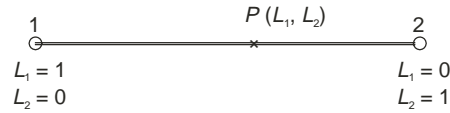


Fig. 5.8

$$u = \alpha_1 L_1 + \alpha_2 L_2 = [L_1 \ L_2] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \quad \dots(5.17)$$

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \quad [\text{Since } L_1 = 1 \text{ and } L_2 = 0 \text{ at node 1 and } L_1 = 0 \text{ and } L_2 = 1 \text{ at node 2}]$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore u = [L_1 \ L_2] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

$$= [L_1 \ L_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [L_1 \ L_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\text{Since } u = [N_1 \ N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \text{ by definition of shape function}$$

$$\therefore N_1 = L_1 \text{ and } N_2 = L_2$$

**Example 5.8:** Derive the expression for shape function for a two noded bar element taking natural coordinate  $\xi$  as varying from -1 to 1.

**Solution:** The typical bar element in the natural coordinate  $\xi$  varying from -1 to 1 is shown in Fig. 5.9 (a). Since there are only two nodal values, a linear polynomial is to be selected. Let displacement at any point  $P(\xi)$  be

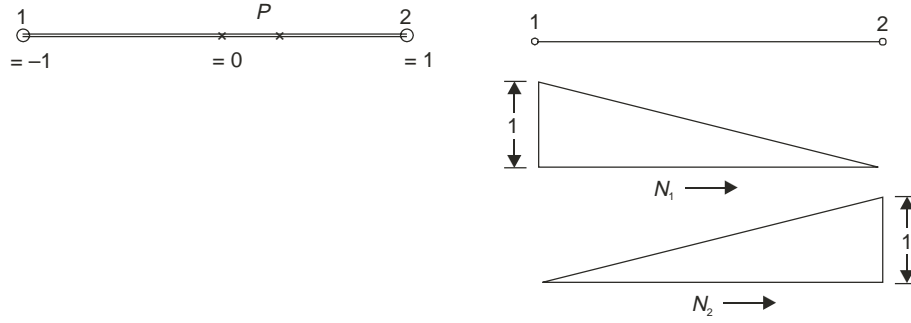


Fig. 5.9 Variation of shape function in bar element

$$\begin{aligned}
 u &= \alpha_1 + \alpha_2 \xi \\
 &= [1 \quad \xi] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \quad \dots(5.18)
 \end{aligned}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \quad \text{Since } \xi = -1 \text{ at node 1 and } \xi = 1 \text{ at node 2}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{1+1} \begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix}^{-T} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore u = [1 \quad \xi] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = [1 \quad \xi] \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2} [1 - \xi \quad 1 + \xi] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} \frac{1 - \xi}{2} & \frac{1 + \xi}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where 
$$N_1 = \frac{1 - \xi}{2} \quad \text{and} \quad N_2 = \frac{1 + \xi}{2} \quad \dots(5.19)$$

Variation of shape functions is shown in Fig. 5.9 (b).

**Example 5.9:** Determine the shape functions for a three noded bar element with natural coordinate system as shown in Fig. 5.10

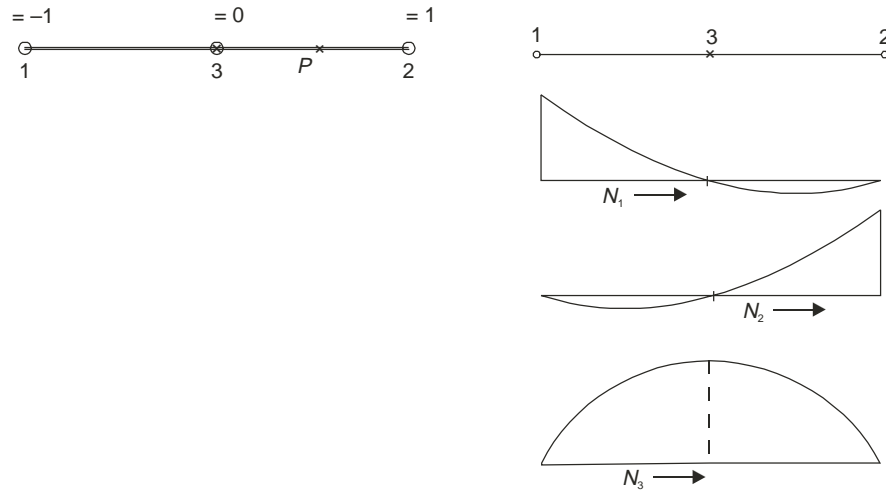


Fig. 5.10 Variation of shape functions in 3 noded bar element

**Solution:** In this case there are three nodal unknowns. Hence a polynomial with 3 generalized coordinates as shown below is selected

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 \quad \dots(5.20)$$

$$\therefore \{\delta\}_e = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

Since  $\xi_1 = -1$ ,  $\xi_2 = 1$  and  $\xi_3 = 0$

$$\begin{aligned} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \\ &= \frac{1}{0 + 1(-1) + 1(-1)} \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & -1 \\ -2 & 0 & 2 \end{bmatrix}^T \{\delta\}_e \\ &= \frac{1}{2} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{bmatrix}^T \{\delta\}_e = \frac{1}{2} \begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \{\delta\}_e \end{aligned}$$

Now,

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & 0 & 2 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix} \{\delta\}_e = \begin{bmatrix} \frac{-\xi + \xi^2}{2} & \frac{\xi + \xi^2}{2} & 1 - \xi^2 \end{bmatrix} \{\delta\}_e$$

$$= \begin{bmatrix} \frac{1}{2} \xi (\xi - 1) & \frac{\xi}{2} (\xi + 1) & 1 - \xi^2 \end{bmatrix} \{\delta\}_e$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \{\delta\}_e \quad \dots(5.21)$$

where  $N_1 = \frac{1}{2} \xi (\xi - 1)$ ,  $N_2 = \frac{1}{2} \xi (\xi + 1)$ ,  $N_3 = 1 - \xi^2$

The variation of the shape functions are as shown in Fig. 5.10 (b).

**Example 5.10:** Determine the shape functions for a Constant Strain Triangular (CST) element in terms of natural coordinate systems.

**Solution:** Let the natural coordinates of nodes 1,2,3 be  $L_1, L_2, L_3$  and shape functions be  $N_1, N_2, N_3$ . The typical CST element is shown in Fig. 5.11 (a). Since the CST element is a linear displacement model, let the displacement function be

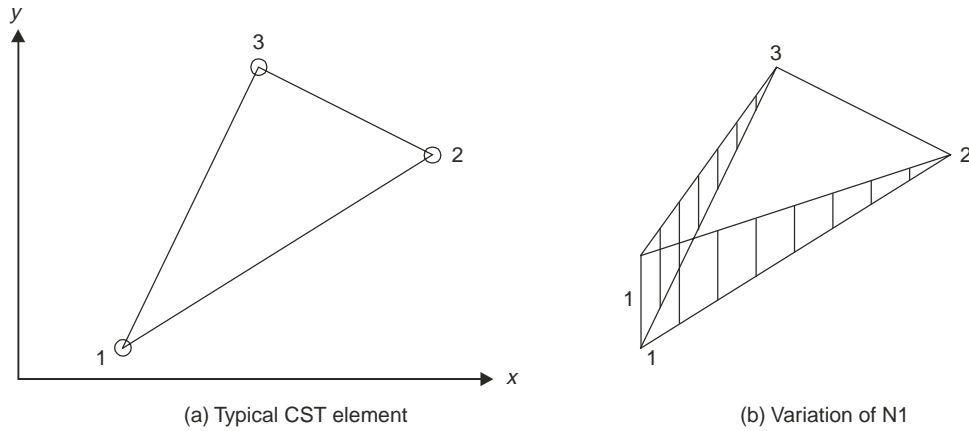


Fig. 5.11

$$u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 \quad \dots(5.22)$$

$$= \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

Since the natural coordinates of node 1,2,3 are (1,0,0), (0,1,0) and (0,0,1) respectively.

$$\{\alpha\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Hence

$$u = [L_1 \quad L_2 \quad L_3] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = [L_1 \quad L_2 \quad L_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= [L_1 \quad L_2 \quad L_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Where  $N_1 = L_1$ ,  $N_2 = L_2$  and  $N_3 = L_3$

Similarly we can show,

$$v = L_1 v_1 + L_2 v_2 + L_3 v_3 = N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$= [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \quad \dots(5.23)$$

$$\therefore \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \{\delta\}_e = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \{\delta\}_e$$

where  $N_1 = L_1$ ,  $N_2 = L_2$  and  $N_3 = L_3$ , variation of  $N_1$  is shown in Fig. 5.11 (b).

**Example 5.11:** Determine the shape function for Linear Strain Triangular (LST) element. Use natural coordinate system.

**Solution:** Figure 5.12 (a) shows the typical LST element.

As there are three nodes along any side, it can be easily seen that displacement varies in the quadratic form (one order higher than the variation of strain). As there are six nodal values we have to pick a polynomial with six constants. Taking all the quadratic terms in natural coordinate system, we can select shape function as:

$$u = \alpha_1 L_1^2 + \alpha_2 L_2^2 + \alpha_3 L_3^2 + \alpha_4 L_1 L_2 + \alpha_5 L_2 L_3 + \alpha_6 L_3 L_1 \quad \dots(5.24)$$

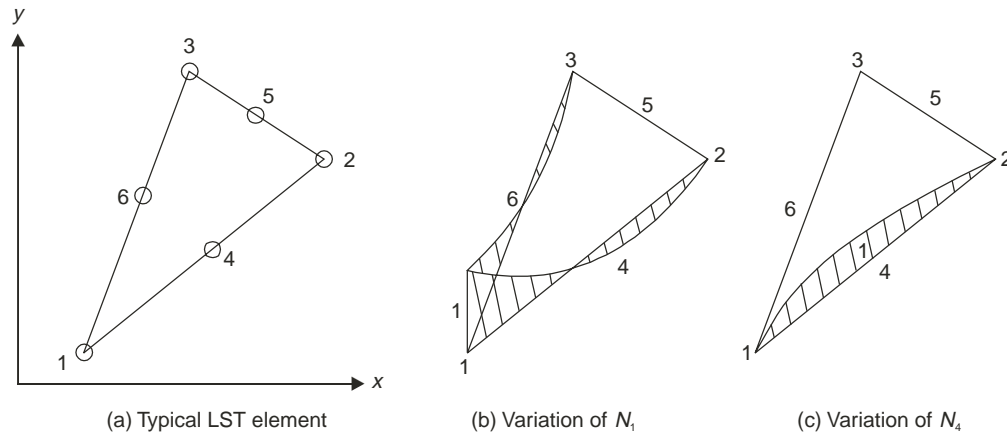


Fig. 5.12

$$\therefore \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

since natural coordinates at nodes 1 to 6 are (1,0,0), (0,1,0), (0,0,1), ( 1/2, 1/2, 0), (0, 1/2, 1/2) and (1/2, 0, 1/2) respectively. Then, we have

$$\{u\}_e = [A] \{\alpha\}$$

where  $\{u\}_e$  is the vector of nodal displacements in  $x$  directions,  $[A]$  is the matrix shown in the above equation and  $\{\alpha\}$  is the vector of generalized coordinates (constants in polynomial)

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 & 4 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 \end{bmatrix}$$

$$\{\alpha\} = [A]^{-1} \{u\}_e$$

and we have started with

$$u = \alpha_1 L_1^2 + \alpha_2 L_2^2 + \alpha_3 L_3^2 + \alpha_4 L_1 L_2 + \alpha_5 L_2 L_3 + \alpha_6 L_3 L_1$$

$$\begin{aligned}
&= \begin{bmatrix} L_1^2 & L_2^2 & L_3^2 & L_1 & L_2 & L_2 & L_3 & L_3 & L_1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \\
&= \begin{bmatrix} L_1^2 & L_2^2 & L_3^2 & L_1 & L_2 & L_2 & L_3 & L_3 & L_1 \end{bmatrix} [A]^{-1} \{u\}_e \\
&= \begin{bmatrix} L_1^2 & L_2^2 & L_3^2 & L_1 & L_2 & L_2 & L_3 & L_3 & L_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 & 4 & 0 \\ -1 & 0 & -1 & 0 & 0 & 4 \end{bmatrix} \{u\}_e \\
&= \left[ (L_1^2 - L_1L_2 - L_3L_1), (L_2^2 - L_1L_2 - L_2L_3), (L_3^2 - L_2L_3 - L_3L_1), 4L_1L_2, 4L_2L_3, 4L_3L_1 \right] \{u\}_e \\
&= [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6] \{u\}_e
\end{aligned}$$

where  $N_1 = L_1^2 - L_1L_2 - L_3L_1$ ,  $N_2 = L_2^2 - L_1L_2 - L_2L_3$ ,  $N_3 = L_3^2 - L_2L_3 - L_3L_1$   
 $N_4 = 4L_1L_2$ ,  $N_5 = 4L_2L_3$ , and  $N_6 = 4L_3L_1$

$$\begin{aligned}
\text{Now, } N_1 &= L_1^2 - L_1L_2 - L_3L_1, \\
&= L_1(L_1 - L_2 - L_3) \\
&= L_1[L_1 - (1 - L_1)] \text{ since } L_1 + L_2 + L_3 = 1 \\
&= L_1(2L_1 - 1)
\end{aligned}$$

Similarly  $N_2 = L_2(2L_2 - 1)$

and  $N_3 = L_3(2L_3 - 1)$

Similarly it can be proved that

$$\begin{aligned}
v &= [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6] \{v\}_e \\
\therefore \begin{Bmatrix} u \\ v \end{Bmatrix} &= \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \{\delta\}_e \quad \dots(5.24)
\end{aligned}$$

where  $\{\delta\}_e^T = [u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6 \ v_1 \ v_2 \ v_3 \ v_4 \ v_5 \ v_6]$

and

$$[N] = [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6] \{\delta\}_e$$

$$N_1 = L_1(2L_1 - 1); N_2 = L_2(2L_2 - 1); N_3 = L_3(2L_3 - 1); N_4 = 4L_1L_2;$$

$$N_5 = 4L_2L_3, \text{ and } N_6 = 4L_3L_1$$

Variation of  $N_1$  and  $N_4$  are shown in Fig. 5.12 (b).

**Example 5.12:** Determine the shape functions for 4 noded rectangular elements. Use natural coordinate system.

**Solution:** The typical 4 noded rectangular element is shown in Fig. 5.13.

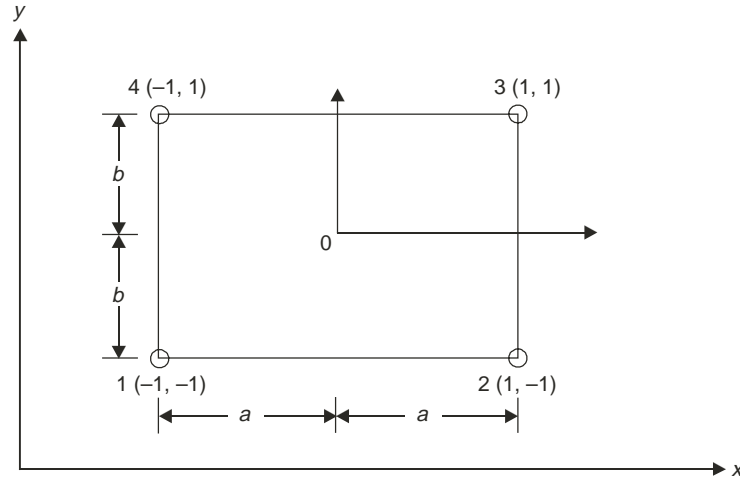


Fig. 5.13 Typical 4 noded rectangular element

Taking the centroid of the rectangle as origin and  $\xi$  and  $\eta$  as natural coordinates, we have

$$\xi = \frac{x - x_c}{a} \text{ and } \eta = \frac{y - y_c}{b}$$

where  $2a \times 2b$  is the size of the element as shown in Fig. 5.13 and  $x_c, y_c$  are the coordinates of the origin.

We need a polynomial in two dimension with 4 constants. Such polynomial is obtained by dropping  $\xi^2$  and  $\eta^2$  terms in second degree polynomial. Such polynomial maintains geometric isotropy also. Thus,

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta \quad \dots(5.25)$$

$$\{u\}_e = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$



$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = [A]^{-1} \{u\}_e$$

It can be shown that

$$[A]^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\therefore u = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta$$

$$\begin{aligned} &= [1 \ \xi \ \eta \ \xi\eta] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = [1 \ \xi \ \eta \ \xi\eta] \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \{u\}_e \\ &= \left[ \frac{(1-\xi)(1-\eta)}{4} \quad \frac{(1+\xi)(1-\eta)}{4} \quad \frac{(1-\xi)(1+\eta)}{4} \quad \frac{(1-\xi)(1+\eta)}{4} \right] \{u\}_e \\ &= [N_1 \ N_2 \ N_3 \ N_4] \{u\}_e \\ &= [N] \{u\}_e \end{aligned}$$

where

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} \quad N_2 = \frac{(1+\xi)(1-\eta)}{4} \quad N_3 = \frac{(1-\xi)(1+\eta)}{4} \quad N_4 = \frac{(1+\xi)(1+\eta)}{4}$$

In short  $N_1$  to  $N_4$  may be written as

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i) \text{ for } i = 1, 2, 3 \text{ and } 4 \quad \dots(5.26)$$

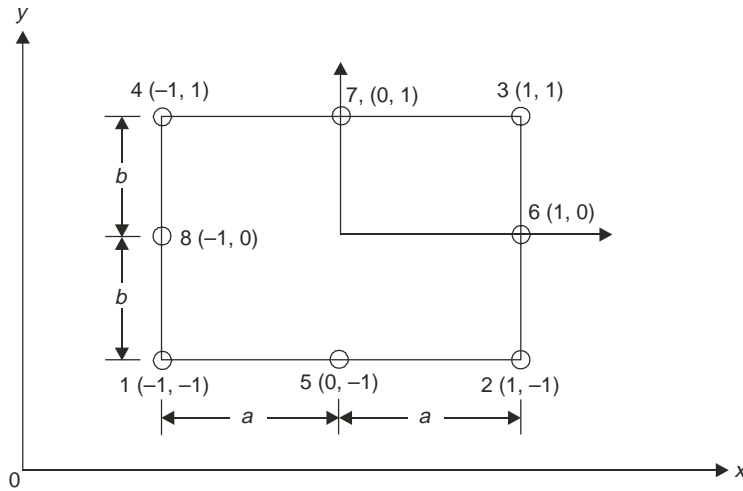
Similarly

$$v = [N_1 \ N_2 \ N_3 \ N_4] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \{\delta\}_e$$

**Note:**  $N_i = 1$  at node  $i$  and is zero at all other nodes.

**Example 5.13:** Determine the shape function for quadratic rectangular element shown in Fig. 5.14



**Fig. 5.14** Typical quadratic rectangular element

**Solution:** There are 8 nodal values for  $u$  and 8 for  $v$ . Hence the displacement function is to be selected with only 8 constant. The polynomial has to maintain geometric isotropy also. This may be obtained by dropping  $\xi^3$  and  $\eta^3$  terms from complete 3rd order polynomial. Thus,

$$u = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi^2 + \alpha_5 \xi \eta + \alpha_6 \eta^2 + \alpha_7 \xi^2 \eta + \alpha_8 \xi \eta^2 \quad \dots(5.27)$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix}_e = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} = [A] \{\alpha\}$$

or

$$\{\alpha\} = [A^{-1}] \{u\}_e$$

It may be shown that,

$$[A]^{-1} = \frac{1}{4} \begin{bmatrix} -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & -2 & 0 & -2 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & -2 & 0 & -2 \\ -1 & -1 & 1 & 1 & 2 & 0 & -2 & 0 \\ -1 & 1 & 1 & -1 & 0 & -2 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \text{But, } u &= [1 \ \xi \ \eta \ \xi^2 \ \xi\eta \ \eta^2 \ \xi^2\eta \ \xi\eta^2] \{\alpha\} \\ &= [1 \ \xi \ \eta \ \xi^2 \ \xi\eta \ \eta^2 \ \xi^2\eta \ \xi\eta^2] [A]^{-1} \{u\}_e \\ &= \left[ \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta)(-\xi-\eta-1), \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1), \right. \\ &\quad \left. \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1), \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1), \right. \\ &\quad \left. \frac{1}{2}(1+\xi)(1-\xi)(\eta-1), \frac{1}{2}(1+\xi)(1+\eta)(1-\eta), \right. \\ &\quad \left. \frac{1}{2}(1+\xi)(1-\xi)(\eta+1), \frac{1}{2}(1-\xi)(1+\eta)(1-\eta) \right] \{u\}_e \\ &= [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6 \ N_7 \ N_8] \{u\}_e \end{aligned} \quad \dots(5.28)$$

Where  $N_1 \dots N_8$  are as defined above.In other words, for corner nodes ( $i = 1, 2, 3, 4$ )

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)(\xi\xi_i + \eta\eta_i - 1) \quad \dots(5.29)$$

For mid side nodes,

$$\text{If } \xi_i = 0 \text{ then } N_i = \frac{1}{2}(1 - \xi^2)(1 + \eta\eta_i) \text{ (i.e., for 5, 7)} \quad \dots(5.30)$$

$$\text{If } \eta_i = 0 \text{ then } N_i = \frac{1}{2}(1 - \eta^2)(1 + \xi\xi_i) \text{ (i.e., for 6, 8)} \quad \dots(5.31)$$

Thus,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & N \end{bmatrix} \begin{Bmatrix} u_e \\ v_e \end{Bmatrix}$$

[Note:  $N_i = 1$  at node  $i$  and is zero at all other nodes, variation is quadratic]

**Example 5.14:** Determine the shape functions for a tetrahedron element.

**Solution:** The typical tetrahedron is as shown in Fig. 5.15.

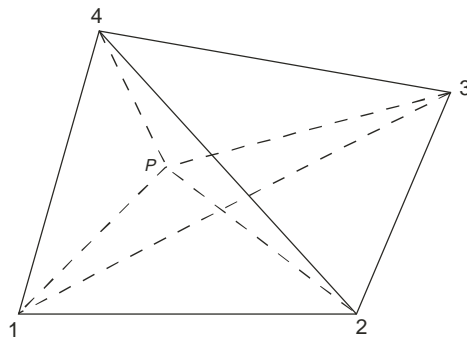


Fig. 5.15 Tetrahedron element

In Chapter 4, we have seen that the natural coordinates for such element are volume coordinates

$$L_i = \frac{V_i}{V}$$

Where  $L_i$  = Natural coordinates

$V_i$  = Volume of sub tetrahedron formed by the point and the nodes except  $i$ th and  $V$  = Volume of the tetrahedron

$\therefore$  Natural coordinates for the node point 1, 2, 3 and 4 are

1(1,0,0,0), 2(0,1,0,0), 3(0,0,1,0) and 4(0,0,0,1)

Let the displacement in  $x$  direction at any point  $P$  in the element be

$$u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4 L_4 \quad \dots(5.32)$$

$$\therefore \{u\}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \{u\}_e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \{u\}_e$$

$$\begin{aligned} \therefore u &= \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4 L_4 \\ &= [L_1 \ L_2 \ L_3 \ L_4] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = [L_1 \ L_2 \ L_3 \ L_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \{u\}_e \\ &= [L_1 \ L_2 \ L_3 \ L_4] \{u\}_e = [N_1 \ N_2 \ N_3 \ N_4] \{u\}_e = [N] \{u\}_e \end{aligned}$$

where  $N_1 = L_1, N_2 = L_2, N_3 = L_3$  and  $N_4 = L_4$  ...(5.34)

Similarly for displacement  $v$  and  $w$ , we get

$$v = [N] \{v\}_e \text{ and } w = [N] \{w\}_e$$

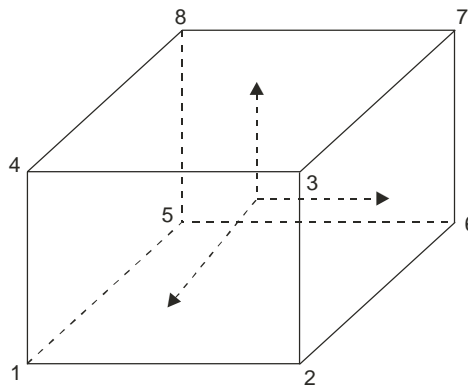
$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N & 0 & 0 \\ 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 0 & N & 0 \\ 1 \times 4 & 1 \times 4 & 1 \times 4 \\ 0 & 0 & N \\ 1 \times 4 & 1 \times 4 & 1 \times 4 \end{bmatrix} \{\delta\}_e$$

where  $\{\delta\}_e^T = [u_1 \ u_2 \ u_3 \ u_4 \ v_1 \ v_2 \ v_3 \ v_4 \ w_1 \ w_2 \ w_3 \ w_4]$

**Note:**  $N_i = 1$  at node  $i$  and is zero at all other nodes. Variation is linear.

**Example 5.15:** Explain the method of finding shape function for a hexahedral element.

**Solution:** The typical element selected is shown in Fig. 5.16. The natural coordinates of various nodal points are



**Fig. 5.16** Typical hexahedron element

- |                |               |              |               |
|----------------|---------------|--------------|---------------|
| 1 (1, -1, -1)  | 2 (1, 1, -1)  | 3(1, 1, 1)   | 4 (1, -1, 1)  |
| 5 (-1, -1, -1) | 6 (-1, 1, -1) | 7 (-1, 1, 1) | 8 (-1, -1, 1) |

There are only eight nodal values for defining displacement inside the element. Hence polynomial with 8 constants is to be selected for shape function. Keeping in view that geometric isotropy is to be maintained the following polynomial is selected.

$$u = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\zeta + \alpha_5\xi\eta + \alpha_6\eta\zeta + \alpha_7\zeta\xi + \alpha_8\xi\eta\zeta \quad \dots(5.35)$$

$$\text{i.e.,} \quad = [1 \ \xi \ \eta\zeta \ \xi\eta \ \eta\zeta \ \zeta\xi \ \xi\eta\zeta] \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_8 \end{Bmatrix}$$

$$\therefore \{u\}_e = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{Bmatrix} = [A] \{\alpha\}$$

$$\therefore \{\alpha\} = [A]^{-1} \{u\}_e$$

$[A]^{-1}$  can be found. Then

$$\begin{aligned} u &= [1 \ \xi \ \eta \ \zeta \ \xi\eta \ \eta\zeta \ \zeta\xi \ \xi\eta\zeta] \{\alpha\} \\ &= [1 \ \xi \ \eta \ \zeta \ \xi\eta \ \eta\zeta \ \zeta\xi \ \xi\eta\zeta] [A]^{-1} \{u\}_e = [N] \{u\}_e \end{aligned}$$

where

$$\begin{aligned} [N] &= [1 \ \xi \ \eta \ \zeta \ \xi\eta \ \eta\zeta \ \zeta\xi \ \xi\eta\zeta] [A]^{-1} \\ &= [N_1 \ N_2 \ N_3 \ N_4 \ N_5 \ N_6 \ N_7 \ N_8] \end{aligned}$$

It will be found that

$$N_i = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(1 + \zeta\zeta_i) \quad \dots(5.36)$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N & 0 & 0 \\ 1 \times 8 & 1 \times 8 & 1 \times 8 \\ 0 & N & 0 \\ 1 \times 8 & 1 \times 8 & 1 \times 8 \\ 0 & 0 & N \\ 1 \times 8 & 1 \times 8 & 1 \times 8 \end{bmatrix} \{\delta\}_e$$

where

$$\{\delta\}_e^T = [u_1 \dots u_8, v_1 \dots v_8, w_1 \dots w_8]$$

**Note:**  $N_i = 1$  at node  $i$  and is zero at all other nodes.

## 5.5 FINDING SHAPE FUNCTIONS USING LAGRANGE POLYNOMIALS

If only continuity of basic unknown (displacement) is to be satisfied, Lagrange polynomials can be used to derive shape functions. Lagrange polynomial in one dimension is defined by

$$L_k(x) = \prod_{\substack{m=1 \\ m \neq k}}^n \frac{x - x_m}{x_k - x_m} \quad \dots(5.37)$$

Thus, if  $n = 5$  and  $k = 3$ ,

$$L_3(x) = \frac{(x - x_1)(x - x_2)(x - x_4)(x - x_5)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)}$$

Obviously equation 5.37 takes the value equal to zero at all points except at point  $k$ . At point  $k$  its value is unity.

This is exactly the property required for the interpolation functions. Hence Lagrange Polynomial can be straight way used as shape functions for one dimensional problems. The following example illustrates it.

**Example 5.16:** Using Lagrange polynomial find shape functions for

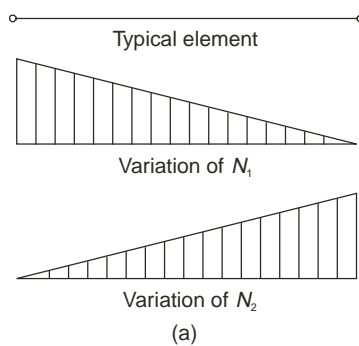
- (i) Two noded bar element
- (ii) Three noded bar element and
- (iii) Five noded bar element

Plot the variation of shape functions.

**Solution:** (i) Two noded bar element

The typical two noded bar elements is shown in Fig. 5.17 (a)

General Lagrange Polynomial is



**Fig. 5.17** (a) Two noded bar element

$$L_k = \frac{(x - x_1)(x - x_2) \dots (x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)}{(x_k - x_1)(x_k - x_2) \dots (x_k - x_{k-1})(x_k - x_{k+1}) \dots (x_k - x_n)}$$

Now  $n = 2$ . Hence when  $k = 1$

$$\therefore N_1 = L_1 = \frac{x - x_2}{x_1 - x_2} = \frac{x_2 - x}{x_2 - x_1}, N_{1 \times 8} \text{ same as found earlier}$$

When  $k = 2$ ,

$$N_2 = L_2 = \frac{x - x_1}{x_2 - x_1}, \text{ same as found earlier variation of } N_1 \text{ and } N_2 \text{ is also shown in Fig. 5.17 (a).}$$

(ii) For Three Noded Element

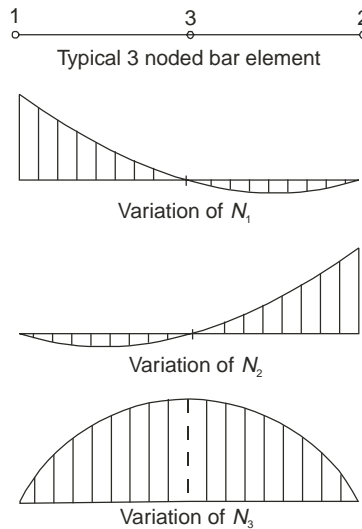
$n = 3$ . Hence when  $k = 1$ ,

$$N_1 = L_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

When  $k = 2$

$$N_2 = L_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

and 
$$N_3 = L_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$



**Fig. 5.17 (contd)** (b) Three noded bar element

The typical element and the variation of its shape functions are shown in Fig. 5.17 (b).

(iii) For Five Noded Element

$n = 5$

$$N_1 = L_1 = \frac{(x - x_2)(x - x_3)(x - x_4)(x - x_5)}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)}$$



$$N_2 = L_2 = \frac{(x - x_1)(x - x_3)(x - x_4)(x - x_5)}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)}$$

$$N_3 = L_3 = \frac{(x - x_1)(x - x_2)(x - x_4)(x - x_5)}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)}$$

$$N_4 = L_4 = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_5)}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)(x_4 - x_5)}$$

$$N_5 = L_5 = \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)(x_5 - x_4)}$$

The five noded bar element and variation of its shape functions are shown in Fig. 5.17 (c)

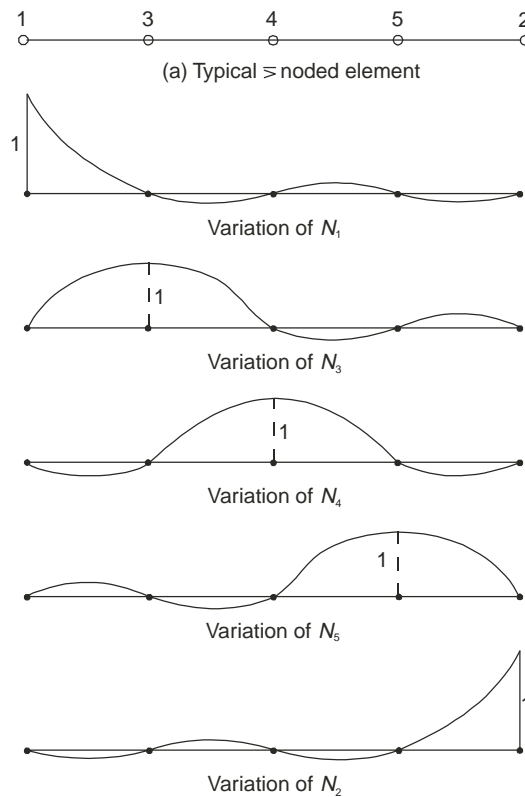


Fig. 5.17 (contd) (c) Typical 5 noded bar element

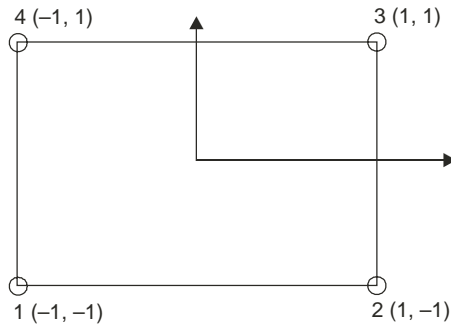
### Lagrange Polynomial Approach for Two Dimensional Elements

Although Lagrangian interpolation functions are for only one dimension, we may extend the concept to two and three dimensions by forming the product of the functions which hold good for the individual one dimensional coordinate directions i.e.,

$$N_1 = L_1(\xi) L_1(\eta) \quad \dots(5.38)$$

Thus for 4 noded rectangular element shown in Fig. 5.18,

$$\begin{aligned} N_1 &= L_1(\xi) L_1(\eta) = \frac{\xi - \xi_2}{\xi_1 - \xi_2} \frac{\eta - \eta_4}{\eta_1 - \eta_4} \\ &= \frac{(\xi - 1)}{-1 - 1} \frac{\eta - 1}{-1 - 1} = \frac{1}{4}(\xi - 1)(\eta - 1) = \frac{1}{4}(1 - \xi)(1 - \eta) \end{aligned}$$



**Fig. 5.18** Typical 4 noded rectangular element

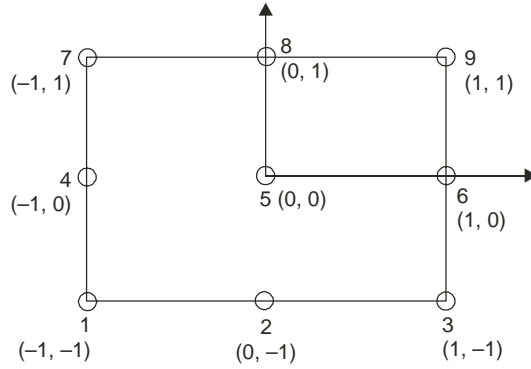
$$\begin{aligned} N_2 &= L_2(\xi) L_2(\eta) = \frac{\xi - \xi_1}{\xi_2 - \xi_1} \frac{\eta - \eta_3}{\eta_2 - \eta_3} \\ &= \frac{(\xi - (-1))}{1 - (-1)} \frac{(\eta - 1)}{-1 - 1} = \frac{(\xi + 1)(\eta - 1)}{(-4)} = \frac{1}{4}(1 + \xi)(1 - \eta) \end{aligned}$$

$$\begin{aligned} N_3 &= L_3(\xi) L_3(\eta) = \frac{\xi - \xi_4}{\xi_3 - \xi_4} \frac{\eta - \eta_2}{\eta_3 - \eta_2} \\ &= \frac{\xi + 1}{1 - (-1)} \frac{\eta + 1}{1 - (-1)} = \frac{(1 + \xi)(1 + \eta)}{4} \end{aligned}$$

$$\begin{aligned} N_4 &= L_4(\xi) L_4(\eta) = \frac{\xi - \xi_3}{\xi_4 - \xi_3} \frac{\eta - \eta_1}{\eta_4 - \eta_1} \\ &= \frac{(\xi - 1)}{-1 - (1)} \frac{\eta + 1}{1 - (-1)} = \frac{(\xi - 1)(\eta + 1)}{-4} = \frac{(1 - \xi)(1 + \eta)}{4} \end{aligned}$$

Thus 
$$N_i = \frac{(1 + \xi\xi_i)(1 + \eta\eta_i)}{4}$$

**Example 5.17:** Using Lagrange functions write shape functions for the nine noded rectangular element shown in Fig. 5.19.



**Fig. 5.19** Typical nine noded rectangular element

**Solution:** The natural coordinates of various nodes are as shown in the figure. For the  $C^0$  continuity element in two dimensions,

$$N_i = L_i(\xi) L_i(\eta)$$

where  $L_i$  refers to Lagrangian function at node  $i$ . In this case there are 3 nodes in each direction. Hence  $n = 3$  in Lagrange function

$$\begin{aligned} N_1 &= \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} \frac{(\eta - \eta_4)(\eta - \eta_7)}{(\eta_1 - \eta_4)(\eta_1 - \eta_7)} \\ &= \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{\xi(\xi - 1)\eta(\eta - 1)}{4} \end{aligned}$$

$$\begin{aligned} N_2 &= \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} \frac{(\eta - \eta_5)(\eta - \eta_8)}{(\eta_2 - \eta_5)(\eta_2 - \eta_8)} \\ &= \frac{(\xi + 1)(\xi - 1)}{(0 + 1)(0 - 1)} \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{(\xi + 1)(\xi - 1)\eta(\eta - 1)}{(-2)} \end{aligned}$$

$$\begin{aligned} N_3 &= \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} \frac{(\eta - \eta_6)(\eta - \eta_9)}{(\eta_3 - \eta_6)(\eta_3 - \eta_9)} \\ &= \frac{(\xi + 1)(\xi - 0)}{(1 + 1)(1 - 0)} \frac{(\eta - 0)(\eta - 1)}{(-1 - 0)(-1 - 1)} = \frac{(\xi + 1)\xi \eta(\eta - 1)}{4} \end{aligned}$$

$$N_4 = \frac{(\xi - \xi_5)(\xi - \xi_6)}{(\xi_4 - \xi_5)(\xi_4 - \xi_6)} \frac{(\eta - \eta_1)(\eta - \eta_7)}{(\eta_4 - \eta_1)(\eta_4 - \eta_7)}$$

$$\begin{aligned}
&= \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} \frac{(\eta + 1)(\eta - 1)}{(0 + 1)(0 - 1)} = \frac{\xi(\xi - 1)(\eta + 1)(\eta - 1)}{-2} \\
N_5 &= \frac{(\xi - \xi_4)(\xi - \xi_6)}{(\xi_5 - \xi_4)(\xi_5 - \xi_6)} \frac{(\eta - \eta_2)(\eta - \eta_8)}{(\eta_5 - \eta_2)(\eta_5 - \eta_8)} \\
&= \frac{(\xi + 1)(\xi - 1)}{(0 + 1)(0 - 1)} \frac{(\eta + 1)(\eta - 1)}{(0 + 1)(0 - 1)} = \frac{(\xi + 1)(\xi - 1)(\eta + 1)(\eta - 1)}{1} \\
N_6 &= \frac{(\xi - \xi_4)(\xi - \xi_5)}{(\xi_6 - \xi_4)(\xi_6 - \xi_5)} \frac{(\eta - \eta_3)(\eta - \eta_9)}{(\eta_6 - \eta_3)(\eta_6 - \eta_9)} \\
&= \frac{(\xi + 1)(\xi - 0)}{(1 + 1)(1 - 0)} \frac{(\eta + 1)(\eta - 1)}{(0 + 1)(0 - 1)} = \frac{(\xi + 1)\xi}{-2} \frac{(\eta + 1)(\eta - 1)}{-2} \\
N_7 &= \frac{(\xi - \xi_8)(\xi - \xi_9)}{(\xi_7 - \xi_8)(\xi_7 - \xi_9)} \frac{(\eta - \eta_1)(\eta - \eta_4)}{(\eta_7 - \eta_1)(\eta_7 - \eta_4)} \\
&= \frac{(\xi - 0)(\xi - 1)}{(-1 - 0)(-1 - 1)} \frac{(\eta + 0)(\eta - 0)}{(1 + 1)(1 - 0)} = \frac{\xi(\xi - 1)(\eta + 1)\eta}{4} \\
N_8 &= \frac{(\xi - \xi_7)(\xi - \xi_9)}{(\xi_8 - \xi_7)(\xi_8 - \xi_9)} \frac{(\eta - \eta_2)(\eta - \eta_5)}{(\eta_8 - \eta_2)(\eta_8 - \eta_5)} \\
&= \frac{(\xi + 1)(\xi - 1)}{(0 + 1)(0 - 1)} \frac{(\eta + 1)(\eta - 0)}{(1 + 1)(1 - 0)} = \frac{(\xi + 1)(\xi - 1)(\eta + 1)(\eta)}{-2} \\
N_9 &= \frac{(\xi - \xi_7)(\xi - \xi_8)}{(\xi_9 - \xi_7)(\xi_9 - \xi_8)} \frac{(\eta - \eta_3)(\eta - \eta_6)}{(\eta_9 - \eta_3)(\eta_9 - \eta_6)} \\
&= \frac{(\xi + 1)(\xi - 0)}{(1 + 1)(1 - 0)} \frac{(\eta + 1)(\eta - 0)}{(1 + 1)(1 - 0)} = \frac{(\xi + 1)\xi}{4} \frac{(\eta + 1)(\eta)}{4}
\end{aligned}$$

Thus in this case, for corner nodes,

$$N_i = \frac{1}{4} \xi \eta (\xi + \xi_i)(\eta + \eta_i) \quad \dots(5.41)$$

For nodes 2 and 8 where  $\xi = 0$  ;

$$N_i = \frac{(\xi + 1)(\xi - 1)}{-2} \eta (\eta + \eta_i)$$

For nodes 4 and 6, where  $\eta_i = 0$

$$N_i = \frac{\xi(\xi + \xi_i)(\eta - 1)(\eta + 1)}{-2}$$

and for central node,

$$N_5 = \frac{(\xi + 1)(\xi - 1)(\eta + 1)(\eta - 1)}{4}$$

**Example 5.18:** Using Lagrange functions, derive shape function for hexahedron (brick) element.

**Solution:** Typical hexahedron element is shown in Fig. 5.20.

The coordinates for various nodal points are

1 (1, -1, -1), 2 (1, 1, -1), 3 (1, 1, 1), 4 (1, -1, 1), 5 (-1, -1, -1), 6 (-1, 1, -1), 7 (-1, 1, 1), 8 (-1, -1, 1)

In general, shape function by Lagrange function for three dimensional case is given by

$$N_i = L_i(\xi) L_i(\eta) L_i(\zeta)$$

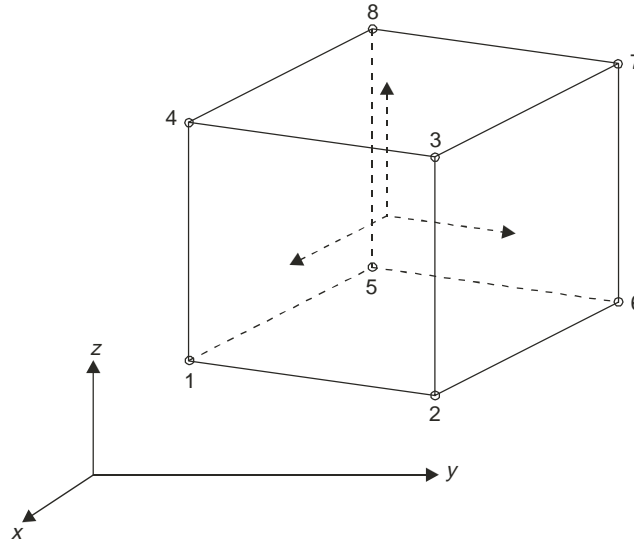


Fig. 5.20 Typical hexahedron element

$$\therefore N_1 = \frac{(\xi - \xi_5)}{\xi_1 - \xi_5} \frac{(\eta - \eta_2)}{\eta_1 - \eta_2} \frac{(\zeta - \zeta_4)}{\zeta_1 - \zeta_4} = \frac{(\xi + 1)(\eta - 1)(\zeta - 1)}{(1 + 1)(-1 - 1)(-1 - 1)} = \frac{(\xi + 1)(\eta - 1)(\zeta - 1)}{8}$$

$$N_2 = L_2(\xi) L_2(\eta) L_2(\zeta)$$

$$= \frac{(\xi - \xi_6)}{(\xi_2 - \xi_6)} \frac{(\eta - \eta_1)}{(\eta_2 - \eta_1)} \frac{(\zeta - \zeta_3)}{(\zeta_2 - \zeta_3)}$$

$$\begin{aligned}
&= \frac{(\xi + 1)(\eta + 1)(\zeta - 1)}{(1 + 1)(1 + 1)(-1 - 1)} = \frac{(\xi + 1)(\eta + 1)(\zeta - 1)}{-8} \\
N_3 &= L_3(\xi) L_3(\eta) L_3(\zeta) \\
&= \frac{(\xi - \xi_7)}{\xi_3 - \xi_7} \frac{\eta - \eta_4}{\eta_3 - \eta_4} \frac{\zeta - \zeta_2}{\zeta_3 - \zeta_2} \\
&= \frac{\xi + 1}{1 + 1} \frac{\eta + 1}{1 + 1} \frac{\zeta + 1}{1 + 1} = \frac{(\xi + 1)(\eta + 1)(\zeta + 1)}{8} \\
N_4 &= L_4(\xi) L_4(\eta) L_4(\zeta) = \frac{\xi - \xi_8}{\xi_4 - \xi_8} \frac{\eta - \eta_3}{\eta_4 - \eta_3} \frac{\zeta - \zeta_1}{\zeta_4 - \zeta_1} \\
&= \frac{\xi + 1}{1 + 1} \frac{\eta - 1}{-1 - 1} \frac{\zeta + 1}{1 + 1} = \frac{(\xi + 1)(\eta - 1)(\zeta + 1)}{-8} \\
N_5 &= L_5(\xi) L_5(\eta) L_5(\zeta) \\
&= \frac{(\xi - \xi_1)}{\xi_5 - \xi_1} \frac{\eta - \eta_6}{\eta_5 - \eta_6} \frac{\zeta - \zeta_8}{\zeta_5 - \zeta_8} = \frac{(\xi - 1)}{-1 - 1} \frac{\eta - 1}{-1 - 1} \frac{\zeta - 1}{-1 - 1} = \frac{(\xi - 1)(\eta - 1)(\zeta - 1)}{-8} \\
N_6 &= L_6(\xi) L_6(\eta) L_6(\zeta) \\
&= \frac{\xi - \xi_2}{\xi_6 - \xi_2} \frac{\eta - \eta_5}{\eta_6 - \eta_5} \frac{\zeta - \zeta_7}{\zeta_6 - \zeta_7} = \frac{\xi - 1}{-1 - 1} \frac{\eta + 1}{1 + 1} \frac{\zeta - 1}{-1 - 1} = \frac{(\xi - 1)(\eta + 1)(\zeta - 1)}{8} \\
N_7 &= L_7(\xi) L_7(\eta) L_7(\zeta) \\
&= \frac{\xi - \xi_3}{\xi_7 - \xi_3} \frac{\eta - \eta_8}{\eta_7 - \eta_8} \frac{\zeta - \zeta_6}{\zeta_8 - \zeta_6} = \frac{\xi - 1}{-1 - 1} \frac{\eta + 1}{1 + 1} \frac{\zeta + 1}{1 + 1} = \frac{(\xi - 1)(\eta + 1)(\zeta + 1)}{-8} \\
N_8 &= L_8(\xi) L_8(\eta) L_8(\zeta) \\
&= \frac{\xi - \xi_4}{\xi_8 - \xi_4} \frac{\eta - \eta_7}{\eta_8 - \eta_7} \frac{\zeta - \zeta_5}{\zeta_8 - \zeta_5} = \frac{(\xi - 1)}{-1 - 1} \frac{(\eta - 1)}{-1 - 1} \frac{\zeta + 1}{1 + 1} = \frac{(\xi - 1)(\eta - 1)(\zeta + 1)}{8}
\end{aligned}$$

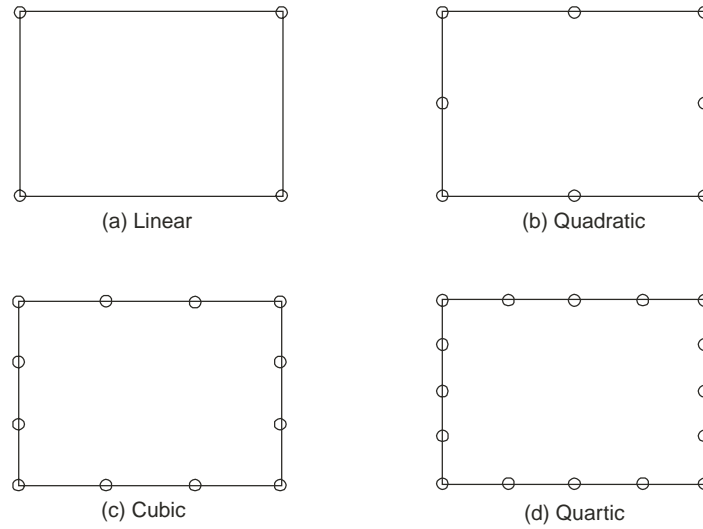
In general it may be noted that,

$$N_i = \frac{1}{8} (1 + \xi \xi_i) (1 + \eta \eta_i) (1 + \zeta \zeta_i) \quad \dots(5.42)$$

## 5.6 SHAPE FUNCTIONS FOR SERENDIPITY FAMILY ELEMENTS

Figure 5.21 shows Serendipity family elements. These elements may be called as boundary node family elements also. In these elements nodes are only on the boundaries. Zienkiewicz called them as ‘Serendip family’ elements by referring to the famous princess of Serendip noted for chance discoveries. The terms linear, quadratic, cubic and quartic are used since the variation of shape functions about a boundary is of that order. The shape functions are found from the consideration that  $N_i = 1$  for  $i$ th node and is zero when referred

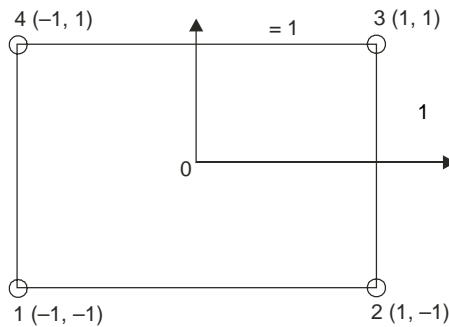
to any other node. Discovery of these elements clubbed with isoparametric concept (explained in ch.13) has made major break through in the finite element analysis. In this article derivation of shape functions for this family of elements is presented, through examples.



**Fig. 5.21** Serendipity family elements

**Example 5.19:** Using ‘serendipity concept’ derive shape functions for 4 noded rectangular element.

**Solution:** Figure 5.22 shows the typical element in natural coordinate system.



**Fig. 5.22** Typical serendipity linear element

$N_1$  has to satisfy the conditions

(a) along  $\xi = 1$ ,  $N_1 = 0$ .

(b) along  $\eta = 1$ ,  $N_1 = 0$

and (c) at  $\xi = -1$ ,  $\eta = -1$   $N_1 = 1$ .

Hence let

$$N_1 = C(1 - \xi)(1 - \eta), \text{ where } C \text{ is arbitrary constant.}$$

Conditions (a) and (b) are satisfied.

Condition (c) gives,

$$1 = C(1 + 1)(1 + 1) \text{ or } C = \frac{1}{4}$$

$$\therefore N_1 = \frac{(1 - \xi)(1 - \eta)}{4}$$

On the same lines we can get,

$$N_2 = \frac{(1 + \xi)(1 - \eta)}{4}$$

$$N_3 = \frac{(1 + \xi)(1 + \eta)}{4}$$

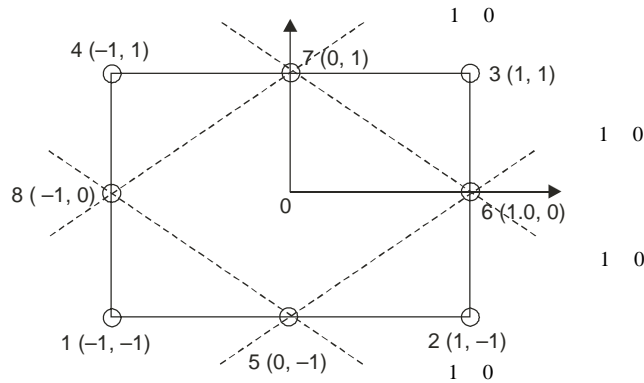
and

$$N_4 = \frac{(1 - \xi)(1 + \eta)}{4}$$

These are same as given in equations 5.26 and 5.40, which were derived from different approaches.

**Example 5.20:** Using serendipity concept find shape functions for quadratic serendipity family element.

**Solution:** Figure 5.23 shows a typical element of this type.



**Fig. 5.23** Typical serendipity quadratic element

The conditions to be satisfied by  $N_1$  are,

(a) along line  $\xi = 1$ ,  $N_1 = 0$

(b) along line  $\eta = 1$ ,  $N_1 = 0$

(c) along line 5-8,  $N_1 = 0$



i.e.,  $\xi + \eta + 1 = 0$ .

(d) At  $\xi = -1$  and  $\eta = -1$ ,  $N_1 = 1$

Hence let

$$N_1 = C(1 - \xi)(1 - \eta)(1 + \xi + \eta)$$

It satisfies the requirements  $a, b, c$ . In other words it ensues  $N_1 = 0$  at nodes 2, 3, ... 8.

From condition (d),

$$1 = C(1 + 1)(1 + 1)(1 - 1 - 1) \therefore C = -\frac{1}{4}$$

$\therefore$  For the corner node

$$N_1 = -\frac{(1 - \xi)(1 - \eta)(1 + \xi + \eta)}{4}$$

Similarly we can show that

$$N_2 = -\frac{(1 + \xi)(1 - \eta)(1 - \xi + \eta)}{4}$$

$$N_3 = -\frac{(1 + \xi)(1 + \eta)(1 - \xi - \eta)}{4}$$

and

$$N_4 = -\frac{(1 - \xi)(1 + \eta)(1 + \xi - \eta)}{4}$$

For mid-side node 5, the conditions to be satisfied are

(a) Along  $\xi = 1$   $N_5 = 0$

(b) Along  $\eta = 1$   $N_5 = 0$

(c) Along  $\xi = -1$   $N_5 = 0$

(d) At node 5 where  $\xi = 0$ ,  $\eta = -1$ ,  $N_5 = 1$

$\therefore$  Let  $N_5 = C(1 - \xi)(1 - \eta)(1 + \xi)$

This form satisfies  $N_1 = 0$  at all nodes other than node 5. From the condition 'd' we get,

$$1 = C(1 - 0)(1 + 1)(1 + 0)$$

$$\therefore C = \frac{1}{2}$$

$$\therefore N_5 = \frac{1}{2}(1 - \xi)(1 - \eta)(1 + \xi) = \frac{(1 - \xi^2)(1 - \eta)}{2}$$

Similarly it may be shown that

$$N_6 = \frac{(1 + \xi)(1 - \eta^2)}{2} \quad \dots(5.44a)$$

$$N_7 = \frac{(1 - \xi^2)(1 + \eta)}{2} \quad \dots(5.44b)$$

and

$$N_8 = \frac{(1 - \xi)(1 - \eta^2)}{2} \quad \dots(5.44c)$$

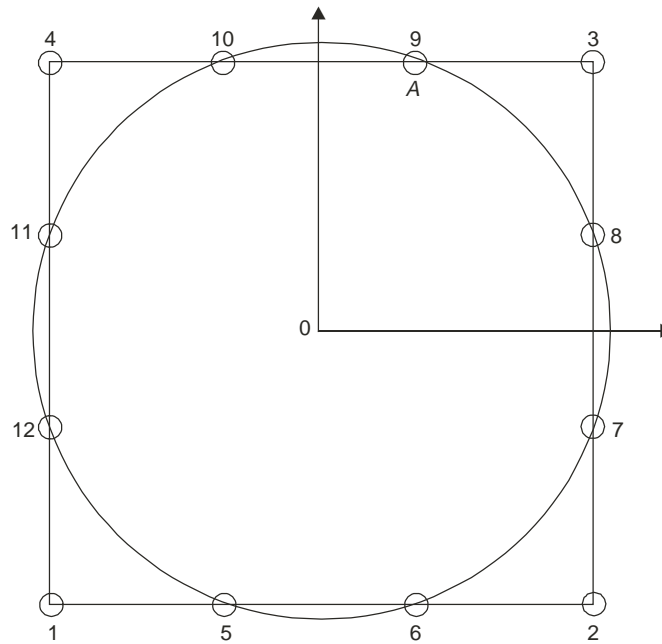
**Example 5.21:** Determine the shape functions for cubic serendipity family element.

**Solution:** Typical element is shown in Fig. 5.24. Shape functions for corner nodes:

$N_1 = 0$  for all nodes except 1 and is 1 for node 1.

$N_1 = 0$  is satisfied for nodes 2, 7, 8, 3 if  $1 - \xi = 0$

$N_1 = 0$  is satisfied for nodes 3, 9, 10, 4 if  $1 - \eta = 0$



**Fig. 5.24** Cubic serendipity family element

The points 5, 6, 7, 8, 9, 10, 11, 12 lie on the circle shown in Figure. The radius of this circle =  $OA$

$$= \sqrt{\left(\frac{1}{3}\right)^2 + 1} \text{ since } A\left(\frac{1}{3}, 1\right) \text{ and } O = (0, 0) = \sqrt{\frac{10}{3}}$$

∴ The equation of the circle is

$$\xi^2 + \eta^2 = \frac{10}{3}$$

$$\xi^2 + \eta^2 - \frac{10}{9} = 0$$

Satisfies  $N_1 = 0$  for nodes 5 to 12.

$$N_1 = C(1 - \xi)(1 - \eta) \left( \xi^2 + \eta^2 - \frac{10}{9} \right) \text{ Satisfies } N_1 = 0 \text{ for all nodes except for node 1.}$$

For node 1  $N_1 = 1$ .

$$\therefore 1 = (1 - \xi_1)(1 - \eta_1) \left( \xi_1^2 + \eta_1^2 - \frac{10}{9} \right)$$

But we know  $\xi_1 = \eta_1 = -1$

$$\therefore 1 = 2 \times 2 \times \left( 1 + 1 - \frac{10}{9} \right) C = \frac{32}{9} C$$

or

$$C = \frac{9}{32}$$

$$\begin{aligned} \therefore N_1 &= \frac{9}{32} (1 - \xi)(1 - \eta) \left( \xi^2 + \eta^2 - \frac{10}{9} \right) \\ &= \frac{1}{32} (1 - \xi)(1 - \eta) \{ 9(\xi^2 + \eta^2) - 10 \} \end{aligned}$$

Similarly it may be shown that

$$N_2 = \frac{1}{32} (1 + \xi)(1 - \eta) \{ 9(\xi^2 + \eta^2) - 10 \}$$

$$N_3 = \frac{1}{32} (1 + \xi)(1 + \eta) \{ 9(\xi^2 + \eta^2) - 10 \}$$

$$N_4 = \frac{1}{32} (1 - \xi)(1 + \eta) \{ 9(\xi^2 + \eta^2) - 10 \}$$

For mid side node 5,

$$1 - \xi = 0 \text{ ensures } N_5 = 0 \text{ at nodes 2, 7, 8, 3}$$

$$1 - \eta = 0 \text{ ensures } N_5 = 0 \text{ at nodes 3, 9, 10, 4}$$

$$1 + \xi = 0 \text{ ensures } N_5 = 0 \text{ at nodes 4, 11, 12, 1.}$$

$$1 - 3\xi = 0 \text{ ensures } N_5 = 0 \text{ at node 6.}$$

$$\therefore \text{ Let } N_5 = C(1 - \xi)(1 - \eta)(1 + \xi)(1 - 3\xi)$$

$$\text{At node 5, } \xi = -\frac{1}{3}, \eta = -1 \text{ and } N_5 = 1$$

$$\therefore 1 = C \times \frac{4}{3} \times 2 \times \frac{2}{3} \times 2$$

i.e.,

$$C = \frac{9}{32}$$

$$\therefore N_5 = \frac{9}{32}(1 - \xi)(1 - \eta)(1 + \xi)(1 - 3\xi)$$

$$= \frac{9}{32}(1 - \xi^2)(1 - \eta)(1 - 3\xi)$$

Similarly,

$$N_6 = \frac{9}{32}(1 - \xi^2)(1 - \eta)(1 + 3\xi)$$

$$N_7 = \frac{9}{32}(1 - \eta^2)(1 + \xi)(1 - 3\eta)$$

$$N_8 = \frac{9}{32}(1 - \eta^2)(1 + \xi)(1 + 3\eta)$$

$$N_9 = \frac{9}{32}(1 - \xi^2)(1 + \eta)(1 + 3\xi)$$

$$N_{10} = \frac{9}{32}(1 - \xi^2)(1 + \eta)(1 - 3\xi)$$

$$N_{11} = \frac{9}{32}(1 - \eta^2)(1 - \xi)(1 + 3\eta)$$

$$N_{12} = \frac{9}{32}(1 - \eta^2)(1 - \xi)(1 - 3\eta)$$

## 5.7 HERMITE POLYNOMIALS AS SHAPE FUNCTIONS

Similar to Lagrangian functions for  $C^0$  continuity elements, there are Hermite polynomials for  $C^n$  continuity elements. Hermite polynomial in one dimension is denoted as  $H^n(x)$ . It is a polynomial of order  $2n + 1$ . Thus  $H^1(x)$  is a first order polynomial which is cubic in  $x$ .  $H^2(x)$  is a second order polynomial and is of 5th order.

The speciality of Hermite polynomials is their values and the values of their derivatives upto order  $n$  are unity or zero at the end points of the interval 0 to 1. The elements of a set of Hermitian polynomials representing these properties may be written as  $H_{mi}^n(x)$  where  $m$  is the order of derivatives,  $i$  the node number and  $n$  is the order of the Hermitian function.

To illustrate it, let us take a two noded beam element (refer Fig. 5.25) in which nodal unknowns are the displacement  $w$  and the slope  $\frac{\partial w}{\partial x}$ . Since the element has four degrees of freedom, we have to select the polynomial with only 4 constants i.e. the first order (cubic) Hermitian polynomials as shape function.

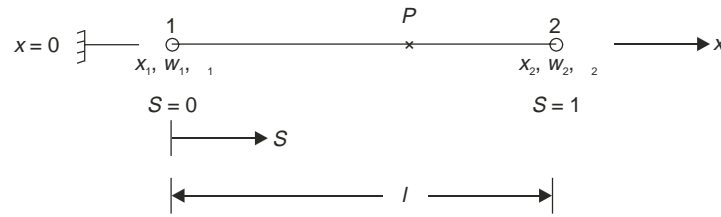


Fig. 5.25 Beam element with natural coordinated varying from zero to one

These are given as

$$H_{01}^1(s) = 1 - 3s^2 + 2s^3$$

$$H_{11}^1(s) = ls(s - 1)^2$$

$$H_{02}^1(s) = s^2(3 - 2s)$$

$$H_{12}^1(s) = ls^2(s - 1)$$

Where  $s = \frac{x - x_1}{x_2 - x_1} = \frac{x - x_1}{l}$  and  $l = x_2 - x_1$

The displacement model for the beam element is

$$w = [N] \{\delta\}_e$$

i.e.,

$$w = [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \quad \dots(5.47)$$

where

$$\theta = \frac{\partial w}{\partial x}$$

Then

$$\frac{\partial w}{\partial x} = \left[ \frac{\partial N_1}{\partial x} \quad \frac{\partial N_2}{\partial x} \quad \frac{\partial N_3}{\partial x} \quad \frac{\partial N_4}{\partial x} \right] \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

We select Hermitian first order polynomial functions as shape functions i.e.,

$$N_1 = H_{01}^1(s) = 1 - 3s^2 + 2s^3$$

$$N_2 = H_{11}^1(s) = ls(s - 1)^2 \quad \dots(5.48)$$

$$N_3 = H_{02}^1(s) = s^2(s - 2s)$$

$$N_4 = H_{12}^1(s) = ls^2(s - 1)$$

Now it may be observed that, at node 1,

$$N_1 = 1, N_2 = N_3 = N_4 = 0$$

and

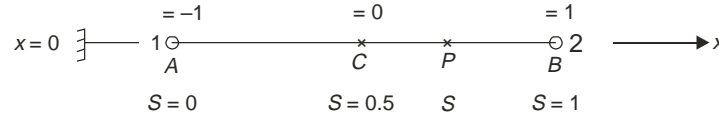
$$\frac{\partial N_1}{\partial s} = 0, \frac{\partial N_2}{\partial s} = 1, \frac{\partial N_3}{\partial s} = \frac{\partial N_4}{\partial s} = 0$$

Similarly at node 2,

$$N_1 = 0, N_2 = 0, N_3 = 1, N_4 = 0$$

$$\frac{\partial N_1}{\partial s} = \frac{\partial N_2}{\partial s} = \frac{\partial N_3}{\partial s} = 0 \text{ and } \frac{\partial N_4}{\partial s} = 1$$

This is exactly the requirement of the shape functions. Hence first order Hermite functions are suitable for beam element in which  $C^1$  continuity is to be satisfied. If we select natural coordinate as shown in Fig. 5.26, it may be observed that



**Fig. 5.26** Beam element with natural coordinates  $\xi$  varying from  $-1$  to  $1$

$$\xi = \frac{PC}{\frac{l}{2}} = \frac{2}{l} (AP - AC) = 2(s - 0.5) = 2s - 1$$

Substituting this in equation 5.47 we get, Hermite polynomial of first order as,

$$N_1 = H_{01}^1 = \frac{2 - 3\xi + \xi^3}{4}$$

$$N_2 = H_{11}^1 = \left(1 - \xi - \xi^2 + \xi^3\right) \frac{l}{8}$$

$$N_3 = H_{02}^1 = \frac{1}{4} (2 + 3\xi - \xi^3) \quad \dots(5.49)$$

$$N_4 = H_{12}^1 = \frac{l}{8} (-1 - \xi + \xi^2 + \xi^3)$$

Hermitian shape functions in two dimensions may be constructed as multiplication of Hermitian functions in  $x$  and  $y$  directions. Thus,

$$N_1 = H_{01}^1(x) H_{01}^1(y) \quad N_2 = H_{11}^1(x) H_{11}^1(y)$$

$$N_3 = H_{02}^1(x) H_{02}^1(y) \quad N_4 = H_{12}^1(x) H_{12}^1(y)$$

### 5.8 CONSTRUCTION OF SHAPE FUNCTIONS BY DEGRADING TECHNIQUE

In the problems in which stress concentration is quite high in some portion and its variation is low in some other portion, a stress analyst prefers to use higher order elements in the area of stress concentration and lower order elements in the other areas. Fig. 5.27 shows a case in which this has been done using LST elements and CST elements. In this problem left side portion is having stress variation fast. Hence LST elements are used while on the right hand side CST elements are used. When this is done we come across few elements which do not belong entirely to CST or LST category. In element No.5, we find there are five nodes while in element No.7, there are four nodes. For these odd noded elements, we can find shape function by degrading higher order (in this case LST) elements. This degrading technique is illustrated in this article.

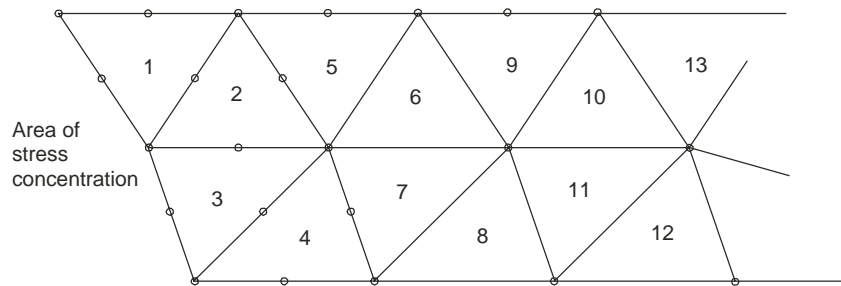


Fig. 5.27 CST and LST elements used in a problem

**(i) Degrading Six Noded Triangular Element to Five Noded Triangular Element**

Let the typical 6 noded and 5 noded triangular elements be as shown in Fig. 5.28. Five noded element is to be obtained by dropping node 6 from LST element. Now for LST element,

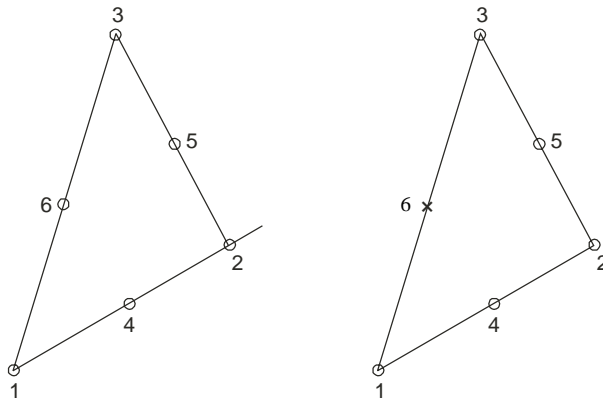


Fig. 5.28 Typical LST and 5-noded triangular elements

$$u = \sum N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 u_5 + N_6 u_6 \quad \dots(5.50)$$

where

$$N_1 = L_1 (2L_1 - 1) \quad N_2 = L_2 (2L_2 - 1) \quad N_3 = L_3 (2L_3 - 1)$$

$$N_4 = 4L_1L_2 \quad N_5 = 4L_2L_3 \quad N_6 = 4L_3L_1$$

In this case variation of displacement is quadratic along all the three sides of the triangle.

In case of five noded triangle, the variation along line 1–3 should be linear, since there are only two nodes along this line. Hence,

$$u'_6 = \frac{u_1 + u_3}{2}$$

Replacing  $u_6$  by  $u'_6$  in equation 5.46, we get

$$\begin{aligned} u &= N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6\left(\frac{u_1 + u_3}{2}\right) \\ &= \left(N_1 + \frac{N_6}{2}\right)u_1 + N_2u_2 + \left(N_3 + \frac{N_6}{2}\right)u_3 + N_4u_4 + N_5u_5 \end{aligned} \quad \dots(5.52a)$$

$$= \sum_{i=1}^5 N_i u_i, \text{ for five noded triangular element.}$$

$$= N'_1u_1 + N'_2u_2 + N'_3u_3 + N'_4u_4 + N'_5u_5 \quad \dots(5.52b)$$

comparing equation 5.52a and 5.52b, we conclude,

$$N'_1 = N_1 + \frac{N_6}{2} = L_1(2L_1 - 1) + \frac{4L_3L_1}{2} = L_1(2L_1 - 1 + 2L_3)$$

$$= L_1[-1 + 2(1 - L_2)] \text{ since } L_1 + L_2 + L_3 = 1 = L_1[1 - 2L_2]$$

$$N'_2 = N_2 = L_2(2L_2 - 1)$$

$$N'_3 = N_3 + \frac{N_6}{2} = L_3(2L_3 - 1) + \frac{4L_3L_1}{2} = L_3[2L_3 - 1 + 2L_1]$$

$$= L_3[-1 + 2(1 - L_2)] = L_3[1 - 2L_2]$$

$$N'_4 = N_4 = 4L_1L_2$$

$$N'_5 = N_5 = 4L_2L_3$$

### (ii) Degrading LST element to 4-Noded Triangular Element

Let the typical LST and 4-Noded triangular elements be as shown in Fig. 5.29.

In this case it is possible to get shape functions for 4-noded triangular element by degrading LST element from which nodes 5 and 6 are to be eliminated. If nodes 5 and 6 do not exist, the variation of displacements along lines 2-3 and 1-3 should be linear. Thus in 4 noded triangle

$$u'_6 = \frac{u_1 + u_3}{2} \text{ and } u'_5 = \frac{u_2 + u_3}{2}$$



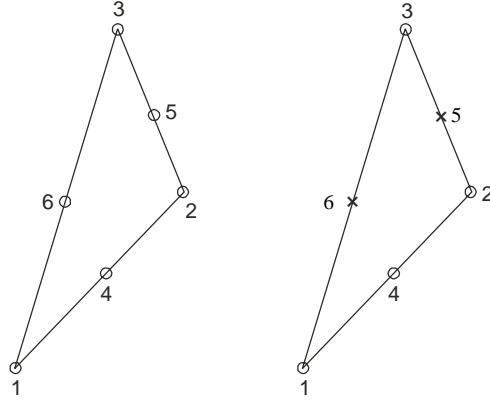


Fig. 5.29 Typical LST and 4 noded triangular element

Substituting these in equation 5.45, we get

$$\begin{aligned}
 u &= N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 \left( \frac{u_2 + u_3}{2} \right) + N_6 \left( \frac{u_1 + u_3}{2} \right) \\
 &= \left( N_1 + \frac{N_6}{2} \right) u_1 + \left( N_2 + \frac{N_5}{2} \right) u_2 + \left( N_3 + \frac{N_5}{2} + \frac{N_6}{2} \right) u_3 + N_4 u_4 \quad \dots(5.34a)
 \end{aligned}$$

But for 4 noded element,

$$u = N'_1 u_1 + N'_2 u_2 + N'_3 u_3 + N'_4 u_4 \quad \dots(5.34b)$$

Comparing 5.52a and 5.52b, we get,

$$\begin{aligned}
 N'_1 &= N_1 + \frac{N_6}{2} = L_1(2L_1 - 1) + \frac{4L_3L_1}{2} = L_1[2L_1 - 1 + 2L_3] \\
 &= L_1[-1 + 2(1 - L_2)], \text{ since } L_1 + L_2 + L_3 = 1 = L_1[1 - 2L_2] \\
 N'_2 &= N_2 + \frac{N_5}{2} = L_2(2L_2 - 1) + \frac{4L_2L_3}{2} = L_2[2L_2 - 1 + 2L_3] \\
 &= L_2[-1 + 2(1 - L_1)], \text{ since } L_1 + L_2 + L_3 = 1 = L_2[1 - 2L_1] \\
 N'_3 &= N_3 + \frac{N_5}{2} + \frac{N_6}{2} = L_3(2L_3 - 1) + \frac{4L_2L_3}{2} + \frac{4L_3L_1}{2} \\
 &= L_3[2L_3 - 1 + 2L_2 + 2L_1] \\
 &= L_3[-1 + 2] \text{ since } L_1 + L_2 + L_3 = 1 = L_3
 \end{aligned}$$

**Example 5.22:** By degrading technique develop shape function for the seven noded rectangular element shown in Fig. 5.30 (b). Given that for eight noded element shown in Fig. 5.30 (a)

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i)(\xi\xi_i + \eta\eta_i - 1) \text{ for nodes 1, 2, 3 and 4.}$$

$$N_i = \frac{1}{2}(1 - \xi^2)(1 + \eta\eta_i) \text{ for nodes 5 and 7}$$

$$N_i = \frac{1}{2}(1 - \eta^2)(1 + \xi\xi_i) \text{ for nodes 6 and 8}$$

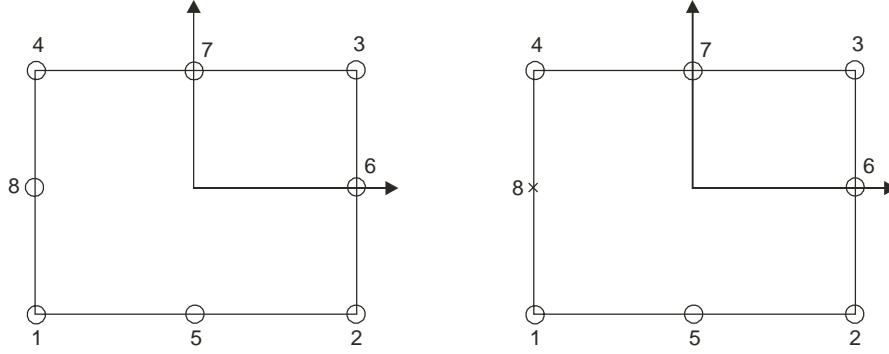


Fig. 5.30 (a) Eight noded element (b) Seven noded element

For eight noded element,

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1)$$

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)(\xi - \eta - 1)$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)(\xi + \eta - 1)$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1)$$

$$N_5 = \frac{1}{2}(1 - \xi^2)(1 - \eta)$$

$$N_6 = \frac{1}{2}(1 + \xi)(1 - \eta^2)$$

$$N_7 = \frac{1}{2}(1 - \xi^2)(1 + \eta)$$

$$N_8 = \frac{1}{2}(1 - \xi)(1 - \eta^2)$$

For seven noded element, variation of displacement along the edge 1–4 is linear. Hence

$$u_8 = \frac{u_1 + u_4}{2}$$

$$\begin{aligned} \therefore u &= N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 u_5 + N_6 u_6 + N_7 u_7 + N_8 \left( \frac{u_1 + u_4}{2} \right) \\ &= \left( N_1 + \frac{N_8}{2} \right) u_1 + N_2 u_2 + N_3 u_3 + \left( N_4 + \frac{N_8}{2} \right) u_4 + N_5 u_5 + N_6 u_6 + N_7 u_7 \quad \dots(5.54a) \end{aligned}$$

If the shape function for 7 noded element is  $N'$ , then,

$$u = N'_1 u_1 + N'_2 u_2 + N'_3 u_3 + N'_4 u_4 + N'_5 u_5 + N'_6 u_6 + N'_7 u_7 \quad \dots(5.54b)$$

comparing equation 5.54a and 5.54b, we conclude

$$\begin{aligned} N'_1 &= N_1 + \frac{N_8}{2} = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1) + \frac{1}{4}(1 - \xi)(1 - \eta^2) \\ &= \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1 + 1 + \eta) = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi) \\ N'_2 &= N_2; \quad N'_3 = N_3 \\ N'_4 &= N_4 + \frac{N_8}{2} = \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1) + \frac{1}{4}(1 - \xi)(1 - \eta^2) \\ &= \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1 + 1 - \eta) \quad \dots(5.55) \\ &= \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi) \\ N'_5 &= N_5, \quad N'_6 = N_6, \quad N'_7 = N_7. \end{aligned}$$

## QUESTIONS

1. Explain the term 'Shape Functions'. Why polynomial terms are preferred for shape functions in finite element method ?
2. State and explain the convergence requirements of polynomial shape functions.
3. Explain the term 'geometric isotropy / geometric Invariance'. Why polynomial shape functions should satisfy these requirement ? How do you check a polynomial for this requirement ?
4. Determine the shape function for a two noded bar element using
  - (i) Cartesian coordinate system
  - (ii) Local coordinate system ranging from 1 to zero.
  - (iii) Local coordinate system with range -1 to 1.
5. Determine the shape function for a three noded bar element
  - (i) using polynomial form in local coordinates
  - (ii) using Lagrangian functions.
 Plot their shapes.

6. Using generalized coordinate approach, determine shape functions for a two noded beam element and apply necessary checks.
7. Determine the shape functions for a CST element. Show that they are nothing but area coordinate
8. Explain the method of finding shape functions for LST element in terms of local coordinates  $L_1$ ,  $L_2$ , and  $L_3$ .
9. Derive shape functions for a rectangular element for plane stress / plane strain analysis starting with
  - (i) Polynomial form
  - (ii) Lagrange functionsApply the checks
10. Explain the method of finding shape function for 8 noded rectangular element to be used for plane stress/plane strain problems.
11. Using Lagrange functions determine the shape function for 9 noded rectangular element. Plot the variation of shape function of a typical corner node, a typical mid side node and the central node.
12. Explain the situations where you need 4 noded and 5 noded triangular elements. By degradation technique derive the shape functions for them. Apply necessary checks.
13. Write short note on Hermite polynomial.
14. Write short note on Lagrange functions.
15. Using Lagrange functions derive the shape functions for a hexahedron element.

# 6

## Strain Displacement Matrix

### 6.1 INTRODUCTION

In the previous chapter we saw the shape function  $[N]$  which established the relationship between displacement at any point in the element with nodal displacements of the element. In this chapter we establish the relationship between strain at any point in the element with nodal displacement. We define strain displacement relation as

$$\{\varepsilon\} = [B]_e \{\delta\}_e \quad \dots(6.1)$$

Where  $\{\varepsilon\}$  is strain at any point in the element.

$\{\delta\}_e$  is displacement vector of nodal values of the element

$[B]_e$  is strain displacement matrix

In this article strain displacement matrix will be found for few standard cases.

### 6.2 STRAIN-DISPLACEMENT MATRIX FOR BAR ELEMENTS

There is only one strain component in such element i.e.  $\varepsilon_x$ . Thus for bar elements [refer Fig. 6.1]

$$\begin{aligned} \{\varepsilon\} &= \varepsilon_x = \frac{\partial u}{\partial x} \\ &= \frac{\partial u}{\partial x} [N] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ &= \frac{\partial u}{\partial x} [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \end{aligned}$$

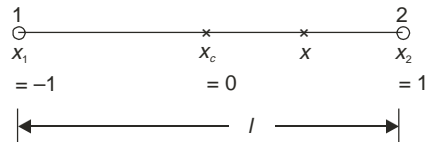


Fig. 6.1

$$\text{But } N_1 = \frac{x_2 - x}{l} \quad N_2 = \frac{x - x_1}{l} \quad (\text{from equation 5.12})$$

$$\therefore \frac{\partial N_1}{\partial x} = -\frac{1}{l} \quad \text{and} \quad \frac{\partial N_2}{\partial x} = \frac{1}{l}$$

$$\begin{aligned} \therefore \{\varepsilon\} = \varepsilon_x &= \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= [B] \{q\} \end{aligned}$$

$$\text{Thus} \quad [B] = \frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix} \quad \dots(6.2)$$

The above relationship may be derived using shape function in terms of natural coordinate ' $\xi$ ' also

$$\{\varepsilon\} = \varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial u}{\partial \xi} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} [N_1 \quad N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\text{where} \quad \xi = \frac{x - x_2}{l/2} = \frac{2}{l} [x - x_2] \quad (\text{from equation 4.5})$$

$$N_1 = \frac{1 - \xi}{2} \quad \text{and} \quad N_2 = \frac{1 + \xi}{2} \quad (\text{from equation 5.19})$$

$$\therefore \frac{\partial \xi}{\partial x} = -\frac{2}{l} \quad \frac{\partial N_1}{\partial \xi} = -\frac{1}{2} \quad \text{and} \quad \frac{\partial N_2}{\partial \xi} = \frac{1}{2}$$

$$\therefore \{\varepsilon\} = \varepsilon_x = \frac{2}{l} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix} \{\delta\}_e = [B] \{\delta\}_e$$

$$\text{where} \quad [B] = \frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

### 6.3 STRAIN DISPLACEMENT MATRIX FOR CST ELEMENT

At any point in the constant strain triangle element [refer Fig. 6.2] used for plane stress or plane strain problems, there are three strain components i.e.  $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ . They are given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

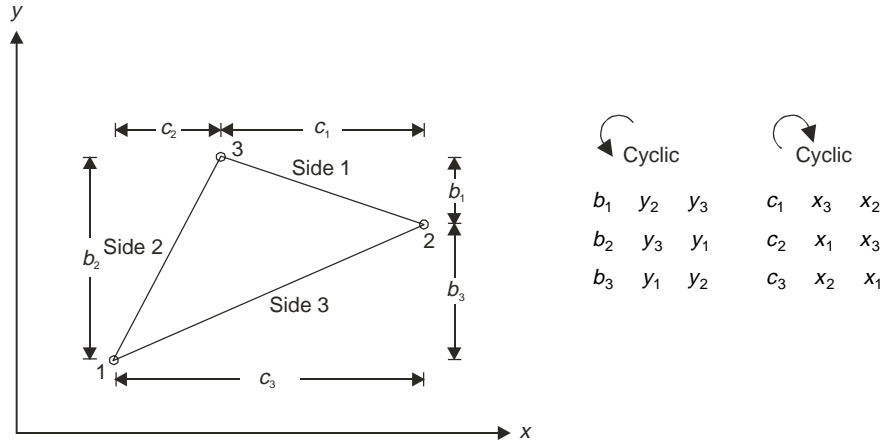


Fig. 6.2

The displacement vector is

$$\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\therefore \{\epsilon\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

But in these elements (from equation 5.15 and 5.16)

$$N_1 = \frac{a_1 + b_1x + c_1y}{2A} \quad N_2 = \frac{a_2 + b_2x + c_2y}{2A} \quad \text{and} \quad N_3 = \frac{a_3 + b_3x + c_3y}{2A}$$

$$\therefore \{\epsilon\} = \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & b_1 & b_2 & b_3 \end{bmatrix} \{\delta\}_e = [B] \{\delta\}_e$$

where

$$\therefore [B] = \frac{1}{2A} \begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & b_1 & b_2 & b_3 \end{bmatrix} \quad \dots(6.3)$$

where  $b_1 = y_2 - y_3$        $c_1 = x_3 - x_2$   
 $b_2 = y_3 - y_1$        $c_2 = x_1 - x_3$   
 $b_3 = y_1 - y_2$        $c_3 = x_2 - x_1$

Note  $b_1, b_2, b_3$  and  $c_1, c_2, c_3$  are in cyclic order and they are constant for the triangle. Hence the strains are constant within the element. Therefore the element is referred as constant strain triangle.

#### 6.4 STRAIN DISPLACEMENT MATRIX FOR A BEAM ELEMENT

We know from constitutive law

$$\{\sigma\} = [D] \{\varepsilon\}$$

From definition of strain displacement matrix

$$\{\varepsilon\} = [B] \{\delta\}_e \quad \dots(6.4)$$

$$\therefore \{\sigma\} = [D] [B] \{\delta\}_e$$

In case of beam, stress resultant is the moment, which is given by the expression,

$$M = EI \frac{\partial^2 v}{\partial x^2} \quad \dots(6.5)$$

where  $v$ -displacement in  $y$ -direction.

Treating the stress resultant  $M$  similar to the stress  $\sigma$  and curvature similar to strain,  $[D]$  and  $[\varepsilon]$  matrix are

$$[D] = EI \quad \dots(6.6)$$

and

$$\{\varepsilon\} = \frac{\partial^2 v}{\partial x^2} \quad \dots(6.7)$$

From equation 5.43, we have

$$v = [N] \{\delta\}_e$$

$$= [N_1 \quad N_2 \quad N_3 \quad N_4] \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

for the typical beam element shown in Fig. 6.3 and from equation 5.44,

$$N_1 = 1 - 3s^2 + 2s^3$$

$$N_2 = ls(s - 1)^2$$



$$N_3 = s^2(3 - 2s)$$

$$N_4 = ls^2(s - 1)$$

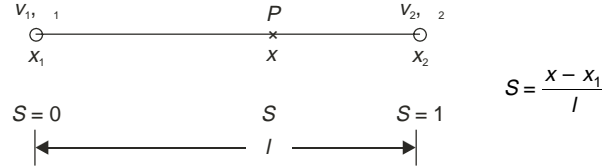


Fig. 6.3

where

$$s = \frac{x - x_1}{l}$$

$$\therefore \{\epsilon\} = \frac{\partial^2 v}{\partial x^2}$$

$$= \frac{\partial^2}{\partial x^2} [N] \{\delta\}_e = \left[ \frac{\partial^2 N_1}{\partial x^2} \quad \frac{\partial^2 N_2}{\partial x^2} \quad \frac{\partial^2 N_3}{\partial x^2} \quad \frac{\partial^2 N_4}{\partial x^2} \right] \{\delta\}_e = [B] \{\delta\}_e$$

$$\therefore [B] = \left[ \frac{\partial^2 N_1}{\partial x^2} \quad \frac{\partial^2 N_2}{\partial x^2} \quad \frac{\partial^2 N_3}{\partial x^2} \quad \frac{\partial^2 N_4}{\partial x^2} \right]$$

Now  $\frac{\partial N_i}{\partial x} = \frac{\partial s}{\partial x} \frac{\partial N_i}{\partial s} = \frac{1}{l} \frac{\partial N_i}{\partial s}$

$$\therefore \frac{\partial^2 N_i}{\partial x^2} = \left( \frac{1}{l} \frac{\partial}{\partial s} \right) \left( \frac{1}{l} \frac{\partial N_i}{\partial s} \right) = \frac{1}{l^2} \frac{\partial^2 N_i}{\partial s^2}$$

$$\therefore [B] = \frac{1}{l^2} \left[ \frac{\partial^2 N_1}{\partial s^2} \quad \frac{\partial^2 N_2}{\partial s^2} \quad \frac{\partial^2 N_3}{\partial s^2} \quad \frac{\partial^2 N_4}{\partial s^2} \right]$$

$$= \frac{1}{l^2} \left[ -6 + 12s \quad l(6s - 4) \quad 6 - 12s \quad l(6s - 2) \right] \quad \dots(6.8)$$

### QUESTIONS

1. Selecting interpolation function for a bar element in its natural coordinates, find the strain matrix.
2. Selecting interpolation function for a CST element in its natural coordinates, find the strain matrix.
3. Determine the strain matrix for a beam element. Use shape functions in natural coordinates form.
4. Determine the natural coordinates (area coordinates) for a CST element. Assemble strain vector.
5. For a CST element shown in Fig. 6.4. Obtain the strain–displacement matrix. Assume Possion’s ratio is zero and Young's modulus is constant.

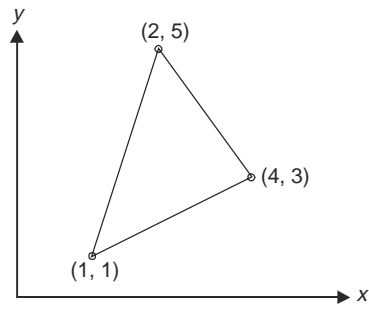


Fig. 6.4

# 7

## Assembling Stiffness Equations—Direct Approach

### 7.1 INTRODUCTION

In assembling the element stiffness equation,  $[k] \{\delta\} = \{F\}$ , the first step is to derive the expression for element stiffness property and nodal force vector. The first method developed in this field was direct approach. Latter on variational approach and Galerkin's weighted residual approaches developed. Direct approach is the extension of matrix displacement approach. For one dimensional elements this is exactly same as explained in Chapter 3. In fact no distinction was seen between matrix and finite element method. The analysts tried to extend the matrix method to two dimensional problems also. The only element in which it could be done successfully was the three noded triangular element (CST–element). This concept is explained in this chapter. Lumped mass concept of assembling the nodal force vector is also presented.

### 7.2 ELEMENT STIFFNESS MATRIX FOR CST ELEMENT BY DIRECT APPROACH

Turner was first to suggest it and that was real starting point of FEM. Consider the typical element shown in Fig. 7.1. It is subjected to constant stresses along its all the three edges. Let the constant stresses be  $\sigma_x, \sigma_y, \tau_{xy} = \tau_{yx}$ . Assembling stiffness matrix means finding nodal equivalent set of forces which are statically equivalent to the constant stress field acting at the edges of the elements.

The equivalent nodal forces to be found are  $F_1, F_2, F_3 \dots F_6$  as shown in Fig. 7.2. We have six unknown nodal forces, but only three equations of equilibrium. Hence it is not possible to determine  $F_1, F_2 \dots F_6$  in terms of  $\sigma_x, \sigma_y, \tau_{xy}$  mathematically. Turner resolved the uniform stress distribution into an equivalent force system at midsides as shown in Fig. 7.3. Note side  $i$  is the side opposite to node  $i$ . With this notation,

$$F_{m1x} = \sigma_x (y_3 - y_2)t + \tau_{xy} (x_2 - x_3)t$$

where  $t$  is the thickness of the element.

$$F_{m1y} = \sigma_y (x_2 - x_3)t + \tau_{xy} (y_3 - y_2)t$$

$$F_{m2x} = -\sigma_x (y_3 - y_1)t + \tau_{xy} (x_3 - x_1)t$$

$$F_{m2y} = \sigma_y (x_3 - x_1)t - \tau_{xy} (y_3 - y_1)t \quad \dots(7.1)$$

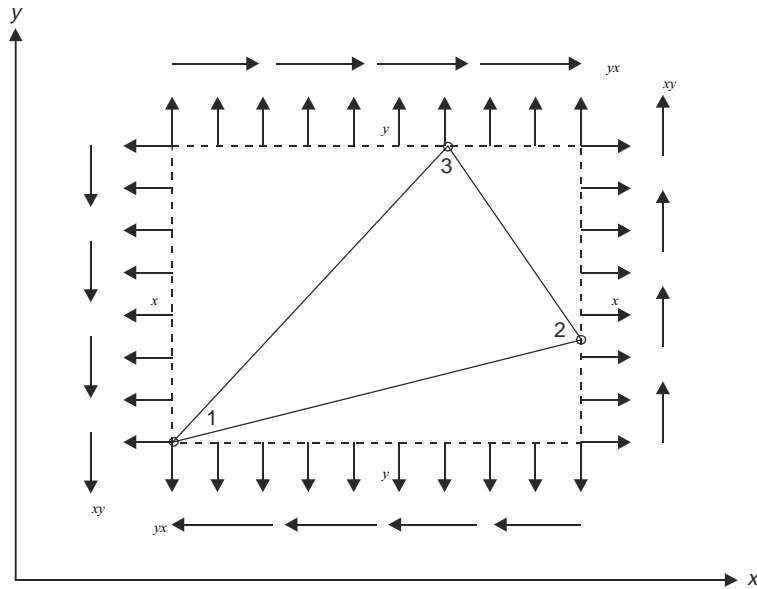


Fig. 7.1 Stress field in CST element

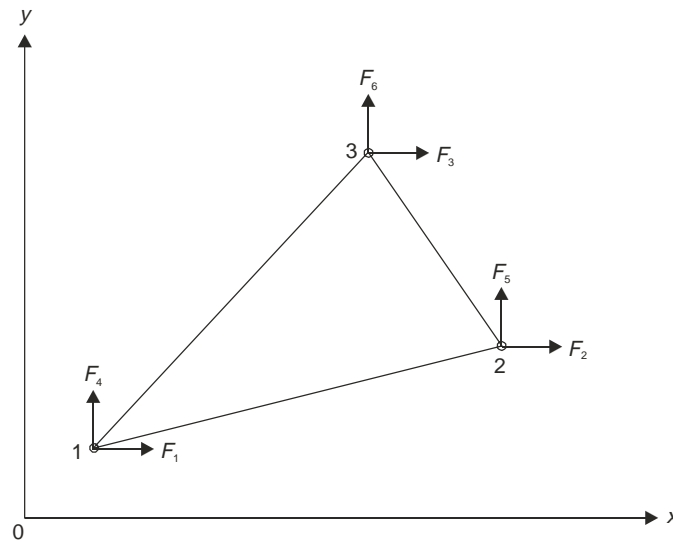
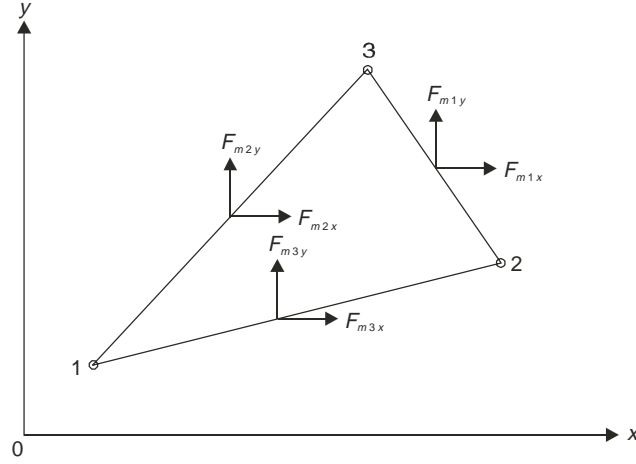


Fig. 7.2 Equivalent nodal forces

$$F_{m3x} = \sigma_x (y_2 - y_1)t - \tau_{xy} (x_2 - x_1)t$$

$$F_{m3y} = -\sigma_y (x_2 - x_1)t + \tau_{xy} (y_2 - y_1)t$$

After this Turner transferred half of mid side forces to nodes at the end of sides to get equivalent nodal forces. Thus he got


**Fig. 7.3** *Equivalent mid-side forces*

$$\begin{aligned}
 F_1 &= \frac{1}{2} (F_{m2x} + F_{m3x}) \\
 &= \frac{1}{2} [-\sigma_x (y_3 - y_1)t + \tau_{xy} (x_3 - x_1)t + \sigma_x (y_2 - y_1)t - \tau_{xy} (x_2 - x_1)t] \\
 &= \frac{t}{2} [\sigma_x (-y_3 + y_1 + y_2 - y_1) + \tau_{xy} (x_3 - x_1 - x_2 + x_1)] \\
 &= \frac{t}{2} [\sigma_x (y_2 - y_3) + \tau_{xy} (x_3 - x_2)] \\
 F_2 &= \frac{1}{2} (F_{m1x} + F_{m3x}) \\
 &= \frac{t}{2} [\sigma_x (y_3 - y_2) + \tau_{xy} (x_2 - x_3) + \sigma_x (y_2 - y_1) - \tau_{xy} (x_2 - x_1)] \\
 &= \frac{t}{2} [\sigma_x (y_3 - y_1) + \tau_{xy} (x_1 - x_3)] \\
 F_3 &= \frac{1}{2} (F_{m1y} + F_{m2y}) \\
 &= \frac{t}{2} [\sigma_x (y_3 - y_2) + \tau_{xy} (x_2 - x_3) - \sigma_x (y_3 - y_1) + \tau_{xy} (x_3 - x_1)] \\
 &= \frac{t}{2} [\sigma_x (y_1 - y_2) + \tau_{xy} (x_2 - x_1)]
 \end{aligned}$$

$$\begin{aligned}
 F_4 &= \frac{1}{2} (F_{m2y} + F_{m3y}) \\
 &= \frac{t}{2} [\sigma_y (x_3 - x_1) - \tau_{xy} (y_3 - y_1) - \sigma_y (x_2 - x_1) + \tau_{xy} (y_2 - y_1)] \\
 &= \frac{t}{2} [\sigma_y (x_3 - x_2) + \tau_{xy} (y_2 - y_3)] \\
 F_5 &= \frac{1}{2} (F_{m1y} + F_{m3y}) \\
 &= \frac{t}{2} [\sigma_y (x_2 - x_3) + \tau_{xy} (y_3 - y_2) - \sigma_y (x_2 - x_1) + \tau_{xy} (y_2 - y_1)] \\
 &= \frac{t}{2} [\sigma_y (x_1 - x_3) + \tau_{xy} (y_3 - y_1)] \\
 F_6 &= \frac{1}{2} (F_{m1y} + F_{m2y}) \\
 &= \frac{t}{2} [\sigma_y (x_2 - x_3) + \tau_{xy} (y_3 - y_2) + \sigma_y (x_3 - x_1) - \tau_{xy} (y_3 - y_1)] \\
 &= \frac{t}{2} [\sigma_y (x_2 - x_1) + \tau_{xy} (y_1 - y_2)]
 \end{aligned}$$

Thus the force vector as derived by Turner is

$$[F] = \frac{t}{2} \begin{bmatrix} b_1 & 0 & c_1 \\ b_2 & 0 & c_2 \\ b_3 & 0 & c_3 \\ 0 & c_1 & b_1 \\ 0 & c_2 & b_2 \\ 0 & c_3 & b_3 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \dots(7.2)$$

where  $b_1, b_2, b_3, c_1, c_2$  and  $c_3$  have the expressions as used while deriving shape function expressions.

$$\text{i.e.} \quad \begin{array}{lll} b_1 = y_2 - y_3 & b_2 = y_3 - y_1 & b_3 = y_1 - y_2 \\ c_1 = x_3 - x_2 & c_2 = x_1 - x_3 & c_3 = x_2 - x_1 \end{array}$$

$$\text{But,} \quad \begin{bmatrix} b_1 & 0 & c_1 \\ b_2 & 0 & c_2 \\ b_3 & 0 & c_3 \\ 0 & c_1 & b_1 \\ 0 & c_2 & b_2 \\ 0 & c_3 & b_3 \end{bmatrix} = 2A[B]^T \quad \dots(7.3)$$

and

$$\{\sigma\} = [D] \{\varepsilon\} = [D][B]\{\delta\}_e$$

We have got

$$\begin{aligned} \{F\} &= \frac{t}{2} 2A [B]^T [D][B]\{\delta\}_e = [B]^T [D][B] t A \{\delta\}_e \quad \dots(7.4) \\ &= [B]^T [D][B] V \{\delta\}_e \end{aligned}$$

where  $V$  is the volume

$$\therefore \{F\} = [k] \{\delta\}_e$$

where

$$\{k\} = [B]^T [D][B] V = \int [B]^T [D][B] dV, \quad \dots(7.5)$$

since  $[B]$  and  $[D]$  are constants.

### 7.3 NODAL LOADS BY DIRECT APPROACH

In the stiffness equation  $[k] \{\delta\} = \{F\}$ , the right hand side term  $\{F\}$  refers to the nodal forces. Generally, while subdividing a structure, nodal locations are selected so as to coincide with the external forces applied. This can be easily done in case of concentrated loads acting on the structure. But in case of distributed loads like self weight, uniformly distributed load, uniformly varying load, we need a technique of transferring the load as nodal loads. There are two procedures for it, namely direct procedure and variational approach. In this chapter the direct procedure is dealt.

This procedure was first to be used in the finite element method. In this procedure classical structural analysis background is utilized or a portion is assigned to each node and load on that region as the nodal load. The latter method is called as **lumped load method**.

Consider the self weight of the uniform bar element shown in Fig. 7.4. Half the bar length may be

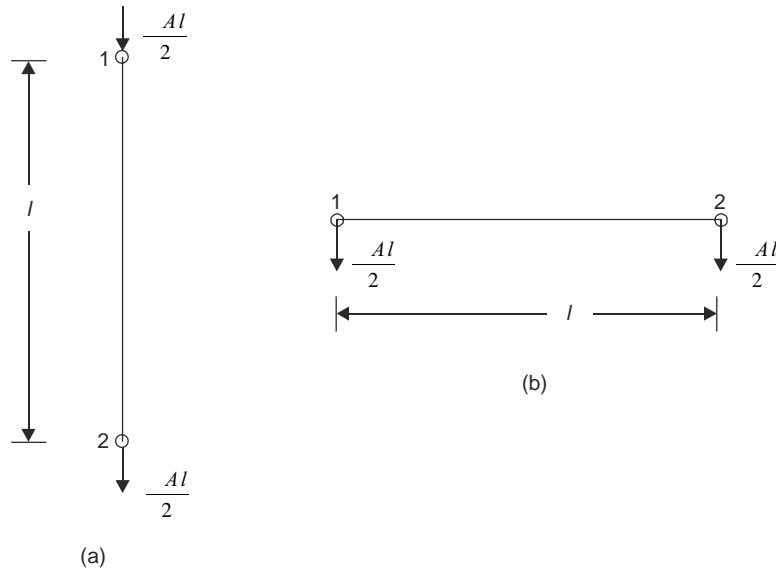


Fig. 7.4

assumed to contribute to each node. Hence at each node vertical downward load is

$$= \frac{\rho A l}{2}$$

Where  $\rho$  is unit weight of the material,  $A$  the cross sectional area and  $l$  is the length of the element.

In case of beam element subject to uniformly distributed load (refer Fig. 7.5(a);

(i) **Lumped load procedure:** half the region is assigned to each node as shown in Fig. 7.5(b). Its

equivalent  $\frac{wl}{2}$  is taken at the centre of gravity of the element as shown in Fig. 7.5(c). Then at the

node, the forces are  $\frac{wl}{2}$  and  $\frac{wl^2}{8}$ . Hence Lumped load vector is

$$\{F\}^T = \begin{bmatrix} \frac{wl}{2} & \frac{wl^2}{8} & \frac{wl}{2} & -\frac{wl^2}{8} \end{bmatrix}$$

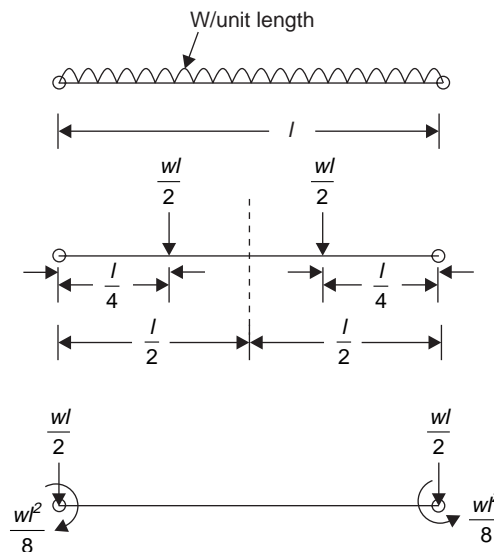


Fig. 7.5

(ii) **Classical Structural Analysis Approach:** In case of beam the end reactions, for a fixed beam [shown in Fig. 7.6.(a)] are

$$\frac{wl}{2} \text{ and } \frac{wl^2}{12}$$

Hence the equivalent nodal loads are

$$\begin{bmatrix} -\frac{wl}{2} & \frac{wl^2}{12} & -\frac{wl}{2} & -\frac{wl^2}{12} \end{bmatrix}$$



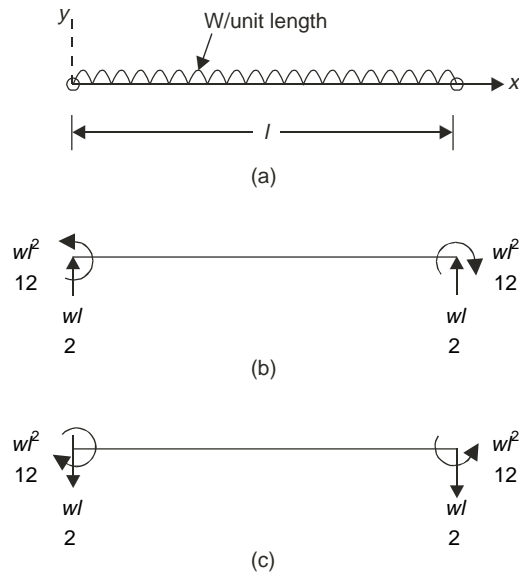


Fig. 7.6

(iii) In case of a CST element  $\frac{1}{3}$ rd area may be assigned to each node and hence equivalent nodal force is  $\frac{1}{3}$ rd the self weight as shown in Fig. 7.7

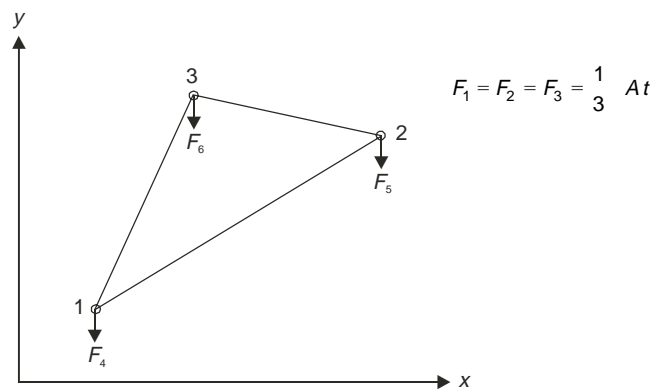


Fig. 7.7

For complex loading and elements this method may not be of much use. The distribution obtained in lumped load approach may be one of the possible distribution. We cannot say confidently that the distribution considered is exact.

Because of difficulties and uncertainties associated with direct approach not much progress could be made with direct approach. The real break through was found in finite element analysis only when variational approach was discovered.

### QUESTIONS

1. Derive stiffness matrix for a CST element by direct approach.
2. Differentiate between the terms 'lumped loads' and consistent loads.

# 8

## Assembling Stiffness Equations—Galerkin's Method, Virtual Work Method

### 8.1 INTRODUCTION

Finite Element Analysis is a versatile method which is used for solving a set of differential equations specified over a region, the solution satisfying specified boundary conditions. In the solid mechanics we try to get the displacements in a structure by solving the equations of equilibrium specified over the structure and the displacements obtained are such that the specified support conditions and the values of loads are satisfied. Galerkin has given a method for solving such differential equations which can be used by the analysts of solid mechanics, fluid mechanics, heat flow, electrical engineering. Galerkin's general method is briefly explained first and then its application in elasticity is presented. It will be found that in elasticity problems this method turns out to be principle of virtual work.

### 8.2 GALERKIN'S METHOD

Let the governing differential equation on a specified region  $V$  be

$$L(u) = P \quad \dots(8.1)$$

Where  $L$  is a differential operator on a basic unknown  $u$ . The value of  $u$  to be found has to satisfy specified values on the boundary of the region. If we take  $\bar{u}$  as approximate solution, then we may get error  $\varepsilon(x)$  at a point  $x$  and

$$\varepsilon(x) = L(\bar{u}) - P \quad \dots(8.2)$$

$\varepsilon(x)$  is called residual at point  $x$ . The solution sought revolve around setting the residual, relative to a weighting function  $w_i$  to zero i.e. to get

$$\oint_V w_i (L\bar{u} - P) dV = 0 \text{ for } i = 1 \text{ to } n \quad \dots(8.3)$$

Depending upon the selection of weighting function, there can be different approaches. In the Galerkin's method the equation 8.3 is taken as,

$$\oint_V \psi (L\bar{u} - P) = 0 \quad \dots(8.4)$$

where  $\psi$  is also chosen from the basis function used for constructing approximate solution function  $\bar{u}$ . Let

$$\bar{u} = \sum_{i=1}^n Q_i G_i \quad \dots(8.5)$$

where  $Q_i$  is the basic unknown vector and  $G_i$  are basis functions.  $G_i$  are usually polynomial in space coordinates  $x, y, z$ . Then in Galerkin's method the weighting function  $\phi$  is taken as

$$\phi = \sum_{i=1}^n \psi_i G_i \quad \dots(8.6)$$

In the above equation  $\phi_i$  are arbitrary, except at the points where boundary conditions are satisfied. Since  $\psi$  is constructed similar to that as  $\bar{u}$ , Galerkin's method leads to simplified method. Thus in Galerkin's method we choose basis function  $G_i$  and determine  $\phi_i$  in  $\bar{u} = \sum_{i=1}^n Q_i G_i$  to satisfy  $\oint_v (L\bar{u}) - PdV = 0$  where coefficient  $\phi_i$  are arbitrary except at specified boundaries.

### 8.3 GALERKIN'S METHOD APPLIED TO ELASTICITY PROBLEMS

In case of three dimensional problems in elasticity the equations of equilibrium are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_b = 0 \quad \dots(8.7) \text{ (Refer 2.2)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y_b = 0$$

and

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z_b = 0$$

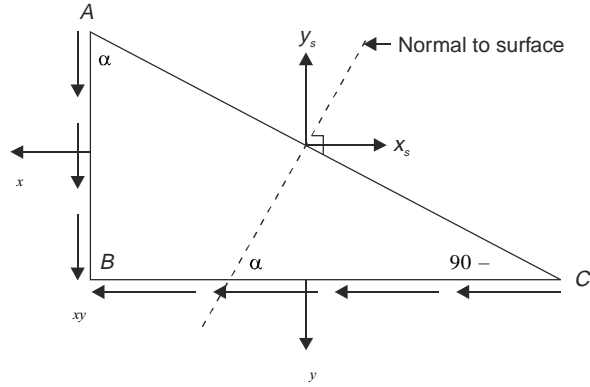
The stresses  $\{\sigma\}$  are in terms of displacements  $\{u\}$ . The displacements are arbitrary in the entire region of structure except at specified boundaries and the solution has to see that specified load values are obtained, wherever loading is existing. For simplicity we take two dimensional problem in elasticity and then extend the results to three dimensional problems. In two dimensional elasticity, the equations of equilibrium is

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X_b = 0$$

and

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y_b = 0 \quad \dots(8.8)$$

If  $X_s$  and  $Y_s$  are the surface forces and  $\alpha$  is the angle made by normal to surface with  $x$ -axis (refer Fig. 8.1), then from the equilibrium of the element we find,


**Fig. 8.1** *Two dimensionally stressed element*

$$X_s ACt = \sigma_x ABt + \tau_{xy} BCt$$

i.e.

$$\begin{aligned} X_s &= \sigma_x \frac{AB}{AC} + \tau_{xy} \frac{BC}{AC} \\ &= \sigma_x \cos \alpha + \tau_{xy} \sin \alpha = \sigma_x l + \tau_{xy} m \end{aligned}$$

where  $l$  and  $m$  are the direction cosines of normal to the surface.

Similarly, considering the equilibrium of the forces on the element in  $y$  direction, we get

$$Y_s = \tau_{xy} l + \sigma_y m$$

Thus the surfaces forces on the element are

$$X_s = \sigma_x l + \tau_{xy} m$$

and

$$Y_s = \tau_{xy} l + \sigma_y m \quad \dots(8.9)$$

Now consider the integral

$$\iint \left[ \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X_b \right) \delta u + \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y_b \right) \delta v \right] dx dy$$

where  $\delta u$  and  $\delta v$  are the elemental displacements in  $x$  and  $y$  directions. The above bounded integral is zero,

since  $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X_b = 0$  and  $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y_b = 0$  (from equation 8.8). Thus,

$$\iint \left[ \left( \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X_b \right) \delta u + \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y_b \right) \delta v \right] dx dy = 0 \quad \dots(8.10)$$

Now we can expand the integral 8.10 using Green's theorem. According to Green's theorem, if  $\phi(x, y)$  and  $\psi(x, y)$  are continuous functions then their first and second partial derivatives also continuous,

$$\iint \left[ \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} \right] dx dy = - \iint \phi \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right] dx dy + \int \phi \left[ \frac{\partial \psi}{\partial x} l + \frac{\partial \psi}{\partial y} m \right] ds \quad \dots(8.11)$$

Now let us consider the integration of the first term in equation 8.11

$$\iint \frac{\partial \sigma_x}{\partial x} \delta u \quad dx dy$$

Assuming  $\phi = \sigma_x \frac{\partial \psi}{\partial x} = \delta u$  and  $\frac{\partial \psi}{\partial y} = 0$ , from Greens function we get

$$\iint \frac{\partial \sigma_x}{\partial x} \delta u \quad dx dy = - \iint \sigma_x \frac{\partial(\delta u)}{\partial x} dx dy + \int \sigma_x l \delta u ds$$

Similarly  $\iint \frac{\partial \sigma_y}{\partial y} \delta u \quad dx dy$  can be found by taking

$$\phi = \sigma_y, \quad \frac{\partial \phi}{\partial x} = 0 \quad \frac{\partial \phi}{\partial y} = \delta v,$$

It leads to

$$\iint \frac{\partial \sigma_y}{\partial y} \delta u \quad dx dy = - \iint \sigma_y \frac{\partial}{\partial y} (\delta u) dx dy + \int \sigma_y m \delta v ds$$

on the same lines, we can get,

$$\iint \frac{\partial \tau_{xy}}{\partial x} \delta v \quad dx dy = - \iint \tau_{xy} \frac{\partial}{\partial x} (\delta v) dx dy + \int \tau_{xy} l \delta v ds$$

and

$$\iint \frac{\partial \tau_{xy}}{\partial y} \delta u \quad dx dy = - \iint \tau_{xy} \frac{\partial}{\partial y} (\delta u) + \int \tau_{xy} m \delta u ds$$

Hence equation 8.10 is equal to

$$\begin{aligned} & - \iint \left[ \sigma_x \frac{\partial}{\partial x} (\delta u) + \sigma_y \frac{\partial}{\partial y} (\delta v) + \tau_{xy} \frac{\partial}{\partial x} (\delta v) + \tau_{xy} \frac{\partial}{\partial y} (\delta u) \right] dx dy \\ & + \int \left[ \sigma_x l \delta u + \sigma_y m \delta v + \tau_{xy} l \delta v + \tau_{xy} m \delta u \right] ds \\ & + \iint X_b \delta u \quad dx dy + \iint Y_b \delta v \quad dx dy = 0 \end{aligned}$$

Regrouping the terms, we get,

$$\begin{aligned} & - \iint \left\{ \sigma_x \frac{\partial}{\partial x} (\delta u) + \sigma_y \frac{\partial}{\partial y} (\delta v) + \tau_{xy} \frac{\partial}{\partial x} (\delta v) + \frac{\partial}{\partial y} (\delta u) \right\} dx dy \\ & + \iint (X_b \delta u + Y_b \delta v) dx dy + \int (\sigma_x l + \tau_{xy} m) \delta u \quad ds + \int (\tau_{xy} l + \sigma_y m) \delta v \quad ds = 0 \quad \dots(8.12) \end{aligned}$$

Now consider the term

$$\iint (X_b \delta u + Y_b \delta v) dx dy \text{ in equation 8.12}$$

$X_b$  is the body force in  $x$  direction and  $\delta u$  is arbitrary (virtual) displacement in  $x$  direction. Similarly  $Y_b$  and  $\delta v$  are the body force and virtual displacements in  $y$  directions. Hence the above term represents virtual work done by the body forces.

Now consider the term,

$$\int [(\sigma_x l + \tau_{xy} m) \delta u + (\tau_{xy} l + \sigma_y m) \delta v] ds$$

from equation 8.9, we have  $X_s = \sigma_x l + \tau_{xy} m$

and

$$Y_s = \tau_{xy} l + \sigma_y m$$

Hence,

$$\begin{aligned} \int [(\sigma_x l + \tau_{xy} m) \delta u + (\tau_{xy} l + \sigma_y m) \delta v] ds \\ = \int [X_s \delta u + Y_s \delta v] ds \end{aligned} \quad \dots(8.13)$$

Thus the above expression represents the virtual work done by surface forces. Barring the first term the other terms in equation 8.12, represents the virtual work done by external forces due to virtual displacements  $\delta u$  and  $\delta v$ .

Now let us try to attach physical meaning to first term in equation 8.12. Consider the element of unit thickness shown in Fig. 8.2. Let a virtual displacement  $\delta u$  be given to the element. Dotted position shows the element with virtual displacement. Hence work done by  $\sigma_x$  stresses are

$$= \sigma_x dy \left[ \delta u + \frac{\partial}{\partial x} (\delta u) dx \right] - \sigma_x dy \delta u = \sigma_x \frac{\partial}{\partial x} (\delta u) dx dy \quad \dots(8.14)$$

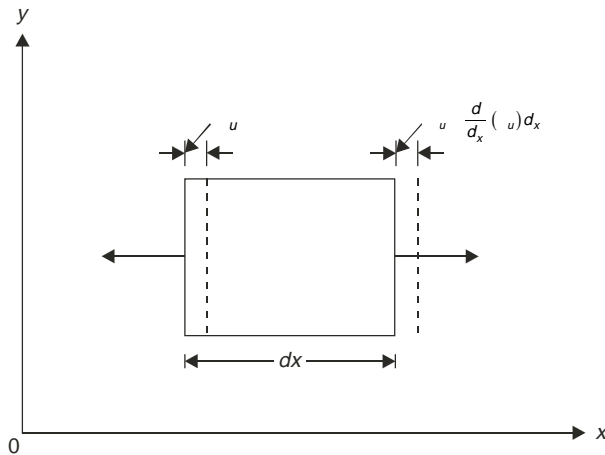


Fig. 8.2 Element with virtual displacement in  $n$ -direction

It may be noted that due to virtual displacement  $\delta u$  the change in strain  $\delta \varepsilon_x$  is given by

$$\delta \varepsilon_x = \frac{\left[ \delta u + \frac{\partial}{\partial x} (\delta u) dx \right] - \delta u}{dx} = \frac{\partial}{\partial x} (\delta u)$$

Substituting it in equation 8.14, we get virtual work done by

$$\sigma_x \text{ stresses} = \sigma_x \delta \varepsilon_x dx dy \quad \dots(8.15)$$

Similarly it can be shown that the terms

$$\iint \sigma_y \frac{\partial}{\partial y} (\delta v) dx dy \quad \text{and} \quad \iint \tau_{xy} \left\{ \frac{\partial}{\partial y} (\delta v) + \frac{\partial}{\partial x} (\delta u) \right\} dx dy$$

i.e. 
$$\iint \sigma_y \delta \varepsilon_y dx dy \quad \text{and} \quad \iint \tau_{xy} \delta \gamma_{xy} dx dy,$$

represent the work done by  $\sigma_y$  and  $\tau_{xy}$  stresses.

$\therefore$  First term in equation 8.12 is

$$\delta U = \iint (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy}) dx dy \quad \dots(8.16)$$

Thus the first term represents the work done by internal forces with negative sign. Therefore equation 8.12 may be looked as,

$$\begin{aligned} -\delta U + \delta W_e &= 0 \\ \delta U &= \delta W_e \end{aligned} \quad \dots(8.17)$$

where  $U$  is internal work done and  $W_e$  external work done. Thus in elasticity problems Galerkin's method turns out to be the **principle of virtual work which may be stated as a deformable body is in equilibrium when the total work done by external forces is equal to the total work done by internal forces**. The work done considered in the above derivation is called virtual, since the forces and deformations considered are not related. The displacements ' $\delta u$ ' and ' $\delta v$ ' are arbitrary. It may also be noted that the principle of virtual work is independent of the material properties.

### Extention to Three Dimensional Problems

The principal of virtual work holds good for the three dimensional problems also. In this case

$$\delta W_e = \iiint_v (X_b \delta u + Y_b \delta v + Z_b \delta w) dV + \iint_s (X_s \delta u + Y_s \delta v + Z_s \delta w) ds \quad \dots(8.18)$$

where  $s$  is the surface on which forces are acting. The above expression in the matrix form is,

$$\delta W_e = \iiint_v \{\delta u\}^T \{X\} dV + \iint_s \{\delta u\}^T \{p\} ds \quad \dots(8.19)$$

where

$$\{\delta u\}^T = [\delta u \quad \delta v \quad \delta w]$$



$$\{X\}^T = [X_b \ Y_b \ Z_b]$$

and

$$\{p\}^T = [X_s \ Y_s \ Z_s]$$

In three dimensional case, internal work expression is,

$$\delta U = \iiint_v (\sigma_x \delta \epsilon_x + \sigma_y \delta \epsilon_y + \sigma_z \delta \epsilon_z + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{xz} \delta \gamma_{xz}) dV$$

or

$$\delta U = \iiint_v \{\delta \epsilon\}^T \{\sigma\} \dots(8.20)$$

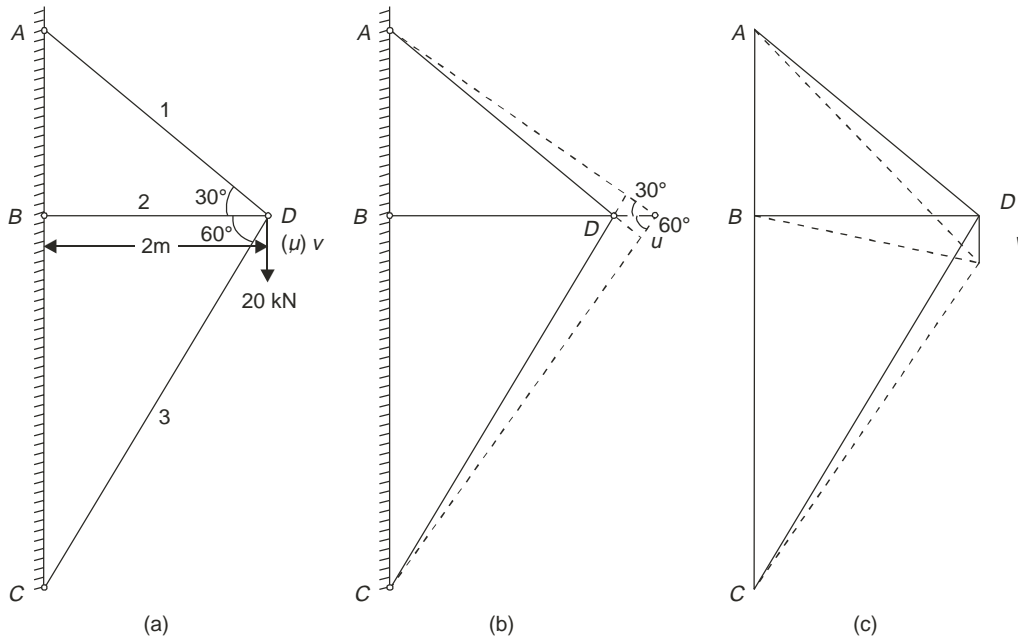
where  $\{\delta \epsilon\}^T = [\delta \epsilon_x \ \delta \epsilon_y \ \delta \epsilon_z \ \gamma_{xy} \ \gamma_{yz} \ \gamma_{xz}]$

and  $\{\sigma\}^T = [\sigma_x \ \sigma_y \ \sigma_z \ \tau_{xy} \ \tau_{yz} \ \tau_{zx}]$

From principal of virtual work,

$$\delta U = \delta W_e$$

**Example 8.1:** Using virtual displacement principle, determine the forces developed in the three bar truss shown in Fig. 8.3 (a).



**Fig. 8.3**

**Solution:** Fig. 8.3 (a) shows the given truss. The length of various members are

$$l_1 = \frac{2}{\cos 30} = 2.3094m \quad l_2 = 2m \quad l_3 = \frac{2}{\cos 60} = 4m$$

Total displacement of point  $D$  is  $u$  in  $x$  direction and  $v$  in  $y$  direction. The displacements  $u$  and  $v$  are separated and shown in Figs 8.2 (b) and 8.3 (c) respectively. From it we can see that total strains in the various members are

$$\varepsilon_1 = \frac{u \cos 30 + v \sin 30}{2.3094} = 0.375u + 0.2165v \quad \dots(a)$$

$$\varepsilon_2 = \frac{u}{2} = 0.5u$$

$$\varepsilon_3 = \frac{u \cos 60 - v \sin 30}{4} = 0.125u - 0.2165v$$

Let a unit virtual displacement be given in  $x$  direction as shown in Fig. 8.4. Then due to  $\delta u = 1$ , the strains introduced are,

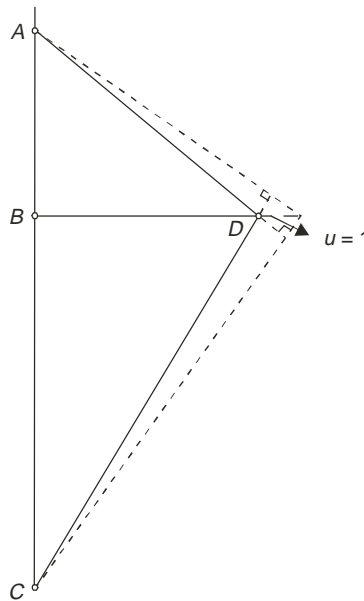


Fig. 8.4

$$\delta\varepsilon'_1 = \frac{1 \times \cos 30}{2.3094} = 0.375$$

$$\delta\varepsilon'_2 = \frac{1}{2} = 0.5$$

$$\delta\varepsilon'_3 = \frac{1 \times \cos 60}{4} = 0.125$$

∴ Internal work done in a member when unit virtual displacement is introduced in x direction is

$$\begin{aligned} \delta U &= \iiint \sigma_x \delta \varepsilon'_x dV \\ &= EAL \varepsilon_x \delta \varepsilon'_x \text{ since } \sigma_x = E\varepsilon_x \text{ and } V = AL \end{aligned}$$

Internal work done in the truss

$$\begin{aligned} &= \delta U_1 + \delta U_2 + \delta U_3 \\ &= EA [2.3094 (0.375u + 0.2165v) 0.375 + 2 \times 0.54 \times 0.5 + 4 (0.125u - 0.2165v) 0.125] \\ &= EA (0.88726u - 0.07924v) \end{aligned}$$

work done by load  $\delta W_e = 0$ , since no displacement in y direction. Equating internal work to external work, we get

$$\begin{aligned} 0.88726u &= 0.07924v \\ \therefore u &= 0.0893v \end{aligned} \quad \dots(b)$$

Now consider virtual work principle when unit virtual displacement is given to point  $D$  in y direction  $\delta v = 1$ . Referring to Fig. 8.5, we can note virtual strains in the members as,

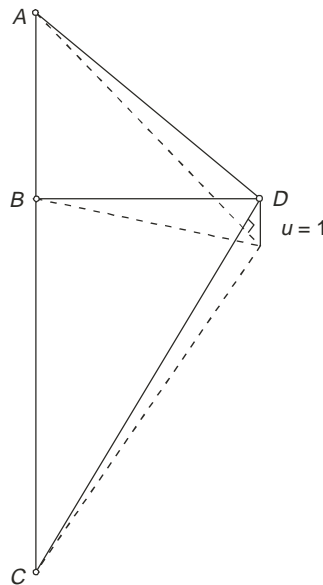


Fig. 8.5

$$\delta \varepsilon'_1 = \frac{1 \times \sin 30}{2.3094} = 0.2165$$

$$\delta \varepsilon'_2 = 0$$

$$\delta \varepsilon'_3 = -\frac{1 \times \sin 60}{4} = -0.1265$$

∴ Work done by internal forces

$$\begin{aligned}\delta U &= \delta U_1 + \delta U_2 + \delta U_3 \\ &= EA [2.3094 (0.375u + 0.2165v) 0.2165 + 0 + 4 (0.125u - 0.2165v) (-0.2165)] \\ &= EA (0.0792u + 0.2957v)\end{aligned}$$

From equation (b)  $u = 0.0893v$

$$\therefore \delta U = EA[0.3028v]$$

work done by external forces

$$\delta W_e = 20 \times 1 = 20$$

Equating internal work to external work, we get

$$0.3028v EA = 20$$

or

$$v = \frac{20}{0.3028EA} = \frac{66.048}{EA}$$

$$\text{Thus } v = \frac{66.048}{EA} \text{ and } u = 0.0893 v = \frac{5.898}{EA}$$

$$\therefore \varepsilon_1 = 0.375u + 0.2165v = \frac{26.045}{EA}$$

$$\varepsilon_2 = 0.5u = 0.5 \times \frac{5.898}{EA} = \frac{2.949}{EA}$$

$$\varepsilon_3 = 0.125u - 0.2165v = -\frac{13.562}{EA}$$

$$\therefore F_1 = A\sigma_x = AE\varepsilon_1 = 26.045kN$$

$$F_2 = AE\varepsilon_2 = 2.949kN \quad \text{Answer}$$

$$F_3 = -13.562kN$$

### Comments on Galerkin's Method

This is a method which can be applied to any problem involving solution of a set of equations subject to specified boundary values. In mechanics of solids it turns out to be virtual work method.

### QUESTIONS

1. Write short note on Galerkin's method.
2. Show that in elasticity problems Galerkin's method turns out to be the principle of virtual work.
3. State and explain the principle of virtual work.

# 9

## Assembling Stiffness Equations—Variational Method

### 9.1 INTRODUCTION

This method of assembling stiffness equations is widely used in elasticity problem. In calculus we know a function has extreme value when its first derivative with respect to variables is zero. The function is maximum, if the second derivative is negative and is minimum, if its second derivative is positive. The first derivative of function of a function is called first variance. The function of a function is termed a functional and the statement that the first variance of functional is zero is termed as first variance attains a stationary value. In many engineering problems there are such functional, the first variance of which attain stationary values. In elasticity problems potential energy of the body of the structure is such functional. In this chapter we will first study general mathematical method of variational approach, then assemble the expression for potential energy in a deformable solid and derive principle of minimum potential energy. Few simple problems are solved to explain the procedure. Then Raileigh–Ritz method is explained, which is useful for complex structural problems, encountered in finite element analysis. The general procedure of assembling stiffness matrix and load vector using the principle of minimum potential energy after expressing potential energy as a function with arbitrary constant using Raileigh – Ritz method is presented. It may be noted that variational approach is possible only if a suitable functional is available, otherwise the Galerkin’s method of weighted residual is to be used.

### 9.2 GENERAL VARIATIONAL METHOD IN ELASTICITY PROBLEMS

Let  $y$  be a function of  $x$ . Then,  $F(y, y', y'')$  is a functional. Say our concern is to find  $y = y(x)$ , such that the first variance of

$$I = \int_{x_1}^{x_2} F(y, y', y'') dx \quad \dots(9.1)$$

is made stationary satisfying the boundary conditions

$$y(x_1) = y_1$$

and

$$y(x_2) = y_2 \quad \dots(9.2)$$

Figure 9.1 shows a typical function  $y = y(x)$ . The continues line shows the exact function and dotted line shows an approximate function. Since to start with we don’t know exact solution  $y = y(x)$ , we have approximate solution  $\bar{y} = \bar{y}(x)$ .

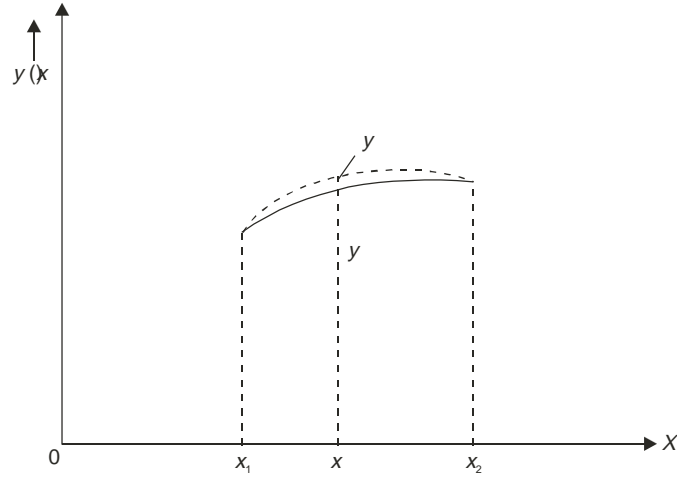


Fig. 9.1

Then, 
$$\delta_r = \bar{y}(x) - y(x). \quad \dots(9.3)$$

We are interested in finding the solution with

$$\delta I = 0 \quad \dots(9.4)$$

i.e. 
$$\int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' \right) dx = 0 \quad \dots(9.5)$$

Let us now find integration of second and third terms by parts.

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \delta y' dx = \left[ \frac{\partial F}{\partial y'} \delta y \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \delta y dx \quad \dots(9.6a)$$

and 
$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y''} \delta y'' dx = \left[ \frac{\partial F}{\partial y''} \delta y' \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \delta y' dx$$

$$= \left[ \frac{\partial F}{\partial y''} \delta y' \right]_{x_1}^{x_2} - \left[ \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \delta y \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) \delta y dx \quad \dots(9.6b)$$

Hence substituting equation 9.6 in equation 9.5, we get

$$\int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) \right) \delta y dx + \left[ \left\{ \frac{\partial F}{\partial y'} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \right\} \delta y \right]_{x_1}^{x_2} + \left[ \frac{\partial F}{\partial y''} \delta y' \right]_{x_1}^{x_2} = 0 \quad \dots(9.7)$$

Since  $\delta y$  is arbitrary, all the three terms in equation 9.7 should be zero. Thus we have

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) = 0 \quad \dots(9.8a)$$

$$\left[ \left\{ \frac{\partial F}{\partial y'} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \right\} \delta y \right]_{x_1}^{x_2} \quad \dots(9.8b)$$

and 
$$\left[ \frac{\partial F}{\partial y''} \delta y' \right]_{x_1}^{x_2} = 0 \quad \dots(9.8c)$$

Equation 9.8 (a) is known as **Euler-Lagrange** equation. Equations 9.8 (b) and 9.8 (c) are known as boundary conditions. To satisfy equation 9.8 (b).

$$\delta y(x_1) = 0 \quad \text{or} \quad \frac{\partial F}{\partial y'} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \Big|_{at \ x_1} = 0$$

Similarly at boundary  $x = x_2$ ,

$$\delta y(x_2) = 0 \quad \text{or} \quad \frac{\partial F}{\partial y'} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \Big|_{at \ x_2} = 0$$

To satisfy equation 9.8 (c),

$$\delta y'(x_1) = 0 \quad \text{or} \quad \frac{\partial F}{\partial y''} \Big|_{at \ x_1} = 0$$

and at  $x = x_2$ ,

$$\delta y'(x_2) = 0 \quad \text{or} \quad \frac{\partial F}{\partial y''} \Big|_{at \ x_2} = 0$$

The conditions like

$$\delta y(x_1) = 0, \delta y(x_2) = 0$$

$$\delta y'(x_1) = 0, \text{ and } \delta y'(x_2) = 0 \quad \dots(9.9)$$

are known as kinematic boundary conditions. In solid mechanics they specify displacement requirement at support points.

The conditions like

$$\frac{\partial F}{\partial y'} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \Big|_{at \ x_1} = 0$$

$$\frac{\partial F}{\partial y'} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \Big|_{at \ x_2} = 0$$

$$\frac{\partial F}{\partial y''} /_{x=x_1} = 0 \text{ and } \frac{\partial F}{\partial y''} /_{at x_2} = 0$$

are known as natural boundary conditions. In solid mechanics, they specify force conditions like moment and shear force conditions at supports.

Thus the first variance of the functional for stationary value yields Euler–Lagrange equation [equation 9.8(a)], kinematics boundary conditions [equation. 9.9(a)] and natural boundary conditions [equation. 9.9(b)]. A finite element analyst who is interested in solving a set of equations subject to a set of boundary values aim at first identifying the functional which satisfy Euler–Lagrange equation and satisfies boundary values specified. Then tries to solve Euler–Lagrange equation.

In solid mechanics it has been identified that total potential energy is suitable functional, the first variance of which yields equation of equilibrium satisfying the boundary conditions. This statement is verified with the following simple problems in solid mechanics.

**Example 9.1:** Show that the condition that first variance of total potential energy is stationary, is equivalent to satisfying equilibrium equation and boundary conditions in case of

- (i) Simply supported beam subjected to udl.
- (ii) Cantilever beam subjected to udl.

**Solution:** Figs 9.2 (a) and 9.2 (b) show the typical beams considered.

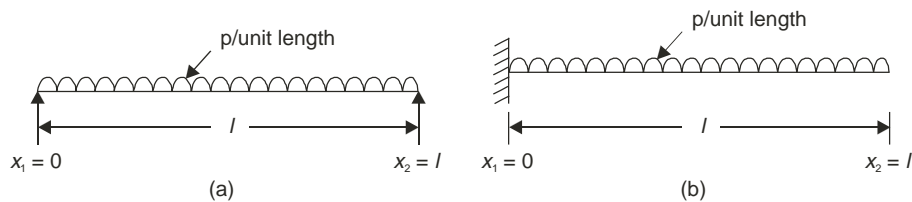


Fig. 9.2

In beams, strain energy due to bending

$$\begin{aligned} &= \int_0^l \frac{M^2}{2EI} dx \\ &= \int_0^l \frac{\left( EI \frac{d^2 y}{dx^2} \right)^2}{2EI} dx \\ &= \int_0^l \frac{EI}{2} \left( \frac{d^2 y}{dx^2} \right)^2 dx \end{aligned}$$



Potential energy due to load

$$= - \int_0^l py dx$$

∴ Total energy is given by

$$\begin{aligned} \Pi &= \int_0^l \frac{EI}{2} \left( \frac{d^2 y}{dx^2} \right)^2 - \int_0^l py dx \\ &= \int_0^l \left[ \frac{EI}{2} \left( \frac{d^2 y}{dx^2} \right)^2 - py \right] dx \\ &= \int_0^l F dx \end{aligned}$$

where

$$F = \frac{EI}{2} \left( \frac{d^2 y}{dx^2} \right)^2 - py$$

From equation 9.8, we know that the first variance of  $\int F dx$  is stationary, means

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial y''} \right) = 0 \quad (a)$$

$$\left[ \frac{\partial F}{\partial y'} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) \right]_{x_1}^{x_2} \delta y = 0 \quad (b)$$

and 
$$\left[ \frac{\partial F}{\partial y''} \delta y' \right]_{x_1}^{x_2} = 0 \quad (c)$$

In this problem we are trying

$$F = \frac{EI}{2} \left( \frac{d^2 y}{dx^2} \right)^2 - py$$

$$\therefore \frac{\partial F}{\partial y} = -p \quad \frac{\partial F}{\partial y'} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y''} = EI \frac{\partial^2 y}{\partial x^2}$$

since

$$y'' = \frac{\partial^2 y}{\partial x^2}$$

Hence Euler–Lagrange equation (equation a) for the beams is,

$$-p - 0 + \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) = 0$$

i.e., 
$$EI \frac{\partial^4 y}{\partial x^4} - p = 0$$

This is the well known equation of equilibrium in structural mechanics. Thus in case of beams Euler–Lagrange equation yields equation of equilibrium. Now let us see whether boundary conditions are satisfied or not.

Noting that  $y$  is exact solution and  $\delta y$  is the difference between exact and approximate solution, we find at supports

$$y = 0 \text{ means } \delta y = 0$$

and 
$$y' = 0 \text{ means } \delta y' = 0$$

In case of simply supported beam,

At  $x = x_1$

$\delta y(x_1) = y(x_1) = 0$  hence equation (b) is satisfied. From equation (c), we have

$$\frac{dF}{dy''} = \frac{EI d^2 y}{dx^2} = M, \text{ where } M \text{ is moment.}$$

At  $x = x_1$  we know moment is zero. Hence equation (c) is also satisfied.

At  $x = x_2$ ,

$$\delta y(x_2) = y(x_2) = 0$$

and 
$$\frac{\partial F}{\partial y''} /_{x=x_2} = M(x_2) = 0$$

Thus the boundary conditions at  $x = x_2$  are also satisfied. Hence the condition that first variance of total potential energy to be stationary, satisfies Euler–Lagrange equation as well as the boundary conditions.

Consider the cantilever beam [refer Fig. 9.2(b)].

At  $x = x_1$ ,

Equation (b) is satisfied since  $\delta y(x_1) = y(x_1) = 0$  and equation (c) is satisfied, since

$$\delta y'(x_1) = y'(x_1) = 0$$

At  $x = x_2$ ,

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left( \frac{\partial F}{\partial y''} \right) = 0$$

yields,

$$0 - \frac{d}{dx} \left( EI \frac{d^2 y}{dx^2} \right) = 0$$

i.e., 
$$EI = \frac{d^3y}{dx^3} = 0$$

i.e., shear force = 0

We know at free end, shear force is zero. Hence equation (c) is satisfied.

Thus, if potential energy is selected as a suitable function, its first variance is stationary, satisfies equilibrium equation as well as the boundary conditions.

### 9.3 POTENTIAL ENERGY IN ELASTIC BODIES

Potential energy is the capacity to do the work by the forces acting on deformable bodies. The forces acting on a body may be classified as external forces and internal forces. External forces are the applied loads while internal forces are the stresses developed in the body. Hence the total potential energy is the sum of internal and external potential energies. We will derive first the potential energy in a spring which is uniaxially stressed member then we will derive the expression for potential energy in a three dimensionally stressed body.

**(a) Potential Energy in a Spring:** Figure 9.3 shows a typical spring. Let its stiffness (load per unit extension) be  $k$  and length  $L$ . Due to a force  $P$  let it extend by  $u$ .

The load  $P$  moves down by distance  $u$ . Hence it loses its capacity to do work by  $Pu$ . Hence the external potential energy in this case

$$W_p = -Pu \quad \dots(9.10)$$

When the load has reached equilibrium position after extension of spring by  $u$ , the force in spring =  $ku$ . But when extension was zero the resisting force

was also zero. Hence the average force during the extension is  $\frac{0 + ku}{2} = \frac{ku}{2}$ .

Hence the energy stored in the spring due to straining,

$$\begin{aligned} &= \text{Average force} \times \text{Extension} \\ &= \frac{1}{2} kuu = \frac{1}{2} ku^2 \end{aligned} \quad \dots(9.11)$$

∴ Total Potential Energy in the Spring

$$\Pi = \frac{1}{2} ku^2 - pu \quad \dots(9.12)$$

**(b) Potential Energy in a Three Dimensional Body:** Consider a body of volume  $V$  subjected to

- (i) body forces  $X_b, Y_b, Z_b$ , in  $x, y, z$  directions.
- (ii) surface forces  $X_s, Y_s, Z_s$ , on surface  $S_1$

Let  $u, v$  and  $w$  be the displacement components. Then the potential energy of the external forces

$$W_p = -\iiint_v (X_b u + Y_b v + Z_b w) dV - \iint_{s_1} (X_s u + Y_s v + Z_s w) ds \quad \dots(9.13a)$$

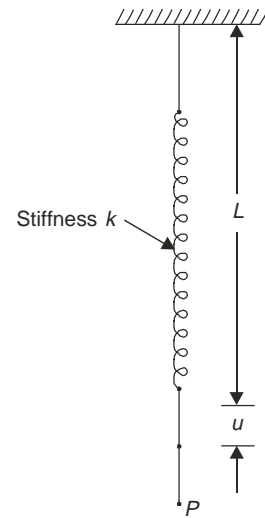


Fig. 9.3

$$= - \iiint_v \{u\}^T \{X_b\} dV - \iint_{s_1} \{u\}^T \{X_s\} ds \quad \dots(9.13b)$$

where

$$\{u\}^T = [u \quad v \quad w]$$

$$\{X_b\}^T = [X_b \quad Y_b \quad Z_b]$$

$$\{X_s\}^T = [X_s \quad Y_s \quad Z_s]$$

To find the internal energy due to straining, let us consider one by one stress component. Figure 9.4 shows a typical element of size  $dx \times dy \times dz$  subjected to  $\sigma_x$  stresses and displacement in  $x$ -directions. The work done by  $\sigma_x$  stresses in the element

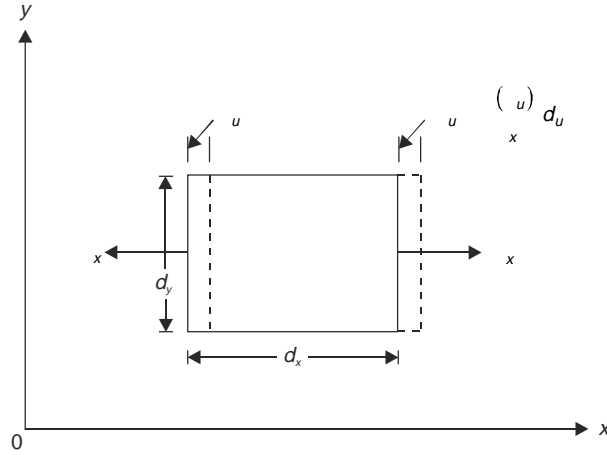


Fig.9.4

$$= \sigma_x dy dz \left( \delta u + \frac{\partial(\delta u)}{\partial x} dx \right) - \sigma_x dy dz \delta u$$

$$= \sigma_x \frac{d(\delta u)}{dx} dx dy dz, \text{ since } \frac{du}{dx} = \epsilon_x$$

$\therefore$  The work done by  $\sigma_x$  stresses in the element is  $= \sigma_x d\epsilon_x dx dy dz$

As the strain increases from zero to the final value  $\epsilon_x$ , the work stored as strain energy is

$$= \int_0^{\epsilon_x} \sigma_x d\epsilon_x du$$

In the three dimensional element, there are six stress components  $\{\sigma\}^T = [\sigma_x \quad \sigma_y \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx}]$ .

Hence the total strain energy in the element

$$= \left[ \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x + \int_0^{\varepsilon_y} \sigma_y d\varepsilon_y + \int_0^{\varepsilon_z} \sigma_z d\varepsilon_z + \int_0^{\gamma_{xy}} \tau_{xy} d\gamma_{xy} + \int_0^{\gamma_{yz}} \tau_{yz} d\gamma_{yz} + \int_0^{\gamma_{zx}} \tau_{zx} d\gamma_{zx} \right] dV$$

The integration is from initial to the final state of stress. Let  $U_0$  be the strain energy per unit volume. Then from the above equation, we get

$$dU_0 = \sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx} \quad (a)$$

But from chain rule of differentiation, we know

$$\partial U_0 = \frac{\partial U_0}{\partial \varepsilon_x} d\varepsilon_x + \frac{\partial U_0}{\partial \varepsilon_y} d\varepsilon_y + \frac{\partial U_0}{\partial \varepsilon_z} d\varepsilon_z + \frac{\partial U_0}{\partial \gamma_{xy}} d\gamma_{xy} + \frac{\partial U_0}{\partial \gamma_{yz}} d\gamma_{yz} + \frac{\partial U_0}{\partial \gamma_{zx}} d\gamma_{zx} \quad (b)$$

Comparing equations (a) and (b) we get,

$$\begin{aligned} \frac{\partial U_0}{\partial \varepsilon_x} &= \sigma_x, \quad \frac{\partial U_0}{\partial \varepsilon_y} = \sigma_y, \quad \frac{\partial U_0}{\partial \varepsilon_z} = \sigma_z \\ \frac{\partial U_0}{\partial \gamma_{xy}} &= \tau_{xy}, \quad \frac{\partial U_0}{\partial \gamma_{yz}} = \tau_{yz} \quad \text{and} \quad \frac{\partial U_0}{\partial \gamma_{zx}} = \tau_{zx} \end{aligned} \quad \dots(9.14)$$

Thus we find that the first derivate of strain energy per unit volume with respect to a strain component is the corresponding stress component i.e.,

$$\begin{aligned} \left\{ \frac{dU_0}{d\varepsilon} \right\} &= \{\sigma\} \\ &= [D] \{\varepsilon\} \end{aligned} \quad \dots(9.15)$$

Integration equation 9.15 we get,

$$U_0 = \frac{1}{2} \{\varepsilon\}^T [D] \{\varepsilon\} = \frac{1}{2} \{\varepsilon\}^T \{\sigma\}$$

To get total strain energy of the solid, the term  $U_0$  be integrated over the entire volume of the solid. Thus,

$$U = \iiint_v U_0 dV = \frac{1}{2} \iiint_v \{\varepsilon\}^T \{\sigma\} dV = \frac{1}{2} \iiint_v \{\varepsilon\}^T [D] \{\varepsilon\} dV$$

$\therefore$  Total potential energy of the solid

$$\Pi = U + Wp \quad \dots(9.16a)$$

$$\text{i.e.,} \quad \Pi = \frac{1}{2} \iiint_v \{\varepsilon\}^T [D] \{\varepsilon\} dV - \iiint_v \{u\}^T \{X_b\} dV - \iint_{s_1} \{u\}^T \{X_s\} ds \quad \dots(9.16b)$$

## 9.4 PRINCIPLE OF MINIMUM POTENTIAL ENERGY

From the expression for total potential energy (9.16.a) we know

$$\Pi = U + Wp$$

$$\therefore \delta \Pi = \delta U + \delta Wp. \quad \dots(9.17)$$

Comparing equation 9.13 (a) with equation 8.18, we find potential energy of external forces  $W_p$  is equal but opposite to total virtual work done by external forces. Thus,

$$W_p = -W_e$$

$$\therefore \delta W_p = -\delta W_e$$

$\therefore$  From Equation 9.17, we conclude

$$\delta \Pi = \delta U - \delta W_e .$$

But from principle of virtual work (equation 8.17) we know,  $\delta U = \delta W_e$

$$\therefore \delta \Pi = 0 \quad \dots(9.18)$$

Hence we can conclude that a deformable body is in equilibrium when the total potential energy is having stationary value. By taking second variance of potential energy, it has been proved by researchers that the value is positive definite. And hence it is concluded that the condition that value of total potential energy is stationary correspond to minimum value. Hence we have principle of minimum potential energy in solid mechanics, which may be stated as **“of all the possible displacement configurations a body can assume which satisfy compatibility and boundary conditions, the configuration satisfying equilibrium makes the potential energy assume a minimum value”**. This is the variation principle in solid mechanics.

**Example 9.2:** Assemble equations of equilibrium for the spring system shown in Fig. 9.5 by direct approach. Show that minimization of potential energy also yields same result.

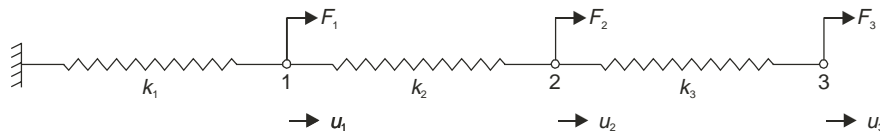


Fig. 9.5

**Solution:** Consider the free body diagram of nodes 1, 2 and 3 shown in Fig. 9.6. Let the displacement of nodes be  $u_1, u_2, u_3$ . Then the extensions of spring 1, 2 and 3 are

$$\delta_1 = u_1 \quad \delta_2 = u_2 - u_1 \quad \text{and} \quad \delta_3 = u_3 - u_2 \quad \dots(1)$$

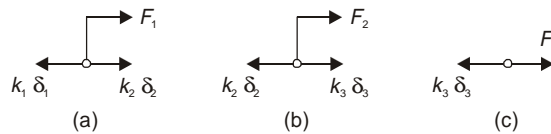


Fig. 9.6

Equations of equilibrium are,

$$-k_1 \delta_1 + k_2 \delta_2 + F_1 = 0 \quad \dots(2a)$$

$$-k_2 \delta_2 + k_3 \delta_3 + F_2 = 0 \quad \dots(2b)$$

$$-k_3 \delta_3 + F_3 = 0 \quad \dots(2c)$$

From equations 1 and 2, we get

$$-k_1 u_1 + k_2 (u_2 - u_1) + F_1 = 0$$

$$-k_2 (u_2 - u_1) + k_3 (u_3 - u_2) + F_2 = 0$$

$$-k_3 (u_3 - u_2) + F_3 = 0$$

i.e.,

$$(k_1 + k_2)u_1 - k_2 u_2 = F_1 \quad \dots(3a)$$

$$-k_2 u_1 + (k_2 + k_3)u_2 - k_3 u_3 = F_2 \quad \dots(3b)$$

and

$$-k_3 u_2 + k_3 u_3 = F_3 \quad \dots(3c)$$

The above equations in the matrix form are

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \dots(4)$$

Now, let us see the potential energy approach. Total potential energy in the system is,

$$\Pi = \frac{1}{2} k_1 \delta_1^2 + \frac{1}{2} k_2 \delta_2^2 + \frac{1}{2} k_3 \delta_3^2 - F_1 u_1 - F_2 u_2 - F_3 u_3$$

$$= \frac{1}{2} k_1 u_1^2 + \frac{1}{2} k_2 (u_2 - u_1)^2 + \frac{1}{2} k_3 (u_3 - u_2)^2 - F_1 u_1 - F_2 u_2 - F_3 u_3$$

$$\therefore \frac{\partial \Pi}{\partial u_1} = 0, \text{ gives}$$

$$k_1 u_1 + k_2 (u_2 - u_1)(-1) - F_1 = 0$$

i.e.,

$$k_1 u_1 - k_2 (u_2 - u_1) - F_1 = 0 \quad \dots(a)$$

i.e.,

$$(k_1 + k_2)u_1 - k_2 u_2 = F_1$$

$$\frac{\partial \Pi}{\partial u_2} = 0 \text{ gives}$$

$$k_2 (u_2 - u_1) + k_3 (u_3 - u_2)(-1) = F_2 \quad \dots(b)$$

$$-k_2 u_1 + (k_2 + k_3)u_2 - k_3 u_3 = F_2$$

and

$$\frac{\partial \Pi}{\partial u_3} = 0 \text{ gives,}$$

$$k_3 (u_3 - u_2) - F_3 = 0 \quad \dots(c)$$

or

$$-k_3 u_2 + k_3 u_3 = F_3$$

In the matrix form,

$$\begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \quad \dots(5)$$

Equations 4 and 5 are exactly same. Thus the condition that potential energy should have extreme value (minimum) leads to equations of equilibrium.

**Example 9.3:** Determine the displacements of nodes 1 and 2 in the spring system shown in Fig. 9.7. Use minimum of potential energy principle to assemble equations of equilibrium.

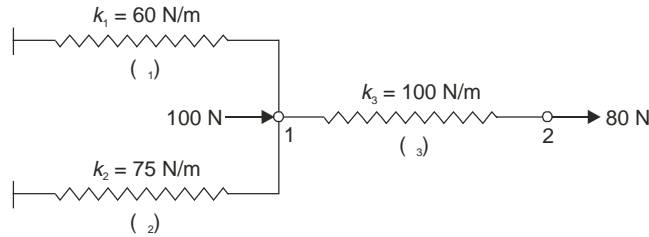


Fig. 9.7

**Solution:** Let  $u_1$  and  $u_2$  be the displacements of nodes 1 and 2. Then the extensions of springs are

$$\delta_1 = u_1 \quad \delta_2 = u_1 \quad \delta_3 = u_2 - u_1$$

$$\Pi = \frac{1}{2} k_1 \delta_1^2 + \frac{1}{2} k_2 \delta_2^2 + \frac{1}{2} k_3 \delta_3^2 - 100u_1 - 80u_2$$

$$= \frac{1}{2} k_1 u_1^2 + \frac{1}{2} k_2 u_1^2 + \frac{1}{2} k_3 (u_2 - u_1)^2 - 100u_1 - 80u_2$$

$$\therefore \frac{\partial \Pi}{\partial u_1} = 0 \rightarrow k_1 u_1 + k_2 u_1 + k_3 (u_2 - u_1)(-1) - 100 = 0. \quad \dots(a)$$

$$\text{i.e.,} \quad (k_1 + k_2 + k_3)u_1 - k_3 u_2 = 100$$

$$\therefore \frac{\partial \Pi}{\partial u_2} = 0 \rightarrow k_3 (u_2 - u_1) - 80 = 0. \quad \dots(b)$$

$$-k_3 u_1 + k_3 u_2 = 80$$

$$\begin{bmatrix} k_1 + k_2 + k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$

Substituting the values of  $k_1, k_2$  and  $k_3$ , we get.

$$\begin{bmatrix} 235 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 80 \end{Bmatrix}$$



$$\therefore \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{23500 - 10000} \begin{bmatrix} 100 & 100 \\ 100 & 235 \end{bmatrix} \begin{Bmatrix} 100 \\ 80 \end{Bmatrix} = \begin{Bmatrix} 1.333 \\ 2.133 \end{Bmatrix} \quad \text{Answer}$$

**Example 9.4:** Solve the three bar truss problem given in Example 8.1 by minimum potential energy principle.

**Solution:** As given in example 8.1,

$$l_1 = 2.309 \text{ m } l_2 = 2 \text{ m and } l_3 = 4.0 \text{ m.}$$

$$\varepsilon_1 = 0.375u + 0.2165v$$

$$\varepsilon_2 = 0.5u$$

and

$$\varepsilon_3 = 0.125u - 0.2165v$$

Strain energy of a bar

$$\begin{aligned} &= \frac{1}{2} \text{ stress} \times \text{strain} \times \text{volume} \\ &= \frac{1}{2} E\varepsilon \times \varepsilon \times AL = \frac{1}{2} EAL\varepsilon^2 \end{aligned}$$

$\therefore$  Potential energy of the structure is

$$\begin{aligned} \Pi &= \sum \frac{1}{2} EAL_i \varepsilon_i^2 - 20v \\ &= \frac{1}{2} EA \left[ 2.3094(0.375u + 0.2165v)^2 + 2(0.5u)^2 + 4(0.125u - 0.2165v)^2 \right] - 20v \end{aligned}$$

Which is exactly same as in example 8.1.

$$\therefore u = 0.0893u$$

$$\frac{\partial \Pi}{\partial v} = 0 \rightarrow \frac{1}{2} EA \left[ 2.3094 \times 2(0.375u + 0.2165v) \times 0.2165 + 4(0.125u - 0.2165v)(-0.2165) \right] - 20 = 0$$

This equation is also exactly same as in example 8.1 for vertical virtual displacement. Hence here also

$$v = \frac{66.048}{EA} \quad \text{and} \quad u = \frac{5.898}{EA}$$

and member forces are

$$F_1 = 26.045 \text{ kN}$$

$$F_2 = 2.949 \text{ kN}$$

and

$$F_3 = -13.562 \text{ kN} \quad \text{Answer}$$

## 9.5 THE RAYLEIGH-RITZ METHOD

The Rayleigh-Ritz method of expressing field variables by approximate method clubbed with minimization of potential energy has made a big break through in finite element analysis. In this article Rayleigh - Ritz method is explained with simple problems.

In 1870 Rayleigh used an approximating field with single degree of freedom for studies on vibration problems. In 1909 he used approximating field with several functions, each function satisfying boundary conditions and associating with separate degree of freedom. Ritz applied this technique to static equilibrium and Eigenvalue problems. The procedure for static equilibrium problem is given below:

Consider an elastic solid subject to a set of loads. The displacements and stresses are to be determined. Let  $u$ ,  $v$  and  $w$  be the displacements in  $x$ ,  $y$  and  $z$  coordinate directions. Then for each of displacement component an approximate solution is taken as

$$\begin{aligned}
 u &= \sum a_i \phi_i(x, y, z) \text{ for } i = 1 \text{ to } m_1 \\
 v &= \sum a_j \phi_j(x, y, z) \text{ for } j = m_1 + 1 \text{ to } m_2 \\
 w &= \sum a_k \phi_k(x, y, z) \text{ for } k = m_2 + 1 \text{ to } m
 \end{aligned}
 \tag{9.12}$$

The function  $\phi_i$  are usually taken as polynomials satisfying the boundary conditions. ‘ $a$ ’ are the amplitudes of the functions. Thus in equation 9.21 there are  $n$  number of unknown ‘ $a$ ’ values. Substituting these expressions for displacement in strain displacements and stress strain relations, potential energy expression 9.16 can be assembled. Then the total potential energy

$$\Pi = \Pi(a_1, a_2 \dots a_{m_1}, a_{m_1+1} \dots a_{m_2}, a_{m_2+1} \dots a_m)$$

From the principle of minimum potential energy,

$$\frac{d\Pi}{da_i} = 0 \text{ for } i = 1 \text{ to } m.
 \tag{9.22}$$

From the solution of  $m$  equation of 9.22, we get the values of all ‘ $a$ ’. With these values of ‘ $a_i$ ’s and  $\phi_i$ ’s satisfying boundary conditions, the displacements are obtained. Then the strains and stresses can be assembled. The Rayleigh – Ritz procedure is illustrated with small problems below:

**Example 9.5:** Using Ragleigh–Ritz method determine the expressions for deflection and bending moments in a simply supported beam subjected to uniformly distributed load over entire span. Find the deflection and moment at midspan and compare with exact solutions.

**Solution:** Figure 9.8 shows the typical beam. The Fourier series  $y = \sum_{m=1,3}^{\alpha} a_m \sin \frac{m\pi x}{l}$  is the ideal function

for simply supported beams since  $y=0$  and  $M = EI \frac{d^2y}{dx^2} = 0$  at  $x=0$  and  $x=l$  are satisfied. For the simplicity

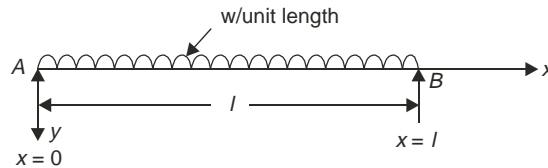


Fig. 9.8

let us consider only two terms in the series i.e. let

$$y = a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l} \quad \dots(a)$$

$$\Pi = \int_0^l \frac{EI}{2} \left( \frac{d^2 y}{dx^2} \right)^2 dx - \int_0^l w y dx \quad \dots(b)$$

Substituting  $y$  in equation (b) we get

$$\begin{aligned} &= \int_0^l \frac{EI}{2} \left[ -\frac{\pi^2}{l^2} a_1 \sin \frac{\pi x}{l} - \frac{9\pi^2}{l^2} a_2 \sin \frac{3\pi x}{l} \right]^2 dx - \int_0^l w \left( a_1 \sin \frac{\pi x}{l} + a_2 \sin \frac{3\pi x}{l} \right) dx \\ &= \frac{EI \pi^4}{2 l^4} \int_0^l \left( a_1^2 \sin^2 \frac{\pi x}{l} + 9a_2^2 \sin^2 \frac{3\pi x}{l} \right) dx - w \left[ -a_1 \frac{l}{\pi} \cos \frac{\pi x}{l} - a_2 \frac{l}{3\pi} \cos \frac{3\pi x}{l} \right]_0^l \\ &= \frac{EI \pi^4}{2 l^4} \int_0^l \left( a_1^2 \sin^2 \frac{\pi x}{l} + 18a_1 a_2 \sin \frac{\pi x}{l} \sin \frac{3\pi x}{l} + 81a_2^2 \sin^2 \frac{3\pi x}{l} \right) dx - \frac{wl}{\pi} \left[ 2a_1 + \frac{2a_2}{3} \right] \end{aligned}$$

Noting that  $\int_0^l \sin^2 \frac{\pi x}{l} dx = \int_0^l \frac{1}{2} \left( 1 - \cos \frac{2\pi x}{l} \right) dx = \frac{l}{2}$

$$\int_0^l \sin \frac{\pi x}{l} \sin \frac{3\pi x}{l} dx = \int_0^l \left( \cos \frac{2\pi x}{l} - \cos \frac{4\pi x}{l} \right) dx = 0$$

and  $\int_0^l \sin^2 \frac{3\pi x}{l} dx = \int_0^l \frac{1}{2} \left( 1 - \cos \frac{6\pi x}{l} \right) dx = \frac{l}{2}$

we get,

$$\begin{aligned} y &= \frac{EI \pi^4}{2 l^4} \left( a_1^2 \frac{l}{2} + 81 \frac{l}{2} a_2^2 \right) - \frac{2wl}{\pi} \left( a_1 + \frac{a_2}{3} \right) \\ &= \frac{EI \pi^4}{4 l^3} \left( a_1^2 + 81a_2^2 \right) - \frac{2wl}{\pi} \left( a_1 + \frac{a_2}{3} \right) \end{aligned}$$

$\Pi$  to be minimum,

$$\frac{\partial \Pi}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial \Pi}{\partial a_2} = 0.$$

i.e., 
$$\frac{EI \pi^4}{4l^3} 2a_1 - \frac{2wl}{\pi} = 0$$

or 
$$a_1 = \frac{4wl^4}{EI \pi^5}$$

and 
$$\frac{EI \pi^4}{4l^3} 81 \times 2a_2 - \frac{2wl}{3\pi} = 0$$

or

$$a_2 = \frac{4wl^4}{243EI\pi^5}$$

$$\therefore y = \frac{4wl^4}{EI\pi^5} \sin \frac{\pi x}{l} + \frac{4wl^4}{243EI\pi^5} \sin \frac{3\pi x}{l}$$

$\therefore$  Max. deflection which occurs at  $x = \frac{l}{2}$  is

$$y_{\max} = \frac{4wl^4}{EI\pi^5} - \frac{4wl^4}{243EI\pi^5} = \frac{wl^4}{76.82EI}$$

we know the exact solution is

$$y_{\max} = \frac{5}{384} \frac{wl^4}{EI} = \frac{wl^4}{76.8EI}$$

Thus the deflection is almost exact.

Now,

$$M_x = EI \frac{d^2 y}{dx^2} = EI \left( -a_1 \frac{\pi^2}{l^2} \sin \frac{\pi x}{l} - a_2 \frac{9\pi^2}{l^2} \sin \frac{3\pi x}{l} \right)$$

$$= EI \left[ -\frac{4wl^2}{EI\pi^3} \sin \frac{\pi x}{l} - \frac{4wl^2 \times 9}{243EI\pi^3} \sin \frac{3\pi x}{l} \right]$$

$$M_{\text{centre}} = \left[ -\frac{4wl^2}{EI\pi^3} + \frac{4wl^2 \times 9}{243EI\pi^3} \right] = \frac{wl^2}{8.05}$$

we know the exact value is  $\frac{wl^2}{8}$ .

By taking more terms in Fourier series more accurate results can be obtained.

**Example 9.6:** Using Rayleigh-Ritz method, determine the expressions for displacement and stress in a fixed bar subject to axial force  $P$  as shown in Fig. 9.9. Draw the displacement and stress variation diagram. Take 3 terms in displacement function.

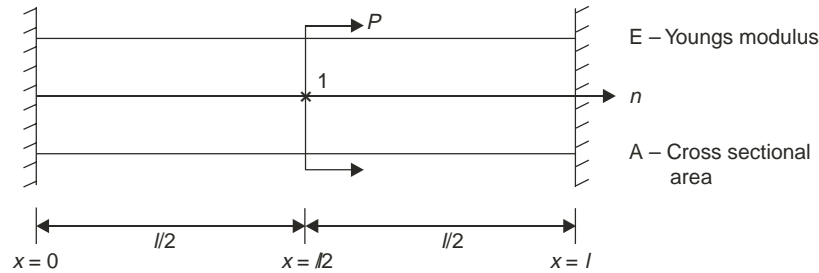


Fig. 9.9

**Solution:** Let the displacement at load point be  $u_1$ . Then the strain energy of the bar

$$U = \frac{1}{2} \int_0^l EA \left( \frac{du}{dx} \right)^2 dx$$

and potential energy due to external forces =  $-Pu_1$

$$\therefore \Pi = \frac{1}{2} \int_0^l EA \left( \frac{du}{dx} \right)^2 dx - Pu_1$$

Let the displacement at any point be given by,

$$u = a_1 + a_2 x + a_3 x^2$$

This function has to satisfy the boundary conditions

(i) at  $x = 0, u = 0$

(ii) at  $x = l, u = 0$

From Boundary condition (i), we get

$$0 = a_1 \tag{1}$$

From Boundary condition (ii), we get,

$$0 = a_1 + a_2 l + a_3 l^2 \tag{2}$$

$\therefore$  From equations (1) and (2) we get

$$0 = a_2 l + a_3 l^2$$

$$a_2 = -a_3 l \tag{3}$$

or

$$\therefore u = -a_3 l x + a_3 x^2 = a_3 [-lx + x^2]$$

At

$$x = \frac{l}{2}$$

$$u = u_1 = a_3 \left[ -l \frac{l}{2} + \frac{l^2}{4} \right]$$

i.e.,

$$u_1 = -\frac{a_3 l^2}{4} \tag{4}$$

Now

$$\frac{du}{dx} = a_3 (-l + 2x)$$

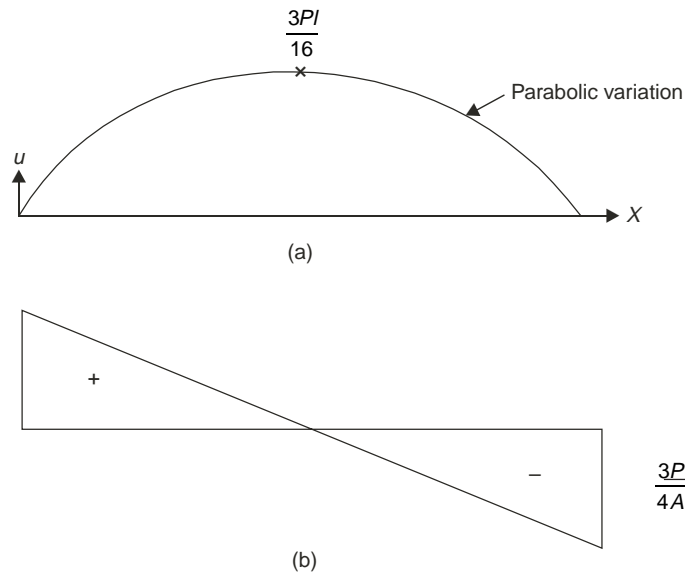
$$\therefore \Pi = \frac{1}{2} \int_0^l EA a_3^2 (-l + 2x)^2 dx + Pa_3 \frac{l^2}{4}$$

$$= \frac{1}{2} EA a_3^2 \int_0^l (l^2 - 4lx + 4x^2) dx + Pa_3 \frac{l^2}{4}$$

$$= \frac{1}{2} EA a_3^2 \left[ l^2 x - 2lx^2 + \frac{4x^3}{3} \right]_0^l + Pa_3 \frac{l^2}{4}$$

$$\begin{aligned}
 &= \frac{1}{2} EA a_3^2 \left( \frac{l^3}{3} \right) + Pa_3 \frac{l^2}{4} \\
 \therefore \frac{d\Pi}{da_3} = 0 &\rightarrow EA a_3 \frac{l^3}{3} + P \frac{l^2}{4} = 0 \\
 a_3 &= -\frac{3p}{4EA} \\
 \therefore u &= -\frac{3p}{4EA} [-lx + x^2] \\
 \therefore u_1 &= -\frac{a_3 l^2}{4} = \frac{3pl}{16} \\
 \sigma &= E \frac{du}{dx} = Ea_3 (-l + 2x) \\
 &= E \frac{-3p}{4EA} [-l + 2x] = \frac{3p}{4A} [l - 2x] \\
 \therefore \sigma_0 = \sigma_{x=0} &= \frac{3p}{4A} \\
 \sigma_1 = \sigma_{x=l/2} &= 0 \\
 \sigma_2 = \sigma_{x=l} &= -\frac{3p}{4A}
 \end{aligned}$$

The variation of displacement and stresses are shown in Fig. 9.10.



**Fig. 9.10** (a) Variation of  $u$  (b) Variation of stress

**Example 9.7:** Determine the displacement and stress in a bar of uniform cross section due to self weight only when held as shown in Fig. 9.11. Use (i) two terms (ii) three terms, for approximating polynomial. Verify the expression for total extension with the exact value.

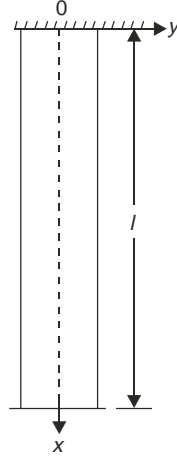


Fig. 9.11

**Solution:** Let ‘ $\rho$ ’ be unit weight and  $E$  Young’s modulus of the material of the bar. If  $A$  is the cross section of the bar then,

$$\begin{aligned}
 U &= \iiint_v \frac{1}{2} \{\epsilon\}^T \{\sigma\} dV \\
 &= \int_0^l \frac{1}{2} \left( \frac{du}{dx} \right)^T \left[ E \frac{du}{dx} \right] A dx \quad \dots(1) \\
 &= \frac{1}{2} \int_0^l EA \left( \frac{du}{dx} \right)^2 dx
 \end{aligned}$$

and

$$\begin{aligned}
 W_e &= - \iiint_v \{u\}^T \{X_b\} dV \\
 &= - \int_0^l u \rho A dx \quad \dots(2)
 \end{aligned}$$

Polynomial function for displacement may be taken as

$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n \quad \dots(3)$$

The boundary condition to be satisfied is

At  $x = 0, u = 0$

From this we get  $0 = a_0$

$$\therefore u = a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

(i) When only two terms of polynomial equations are used,

$$u = a_1 x \quad \therefore \frac{du}{dx} = a_1$$

$$\therefore U = \frac{1}{2} \int_0^l EA \left( \frac{du}{dx} \right)^2 dx = \frac{1}{2} \int_0^l EA a_1^2 dx = \frac{1}{2} EA a_1^2 l$$

$$W_p = - \int_0^l u \rho A dx$$

$$= - \int_0^l a_1 x \rho A dx = - \rho A a_1 \left[ \frac{x^2}{2} \right]_0^l$$

$$= - \rho A a_1 \frac{l^2}{2}$$

$$\therefore \Pi = \frac{1}{2} EA a_1^2 l - \rho A a_1 \frac{l^2}{2}$$

From minimization condition, we get

$$\frac{d\Pi}{da_1} = 0 \text{ i.e., } 0 = EA a_1 l - \rho A \frac{l^2}{2}$$

or

$$a_1 = \frac{\rho l}{2E} \quad \dots(4)$$

$$\therefore u = \frac{\rho l}{2E} x \quad \dots(5)$$

and

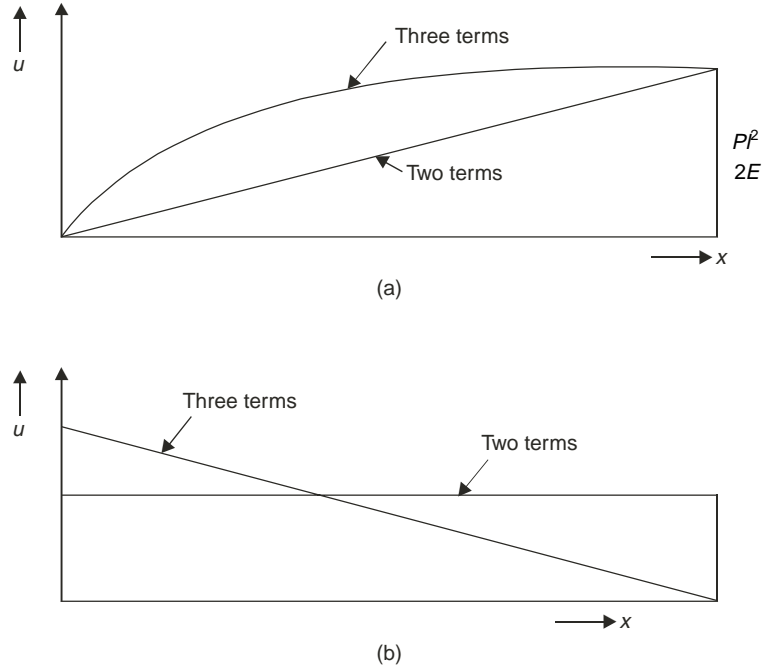
$$\sigma = E \frac{du}{dx} = \frac{\rho l}{2} \quad \dots(6)$$

The displacement and stress variations are shown in Fig. 9.12.

Extension of the bar =  $u_l - u_0$

$$= \frac{\rho l}{2E} l - 0 = \frac{\rho l^2}{2E} \quad \dots(7)$$





**Fig. 9.12** (a) Variation of displacement (b) Variation of stress

(ii) When three terms are considered for displacement in equation 3:

$$u = a_1 x + a_2 x^2 \quad \therefore \frac{du}{dx} = a_1 + 2a_2 x$$

$$\Pi = U + W_p$$

$$= \frac{1}{2} \int_0^l EA \left( \frac{du}{dx} \right)^2 dx - \int_0^l u \rho A dx$$

$$= \frac{1}{2} EA \int_0^l (a_1 + 2a_2 x)^2 dx - \rho A \int_0^l (a_1 x + a_2 x^2) dx$$

$$= \frac{1}{2} EA \left[ a_1^2 x + 4a_1 a_2 \frac{x^2}{2} + 4a_2^2 \frac{x^3}{3} \right]_0^l - \rho A \left( \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} \right)_0^l$$

$$= \frac{1}{2} EA \left[ a_1^2 l + 2a_1 a_2 l^2 + \frac{4}{3} a_2^2 l^3 \right] - \rho A \left( a_1 \frac{l^2}{2} + a_2 \frac{l^3}{3} \right)$$

$$\therefore \frac{d\Pi}{da_1} = 0 \rightarrow \frac{1}{2} EA [2a_1 l + 2a_2 l^2] - \rho A \frac{l^2}{2} = 0$$

$$\text{i.e.,} \quad a_1 + a_2 l = \frac{\rho l}{2E} \quad \dots(8)$$

$$\frac{d\Pi}{da_2} = 0 \rightarrow \frac{1}{2}EA \left[ 2a_1 l^2 + \frac{8}{3}a_2 l^3 \right] - \rho A \frac{l^3}{3} = 0$$

$$\text{i.e.,} \quad a_1 + \frac{4}{3}a_2 l = \frac{\rho l}{3E} \quad \dots(9)$$

From equation 8 and 9 we get,

$$\frac{1}{3}a_2 l = \frac{\rho l}{3E} - \frac{\rho l}{2E} = -\frac{\rho l}{6E}$$

$$\therefore a_2 = -\frac{\rho}{2E}$$

Substituting it in equation 8, we get

$$a_1 = \frac{\rho l}{2E} - \left( -\frac{\rho}{2E} l \right) = \frac{\rho l}{E}$$

$$\therefore u = a_1 x + a_2 x^2 = \frac{\rho l}{E} x - \frac{\rho}{2E} x^2 = \frac{\rho}{E} \left[ lx - \frac{x^2}{2} \right]$$

$$\sigma = E \frac{du}{dx} = \rho[l - x]$$

The variations of displacement and stress in this case also are shown in Fig. 9.12.

$$\text{Extension of the bar } u_1 - u_0 = \frac{\rho}{E} \left[ l^2 - \frac{l^2}{2} \right] = \frac{\rho l^2}{2E}$$

Actual extension of the bar [refer Fig. 9.13]

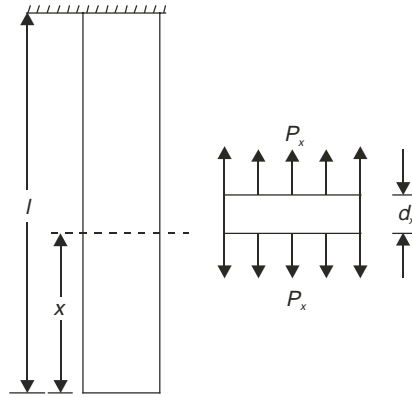


Fig. 9.13

$$= \int_0^l \frac{P_x dx}{AE} = \int_0^l \frac{\rho A x dx}{AE} = \frac{\rho}{E} \left[ \frac{x^2}{2} \right]_0^l = \frac{\rho l^2}{2E}$$

Thus total extension of the bar obtained is exact in both the cases.

### Comments on Rayleigh–Ritz Method

In this method the approximating functions must satisfy the boundary conditions and should be easy to use. Polynomials are normally used. Some times sine-cosine terms are also used.

Results can be obtained for complex problems. But for complex problems it is difficult to say whether the results obtained are accurate enough to use. The doubt will arise due to the following two reasons

- (i) Whether this is the only function which can be used
- (ii) How many terms in the function are to be used.

The best way to ensure the accuracy is to get result using a certain number of terms and then use additional terms to get the results. If the difference is negligible, we can conclude that the satisfactory result is obtained. However it may be noted that the lowest terms in the series should not be omitted in the approximating functions.

## 9.6 VARIATIONAL FORMULATION IN FINITE ELEMENT ANALYSIS

From variational principle, we have concluded that a body is in equilibrium when potential energy is minimum. From Rayleigh-Ritz method we have found approximating functions satisfying the boundary conditions can be used to get the solutions. In finite element analysis we use approximating functions for the elements but not for entire body and use the principle of minimizing potential energy to get the solutions for complex structures. The similarity and the differences in Rayleigh-Ritz method and finite element method are as listed below:

### (a) Similarity:

- (i) Both methods use approximating functions as trial solution
- (ii) Both methods take linear combinations of trial functions.
- (iii) In both methods completeness condition of the function should be satisfied
- (iv) In both methods solution is sought by making a functional stationary.

### Difference

- (i) Rayleigh-Ritz method assumes trial functions over entire structure, while finite element method uses trial functions only over an element.
- (ii) The assumed functions in Rayleigh-Ritz method have to satisfy boundary conditions over entire structure while in finite element analysis, they have to satisfy continuity conditions at nodes and some times along the boundaries of the element. However completeness condition should be satisfied in both methods.

Now let us see the variational method in finite element analysis in detail. The potential energy of a structure is

$$\begin{aligned} \Pi &= \text{Internal potential energy} - \text{External potential energy} \\ &= \text{Strain energy} - \text{Work done by external forces} \\ &= U - W \end{aligned}$$

These expressions involve integration terms.

The above expression refers to entire structure. Since the integration of the summation is the same as the sum of the individual integrals, we can apply the principle to each element separately. Thus

$$\begin{aligned}\Pi &= \sum_{e=1}^n \Pi_e = \sum_{e=1}^n U_e - \sum_{e=1}^n W_e \\ \therefore \frac{d\Pi}{d\delta} &= 0 = \sum \frac{dU_e}{d\{\delta\}_e} - \sum \frac{dW_e}{d\{\delta\}_e}\end{aligned}\quad \dots(9.23)$$

Where  $\{\delta\}$  is vector of nodal displacement in the structure.

And  $\{\delta\}_e$  is the vector of nodal displacements in the element.

Now,  $U_e$  = Strain energy of the element

$$= \iiint_v \frac{1}{2} \{\varepsilon\}^T [\sigma] dV$$

But

$$\{\varepsilon\} = [B]\{\delta\}$$

and

$$\delta = [D]\{\varepsilon\} = [D][B]\{\delta\}$$

$$\begin{aligned}\therefore U_e &= \iiint_v \frac{1}{2} ([B]\{\delta\}_e)^T [D][B]\{\delta\}_e dV \\ &= \frac{1}{2} \iiint_v \{\delta\}_e^T [B]^T [D][B]\{\delta\}_e dV,\end{aligned}\quad \dots(9.24)$$

If  $\{X\}^T = \{X_b \ Y_b \ Z_b\}$  and  $\{X_s\}^T = \{X_s \ Y_s \ Z_s\}$ , then the work done by these forces is given by equation 9.13 as

$$W_p = - \iiint_v \{u\}^T \{X_b\} dV - \iint_{s_1} \{u\}^T \{X_s\} ds \quad \dots(9.25)$$

where

$$\{u\}^T = \{u \ v \ w\}$$

But

$$u = [N]\{\delta\}_e$$

where  $\delta_e$  is the nodal displacement vector of the element.

$$\begin{aligned}\therefore W &= - \iiint_v ([N]\{\delta\}_e)^T \{X_b\} dV - \iint_{s_1} ([N]\{\delta\}_e)^T \{X_s\} ds \\ &= - \iiint_v \{\delta\}_e^T [N]^T \{X_b\} dV - \iint_{s_1} \{\delta\}_e^T [N]^T \{X_s\} ds\end{aligned}$$

Hence,

$$\begin{aligned}\Pi e &= Ue + Wp \\ &= \frac{1}{2} \iiint_v \{\delta\}_e^T [B]^T [D][B] \{\delta\}_e dV - \iiint_v \{\delta\}_e^T N^T [X_b] dV - \iint_{s_1} \{\delta\}_e^T [N]^T \{X_s\} ds.\end{aligned}$$

From principle of minimum potential energy (equation 9.23),

$$0 = \iiint_v \frac{1}{2} 2[B]^T [D][B] \{\delta\}_e dV - \iiint_v [N]^T \{X_b\} dV - \iint_{s_1} [N]^T \{X_s\} ds$$

i.e.,

$$\iiint_v [B]^T [D][B] \{\delta\}_e dV = \iiint_v [N]^T \{X_b\} dV + \iint_{s_1} [N]^T \{X_s\} ds$$

Thus element equilibrium equation is

$$[k]_e \{\delta\}_e = (F)_e$$

where

$$[k]_e = \iiint_v [B]^T [D][B] dV \quad \dots(9.26)$$

and

$$\{F\}_e = \iiint_v [N]^T \{X_b\} dV + \iint_{s_1} [N]^T \{X_s\} ds$$

The matrix  $[k]_e = \iiint_v [B]^T [D][B] dV$  is called stiffness matrix of the element and the load vector  $\{F\}_e$

is called consistent load. Thus

$$[k]_e = [B]^T [D][B] \text{ is stiffness matrix of the element and } \{F\}_e = \iiint_v [N]^T \{X_b\} dV + \iint_{s_1} [N]^T \{X_s\} ds$$

is called consistent load vector.

The above equation of equilibrium is to be assembled for entire structure and boundary conditions are to be introduced. Then the solution of equilibrium equations result into nodal displacements of all the nodal points. Once these basic unknowns are known, then displacement at any point may be obtained by the relation  $u = [N]\{\delta\}_e$ . The strains are assembled using the relation  $\{\epsilon\} = [B]\{\delta\}_e$  and then stresses also can be found  $\{\sigma\} = [D]\{\epsilon\}_e$ .

### Comments on Variational Method

It is a versatile method. Using this method stiffness matrices and consistent load vectors can be assembled easily. This method has made finite element analysis a versatile method. All complex problems can be solved. This is universally used method in solid mechanics.

## QUESTIONS

1. State and explain the principle of minimum potential energy.
2. Derive the general equation for determining the stiffness of an element with usual notations in the form

$$[k]_e = \int [B]^T [D] [B] dV .$$

- (i) Explain the principle of Rayleigh–Ritz method.
- (ii) Write short note on variational principles.
- (iii) Derive Euler–Lagrange equation for an integral function using variational principle.
- (iv) Using principle of minimum potential energy derive the expressions for consistent loads for body forces and surface forces.

# 10

## Discretization of Structures

### 10.1 INTRODUCTION

The process of modeling a structure using suitable number, shape and size of the elements is called discretization. The modeling should be good enough to get the results as close to actual behavior of the structure as possible. In this chapter various aspects of discretization of structures are discussed.

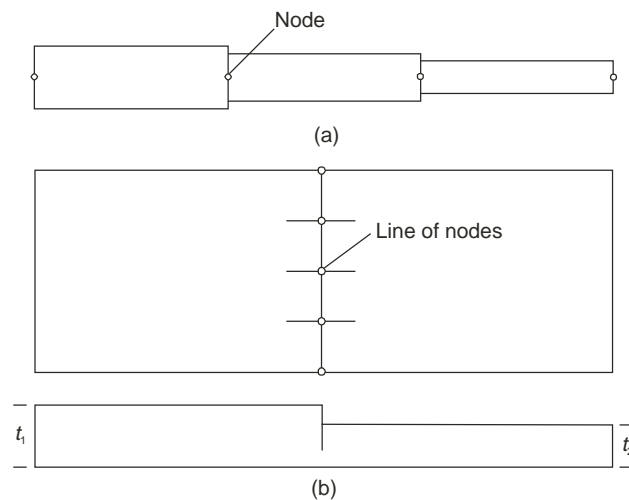
### 10.2 NODES AT DISCONTINUITIES

In a structure we come across the following types of discontinuities:

- (a) Geometric
- (b) Load
- (c) Boundary conditions
- (d) Material.

#### (a) Geometric Discontinuities

Whenever there is sudden change in shape and size of the structure there should be a node or line of nodes. Figure 10.1 shows some of such situations.



**Fig. 10.1** (a) Bar subject to axial forces (b) Plate with varying

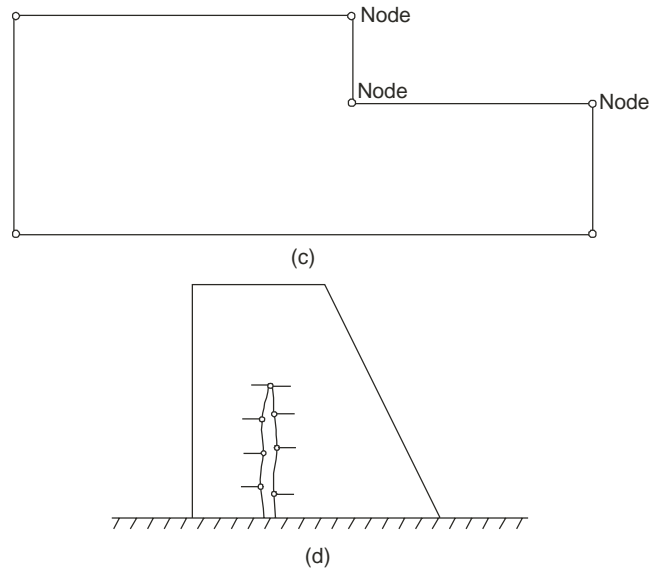


Fig. 10.1 (contd) (c) Slab (d) Dam with a crack

**(b) Discontinuity of Loads**

Concentrated loads and sudden change in the intensity of uniformly distributed loads are the sources of discontinuity of loads. A node or a line of nodes should be there to model the structure. Some of these situations are shown in Fig. 10.2.

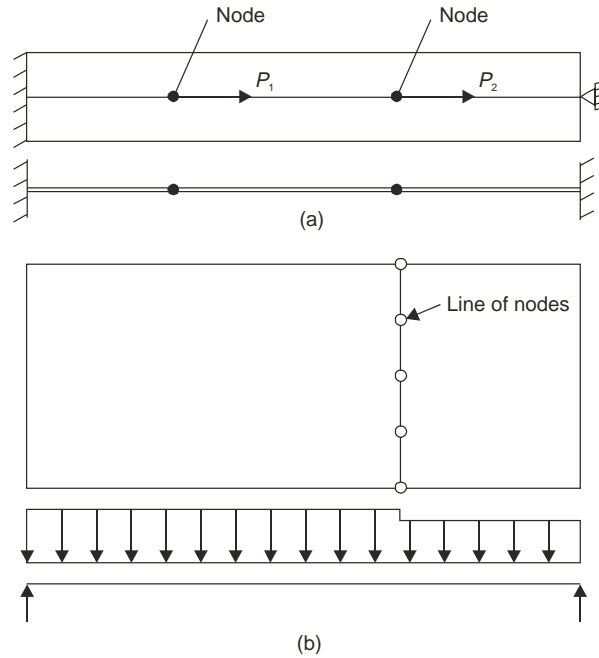
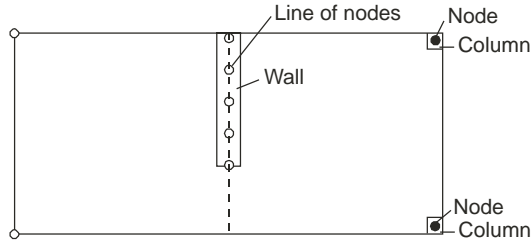


Fig. 10.2 (a) FEM model (b) Slab with different udls



**(c) Discontinuity of Boundary conditions**

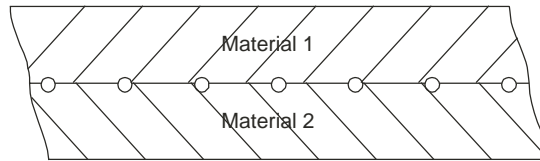
If the boundary condition for a structure suddenly change we have to discretize such that there is node or a line of nodes. This type of situations are shown in Fig. 10.3.



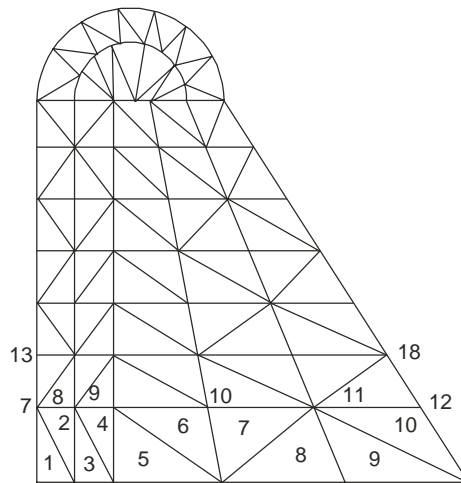
**Fig. 10.3** *Slab with intermediate wall and columns*

**(d) Material Discontinuity**

Node or node lines should appear at the places where material discontinuity is seen.



**Fig. 10.4** *Material discontinuity*



**Fig. 10.5** *Refined mesh near curved boundary of a dam*

**10.3 REFINING MESH**

To get better results the finite element mesh should be refined in the following situations

- (a) To approximate curved boundary of the structure
- (b) At the places of high stress gradients.

Such situations are shown in Fig. 10.5 and Fig. 1.2.

## 10.4 USE OF SYMMETRY

Wherever there is symmetry in the problem it should be made use. By doing so lot of memory requirement is reduced or in other words we can use more elements (refined mesh) for the same capacity of computer memory. When symmetry is to be used, it is to be noted that at right angles to the line of symmetry displacement is zero. In the tension bar example shown in Fig. 1.2, biaxial symmetry of the problem is utilized and only quarter of the bar is taken for the analysis.

## 10.5 FINITE REPRESENTATION OF INFINITE BODIES

Soil is a typical example of infinite body. Wherever settlement of soil is to be studied or study is required about soil structure interaction, the mass of soil is to be modeled. It is well known fact that the soil mass away from footing is not affected. However the question remains how much mass of soil is to be considered. Best way to handle this type of problems is to consider a certain mass of soil and determine the settlement under the load. Then increase or decrease the mass of soil, analyses and again compare the results. This type of preliminary study helps in identifying the mass of soil to be considered. Figure 10.6 shows the finite element idealization of one such problem. There are research reports that for homogeneous soil mass,  $H$  should be 4 to 6 times and  $V$  should be 10 to 12 times the footing width or diameter.

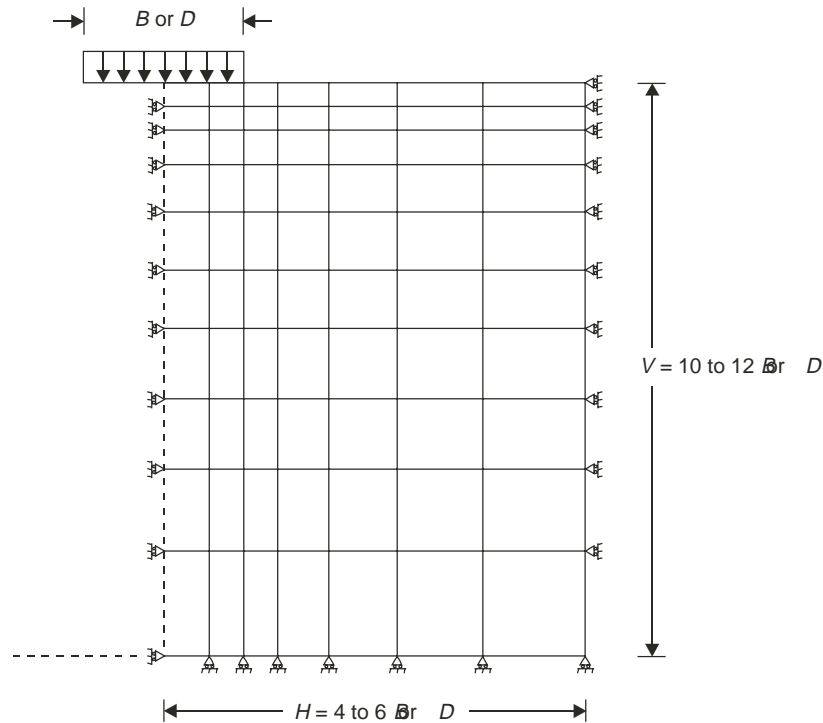
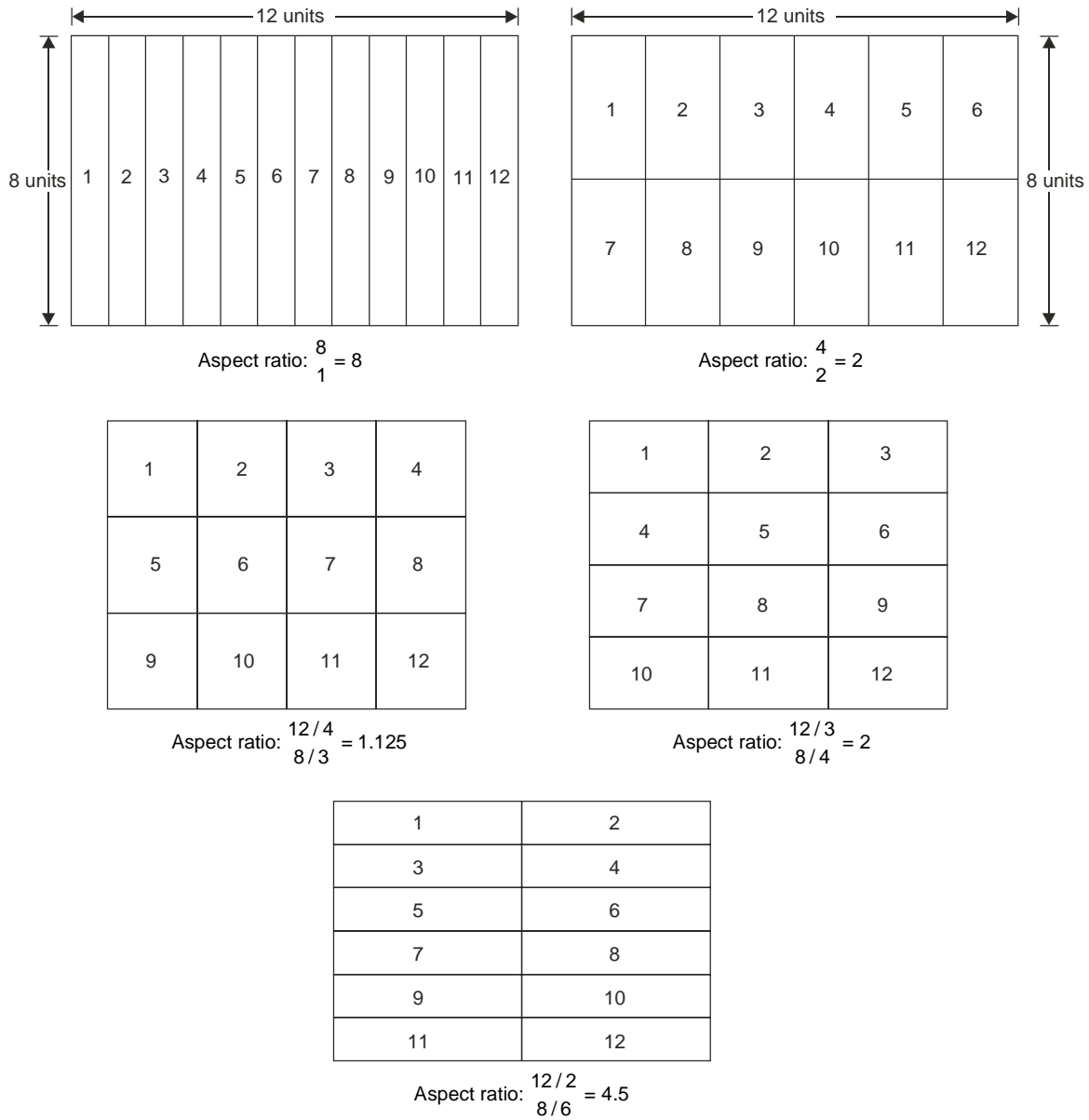


Fig. 10.6

### 10.6 ELEMENT ASPECT RATIO

The shape of the element also affects the accuracy of analysis. Defining the aspect ratio as ratio of largest to smallest size in an element, the conclusion of many researchers is aspect ratio should be as close to unity as possible. For a two dimensional rectangular element, the aspect ratio is conveniently defined as length to breadth ratio. To study the effect of aspect ratio on the accuracy of results, Desai and Abel analyzed a beam with 12 elements of different aspect ratios as shown in Figs. 10.7 and 10.8 shows the plot of inaccuracy of the displacement verses the aspect ratio. From this it can be concluded that the aspect ratio closer to unity yields better results.



**Fig. 10.7**

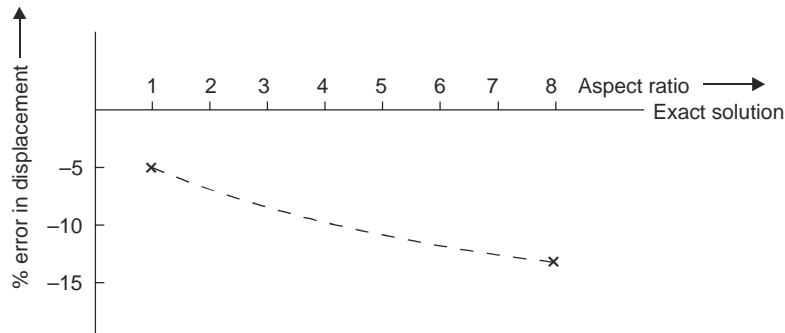


Fig. 10.8 Effect of aspect ratio on accuracy of result

## 10.7 HIGHER ORDER ELEMENTS VS REFINED MESH

Accuracy of calculation increases if higher order elements are used. Accuracy can also be increased by using more number of elements. Limitation on use of number of elements comes from the total degrees of freedom the computer can handle. The limitation may be due to cost of computation time also. Hence to use higher order elements we have to use less number of such elements. The question arises whether to use less number of higher order elements or more number of lower order elements for the same total degree of freedom. There are some studies in this matter keeping degree of accuracy per unit cost as the selection criteria. However the cost of calculation is coming down so much that such studies are not relevant today. Accuracy alone should be selection criteria which may be carried out initially on the simplified problem and based on it element may be selection for detailed study.

## 10.8 NUMBERING SYSTEM TO REDUCE BAND WIDTH

Storing global stiffness matrix in the computer memory imposes a serious limitation on the number of elements/degrees of freedom to be used. In elasticity problem the stiffness matrix is symmetric and banded. In a problem with 1000 degree of freedom the size of stiffness matrix is  $1000 \times 1000$ . If it has semi band width of 28, we can store only  $1000 \times 28$  elements and handle the solution using suitable programming technique. The size of semi band width of stiffness matrix depends upon the numbering system adopted for nodes. The semi band width  $B$  is given by the expression

$$B = (D + 1)f \quad \dots(10.1)$$

where  $D$  is maximum difference in node number in an element after considering all elements  
 $f$ -degrees of freedom per node.

The semi band widths for various types of numbering for a two dimensional problems are shown in Fig. 10.9. It may be concluded that the semi band width is minimum if numbering is in shorter direction and restarted from the initial end after reaching other end.

In many standard packages numbering is done automatically to keep the semi band width least.

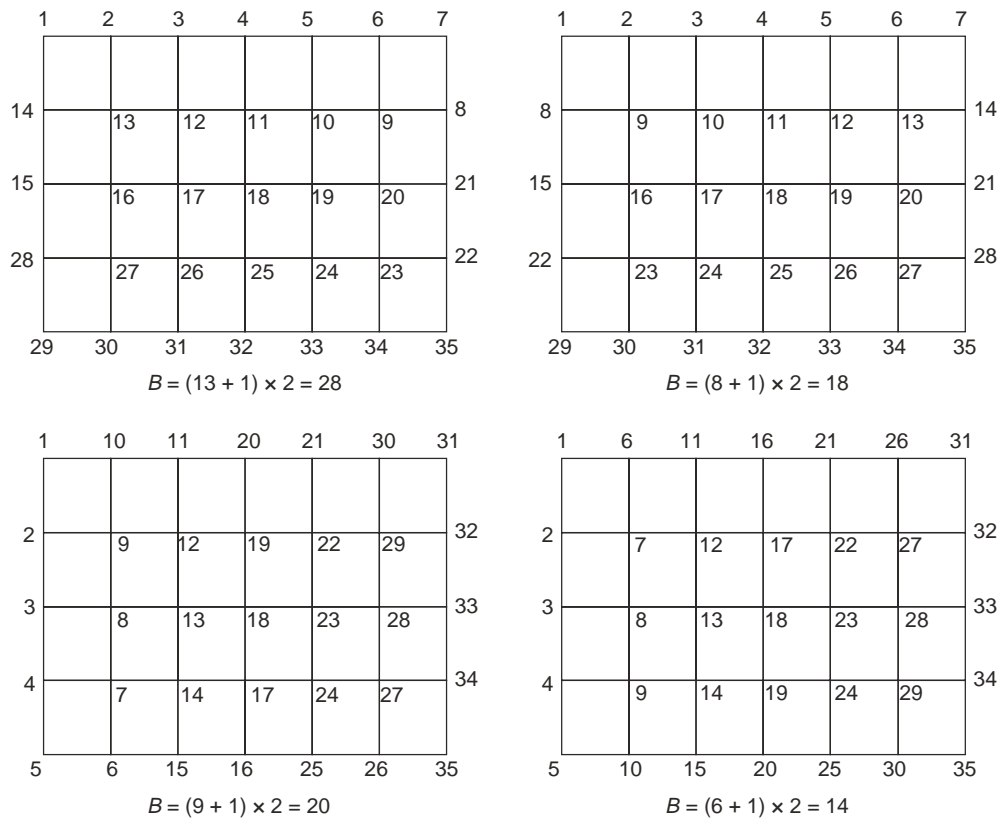


Fig. 10.9

**QUESTIONS**

1. Write short notes on:
  - (a) Effect of element aspect ratio on accuracy
  - (b) Numbering nodes for band width minimization
  - (c) Mesh refinement vs higher order elements.
2. Discuss the various points to be considered while discretizing a structure for finite element analysis.
3. Briefly explain how problem involving infinite bodies are handled in finite element analysis.

# 11

## Finite Element Analysis—Bars and Trusses

### 11.1 INTRODUCTION

Under ‘bars’ we consider the analysis of members subject to axial forces only. These members are having one dimension (length) considerably large compared to cross sectional dimensions. Tension bars and columns fall under this category. In case of pin connected frames (trusses), members can be assumed to have only axial forces. In this chapter the analysis of the following three types of members is explained:

- (i) Tension bars/columns
- (ii) Two dimensional trusses (plane trusses)
- (iii) Three dimensional trusses (space trusses)

Various steps involved in finite element analysis is thoroughly presented and few simple problems are solved with hand calculations.

### 11.2 TENSION BARS/COLUMNS

The typical member considered for explaining the procedure is shown in Fig.11.1. In this problem we see cross section varies in 3 steps  $A_1$ ,  $A_2$  and  $A_3$ . There are three point loads  $P_1$ ,  $P_2$  and  $P_3$ . The surface forces are  $x_{s1}$ ,  $x_{s2}$ , and  $x_{s3}$  and  $X_b$  is the body force. The surface forces may be due to frictional forces, viscous drag or surface shear. The body force is due to self weight. The material of the bar is same throughout.

**Step 1:** Selecting suitable field variables and elements:

In all stress analysis problems, displacements are selected as field variables. In the tension bar or columns at any point there is only one component of displacement to be considered, i.e., the displacement in  $x$  direction.

Since there is only one degree of freedom and it needs only  $C^0$  continuity, we select bar element shown in Fig. 11.2. In this case there are only two nodes.

**Step 2:** Discretise the continua

In this problem there are geometric discontinuities at  $x = 200$  mm,  $500$  mm and  $650$  mm. There is additional point of discontinuity at  $x = 350$  mm, where concentrated load  $P_1$  is acting. Hence we discretise the continua as shown in Fig. 11.3 using four bar elements.

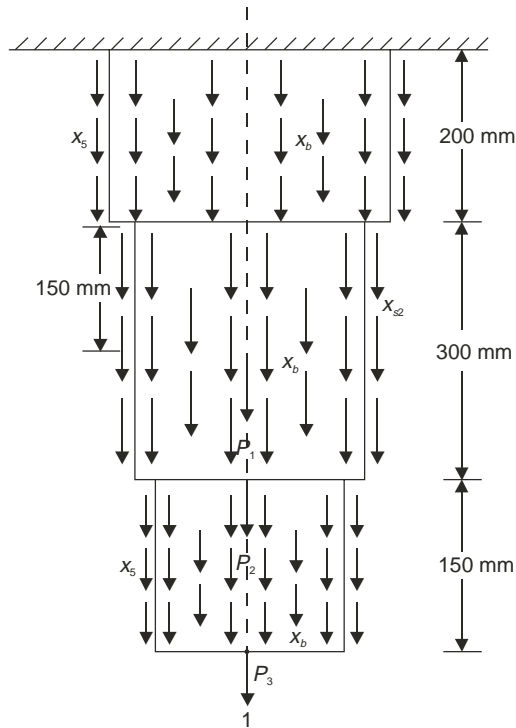


Fig. 11.1

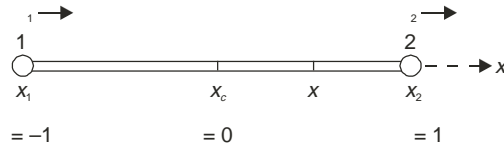


Fig. 11.2

Hence nodal displacement vector is

$$\{\delta\} = \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

In finite element analysis the nodes may be numbered in any fashion, but to keep the band width minimum we number the nodes continuously. In this problem there are five nodes and in all such problem there is definite relationship between number of nodes and number of element i.e. Number of node = Number of elements + 1.

There is only one degree of freedom at each node. Hence total degree of freedom in the problem is  
 = Number of nodes  $\times$  degree of freedom at each node  
 =  $5 \times 1 = 5$

i.e.  $\{\delta\}^T = [\delta_1 \ \delta_2 \ \delta_3 \ \delta_4 \ \delta_5]$  ... (11.2)

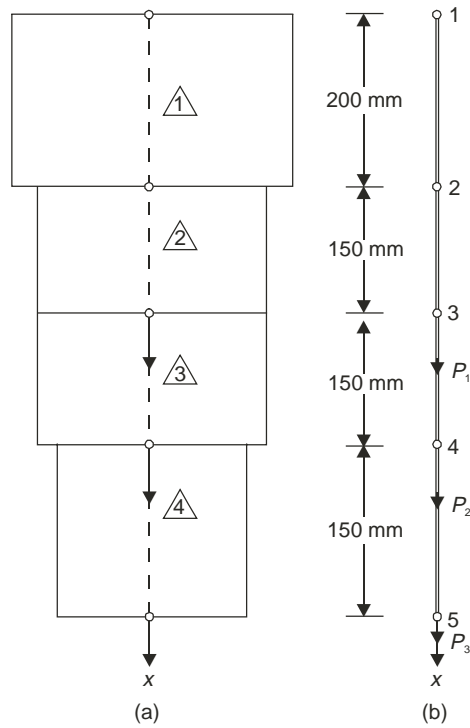


Fig. 11.3

For any element local node number is 1 and 2 only, but global coordinate numbers for each element are different. For example, local coordinate numbers 1 and 2 for element 3 refers to global numbering system 3 and 4 respectively. The relation between the local and global node number is called connectivity details. In this problem the connectivity detail is as shown in Fig. 11.4. From this Figure it can be seen that the connectivity detail can be easily generated also. Thus

For element ( $i$ ),

Local node number 1 =  $i$

Local node number 2 =  $i + 1$

Element	Nodes		Local numbers
	1	2	
1	1	2	Global numbers
2	2	3	
3	3	4	
4	4	5	

Fig. 11.4



**Step 3:** Select Interpolation Functions

In chapter 5 we have seen interpolation functions  $[N]$  is given by

$$\{u\} = [N]\{\delta\}_e \quad \dots(11.3)$$

and for bar elements

$$[N] = [N_1 \ N_2], \text{ where}$$

$$N_1 = \frac{x_2 - x}{l_e} = \frac{1 - \xi}{2}$$

$$\text{and} \quad N_2 = \frac{x - x_1}{l_e} = \frac{1 + \xi}{2} \quad \dots(11.4)$$

**Step 4:** Element Properties

In this step we assemble element stiffness matrix and nodal force vector of the element. At any point in the element,

$$\{u\} = u \quad \{\varepsilon\} = \varepsilon \quad \text{and} \quad \{\sigma\} = \sigma, \text{ all in } x \text{ direction, which is the only direction for these elements.}$$

From strain displacement relations,

$$\{\varepsilon\} = \varepsilon = \frac{du}{dx} = \frac{d}{dx} [N_1 \ N_2] \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \quad \dots(11.5)$$

$$= \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

$$= [B] \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}, \text{ where } [B] = \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\{\sigma\} = \sigma = [D]\{\varepsilon\}$$

$$= E \varepsilon, \text{ since } D = E \quad \dots(11.6)$$

Element stiffness matrix

$$\begin{aligned} [k]_e &= \iiint_v [B]^T [D][B] dV \\ &= \int_0^l \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} E \frac{1}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} A dx = \frac{EA}{l_e^2} \int \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \\ &= \frac{EA}{l_e^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} [x]_0^l = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \dots(11.7) \end{aligned}$$

### Consistant Load

Equivalent nodal loads are to be calculated for each type of load acting on the element

**(i) Body Force:**  $X_b$  is the only body force in this case. In case of self weight  $X_b = \rho$  where  $\rho$  is unit weight of the material. From equation 9.26 the consistant load due to this body force is given by

$$\{F\}_e = \iiint_v [N]^T \{X_b\} dV = \int_0^{l_e} \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \rho_b A dx$$

since

$$\xi = \frac{x - x_c}{\frac{l_e}{2}} = \frac{2}{l_e} (x - x_c)$$

we get

$$d\xi = \frac{2}{l_e} dx \text{ or } dx = \frac{l_e}{2} d\xi$$

and limits of integration will be from  $-1$  to  $1$

$$\{F\}_e = \int_{-1}^1 \begin{Bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{Bmatrix} \rho_b A \frac{l_e}{2} d\xi$$

Now 
$$\int \frac{1-\xi}{2} \rho_b A \frac{l_e}{2} d\xi = \frac{l_e}{4} A \rho_b \left[ \xi - \frac{\xi^2}{2} \right]_{-1}^1 = \frac{l_e}{2} A \rho_b$$

Similarly 
$$\int \frac{1+\xi}{2} \rho_b A \frac{l_e}{2} d\xi = \frac{1}{2} A l_e \rho_b$$

$$\therefore \{F\}_e = \frac{A l_e \rho_b}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \dots(11.8)$$

Noting that  $A l_e$  is volume of the element, we find that half the self weight goes to each node.

**(ii) Surface Load:** If  $X_s$  is the intensity of surface load,  $T = X_s \times$  perimeter is the load per unit length of the element. Then consistant load corresponding to it is

$$\begin{aligned} \{F\}_e &= \iint \{N\}^T X_s ds \\ &= \int_0^l \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} T dx = \int_{-1}^1 \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} T_s \frac{l_e}{2} d\xi \\ &= \int_{-1}^1 \begin{Bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{Bmatrix} T_s \frac{l_e}{2} d\xi = \frac{T l_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \end{aligned} \quad \dots(11.9)$$

Thus the constant load for such surface traction is also half the total load at each node.

**(iii) Point Load:** Point loads can be directly added to nodal force vector.

After finding constant load due to all types of loads, element nodal force vector  $\{F\}_e = \begin{Bmatrix} F_{e1} \\ F_{e2} \end{Bmatrix}$  can be assembled. ...(11.10)

**Step 5: Global Properties**

From step 3, we have

$$[k]_{e_1} = \frac{EA_1}{l_{e1}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \quad [k]_{e_2} = \frac{EA_2}{l_{e2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

$$[k]_{e_3} = \frac{EA_3}{l_{e3}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix} \quad [k]_{e_4} = \frac{EA_4}{l_{e4}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} 4 \\ 5 \end{matrix}$$

For each element their position corresponding to global rows and columns are indicated above. Now global stiffness matrix  $\{k\}$  of size  $5 \times 5$  is to be assembled. First this is made a null matrix and then one by one element stiffness matrix is added to corresponding element in global matrix. After first element stiffness matrix is placed in global stiffness matrix, it looks as-

$$E \begin{bmatrix} \frac{A_1}{l_{e1}} & -\frac{A_1}{l_{e1}} & 0 & 0 & 0 \\ -\frac{A_1}{l_{e1}} & \frac{A_1}{l_{e1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

After second element stiffness is placed in global stiffness matrix, it looks as

$$E \begin{bmatrix} \frac{A_1}{l_{e1}} & -\frac{A_1}{l_{e1}} & 0 & 0 & 0 \\ -\frac{A_1}{l_{e1}} & \frac{A_1}{l_{e1}} + \frac{A_2}{l_{e1}} & -\frac{A_2}{l_{e2}} & 0 & 0 \\ 0 & -\frac{A_2}{l_{e2}} & \frac{A_2}{l_{e2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Final stiffness matrix in global system

$$K = E \begin{bmatrix} \frac{A_1}{l_{e1}} & -\frac{A_1}{l_{e1}} & 0 & 0 & 0 \\ -\frac{A_1}{l_{e1}} & \frac{A_1}{l_{e1}} + \frac{A_2}{l_{e2}} & -\frac{A_2}{l_{e2}} & 0 & 0 \\ 0 & -\frac{A_2}{l_{e2}} & \frac{A_2}{l_{e2}} + \frac{A_3}{l_{e3}} & -\frac{A_3}{l_{e3}} & 0 \\ 0 & 0 & -\frac{A_3}{l_{e3}} & \frac{A_3}{l_{e3}} + \frac{A_4}{l_{e4}} & -\frac{A_4}{l_{e4}} \\ 0 & 0 & 0 & -\frac{A_4}{l_{e4}} & \frac{A_4}{l_{e4}} \end{bmatrix} \quad \dots(11.11)$$

Thus we find the stiffness matrix is a symmetric matrix and its half the band width is equal to maximum difference in nodes of any element multiplied by degrees of freedom at each node plus 1, that is 2 in this problem

Load Vector {F}

Load vector  $\{F\}^T = [F_1 \ F_2 \ F_3 \ F_4 \ F_5]$

Let the element load vectors be

$$\begin{aligned} \{F\}_{e1} &= \begin{Bmatrix} F_{11} \\ F_{12} \end{Bmatrix}; & \{F\}_{e2} &= \begin{Bmatrix} F_{21} \\ F_{22} \end{Bmatrix} \\ \{F\}_{e3} &= \begin{Bmatrix} F_{31} \\ F_{32} \end{Bmatrix}; & \{F\}_{e4} &= \begin{Bmatrix} F_{41} \\ F_{42} \end{Bmatrix} \end{aligned}$$

Then global load vector {F} is given by

$$\{F\} = \begin{Bmatrix} F_{11} \\ F_{12} + F_{21} \\ F_{22} + F_{31} \\ F_{32} + F_{41} \\ F_{42} \end{Bmatrix} \quad \dots(11.12)$$

Thus we can assemble global / structure stiffness equation as

$$\underset{5 \times 5}{[k]} \underset{5 \times 1}{\{\delta\}} = \underset{5 \times 1}{\{F\}} \quad \dots(11.13)$$

#### Step 6: Boundary Conditions

In this problem there is only one boundary condition i.e.  $\delta_1 = 0$  or it may have specified value. There are two methods of imposing the boundary conditions:

- (i) Elimination Approach
- (ii) Penalty Approach

**(i) Elimination Approach**

In this method the known displacement is removed from the list of unknowns and the equations are reduced.

If  $\delta_1$  is known displacement, then

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}$$

$$\begin{bmatrix} k_{22} & k_{23} & k_{24} & k_{25} \\ k_{32} & k_{33} & k_{34} & k_{35} \\ k_{42} & k_{43} & k_{44} & k_{45} \\ k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \end{Bmatrix} = \begin{Bmatrix} F_2 - k_{21}\delta_1 \\ F_3 - k_{31}\delta_1 \\ F_4 - k_{41}\delta_1 \\ F_5 - k_{51}\delta_1 \end{Bmatrix} \quad \dots(11.14)$$

Note elements corresponding to row and column of  $k_{11}$  are eliminated. If  $\delta_1$  is zero  $[F]^T = [F_2 \ F_3 \ F_4 \ F_5]$ .

This method is useful when hand calculations are made. If computers are used, computer coding becomes too lengthy.

**(ii) Penalty Approach**

The round off errors involved in computations are advantageously used in imposing boundary conditions. It involve adding a very large number to the diagonal element and right hand side vector corresponding to the displacement on which boundary condition is specified. Thus to impose  $\delta_1 = a_1$ , the modified equation will be

$$\begin{bmatrix} k_{11} + C & k_{12} & k_{13} & k_{14} & k_{15} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \delta_5 \end{Bmatrix} = \begin{Bmatrix} F_1 + Ca_1 \\ F_2 \\ \cdot \\ \cdot \\ F_5 \end{Bmatrix} \quad \dots(11.15)$$

Because of the modified equation 1,  $\delta_1$  results into  $\delta_1 = a_1$

Other values are obtained as usual. Thus the required result is achieved without much changes in computer coding. The value of C selected should be much larger than  $k_{11}$ , not less than  $10^8$  times  $k_{11}$  so that with round off errors  $\delta_1$  comes out to be  $a_1$ . The author used  $C = 1 \times 10^{20}$  to  $1 \times 10^{30}$  and got satisfactory results.

**Step 7: Solution of Simultaneous Equations**

After imposing the boundary conditions, the simultaneous equations 11.13 are to be solved. Any method of solving simultaneous equations can be employed. Gauss elimination is commonly employed. In many programs to save the memory in storing stiffness matrix  $k$ , half the band width of the matrix is stored and Choleski's decomposition method employed. The solution gives the unknown nodal values.

**Step 8: Additional Calculations**

The additional calculations required may be to find strains and stresses at various points. These calculations are carried out element by element. From the list of global nodal values  $\delta$ , for each element nodal values  $\delta_1$  and  $\delta_2$  of the element under consideration is picked up. Then displacement within the element.

$$u = [N]\{\delta\}_e = [N_1 \ N_1] \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

since '  $\xi$  ' coordinate of the point under consideration is known '  $u$  ' can be found. Then

$$\{\varepsilon\} = \varepsilon = [B]\{\delta\}_e$$

and

$$\{\sigma\} = \sigma = [D]\{\varepsilon\}_e = E\varepsilon \quad \dots(11.16)$$

$$= E[B]\{\delta\}_e \quad \dots(11.17)$$

### Calculation of Reactions

Another important stress resultant required in the stress analysis is the reactions at support. This can be found from the equilibrium conditions of the support. For example, in this problem support is at node 1 and at this point displacement  $\delta_1$  is zero. Hence if  $R_1$  is the reaction of the support in direction 1, then

$$k_{11}\delta_1 + k_{12}\delta_2 + k_{13}\delta_3 + k_{14}\delta_4 + k_{15}\delta_5 = F_1 + R_1$$

or

$$R_1 = k_{11}\delta_1 + k_{12}\delta_2 + k_{13}\delta_3 + k_{14}\delta_4 + k_{15}\delta_5 - F_1$$

$$\text{In general } R_1 = k_{11}\delta_1 + k_{12}\delta_2 + \dots + k_{1N}\delta_N - F_1 \quad \dots(11.18)$$

Where  $N$  is total number of nodal displacements

**Example 11.1:** The thin plate of uniform thickness 20 mm, is as shown in Fig. 11.5(a). In addition to the self weight, the plate is subjected to a point load of 400N at mid-depth. The Young's modulus  $E = 2 \times 10^5 \text{ N/mm}^2$  and unit weight  $\rho = 0.8 \times 10^{-4} \text{ N/mm}^2$ . Analyse the plate after modeling it with two elements and find the stresses in each element. Determine the support reactions also.

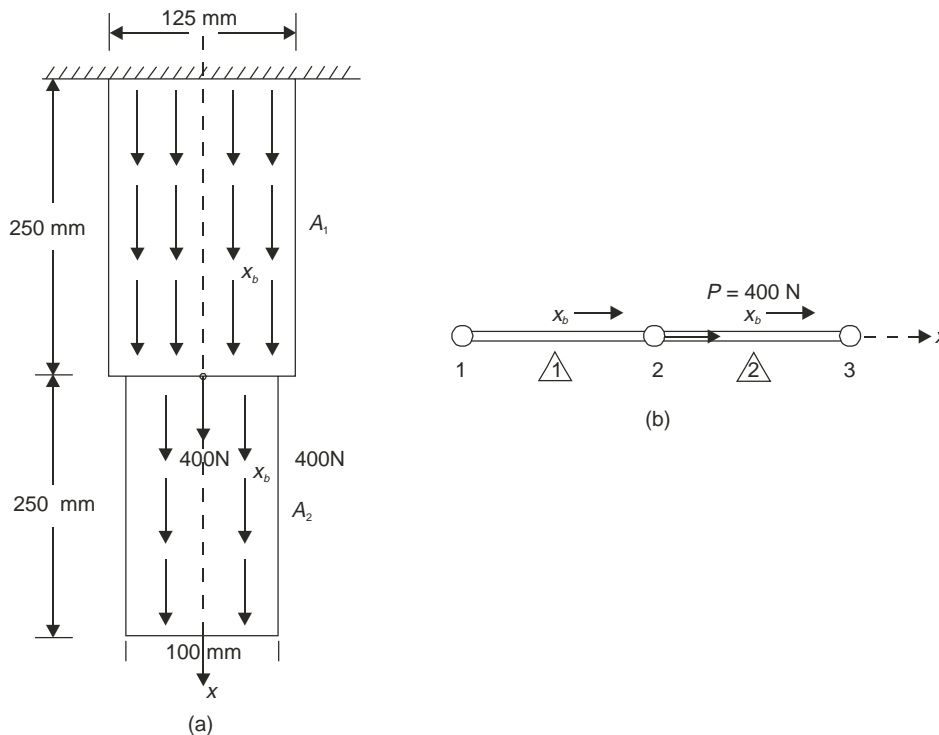


Fig. 11.5

**Solution:**

$$A_1 = 125 \times 20 = 2500 \text{ mm}^2$$

$$A_2 = 100 \times 20 = 2000 \text{ mm}^2$$

The plate is modeled with two elements as shown in Fig. 11.5 (b)

$$[k]_{e1} = \frac{EA_1}{l_{e1}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2 \times 10^5 \times 2500}{250} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2 \times 10^5 \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}$$

$$[k]_{e2} = \frac{EA_2}{l_{e2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2 \times 10^5 \times 2000}{250} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 2 \times 10^5 \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

$$\therefore [k] = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 10 + 8 & -8 \\ & -8 & 8 \end{bmatrix} = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 18 & -8 \\ 0 & -8 & 8 \end{bmatrix}$$

Constant Loads: Due to body force only

$$\{F\}_e = \begin{Bmatrix} F_{e1} \\ F_{e2} \end{Bmatrix} = X_b \frac{Al_e}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\{F\}_{e1} = \begin{Bmatrix} F_{11} \\ F_{12} \end{Bmatrix} = \frac{0.8 \times 10^{-4} \times 2500 \times 250}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 25 \end{Bmatrix}$$

$$\{F\}_{e2} = \begin{Bmatrix} F_{21} \\ F_{22} \end{Bmatrix} = \frac{0.8 \times 10^{-4} \times 2000 \times 250}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 20 \end{Bmatrix}$$

Apart from these there is a 400N concentrated load at node 2. Hence,

$$\{F\} = \begin{Bmatrix} 25 \\ 25 + 20 + 400 \\ 20 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 445 \\ 20 \end{Bmatrix}$$

Hence the stiffness equation is,

$$2 \times 10^5 \begin{bmatrix} 10 & -10 & 0 \\ -10 & 18 & -8 \\ 0 & -8 & 8 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 445 \\ 20 \end{Bmatrix}$$

The boundary condition is  $\delta_1 = 0$ . Hence the reduced equation is,

$$2 \times 10^5 \begin{bmatrix} 18 & -8 \\ -8 & 8 \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} 445 - 10 \times 0 \\ 20 - 0 \times 0 \end{Bmatrix} = \begin{Bmatrix} 445 \\ 20 \end{Bmatrix}$$

$$2 \times 10^5 \begin{bmatrix} 18 & -8 \\ 0 & 8 - \frac{8}{18} \times 8 \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} 445 \\ 20 + \frac{8}{18} \times 445 \end{Bmatrix}$$

i.e.

$$2 \times 10^5 \begin{bmatrix} 18 & -8 \\ 0 & 4.444 \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{bmatrix} 445 \\ 217.778 \end{bmatrix}$$

$$\therefore \delta_3 = \frac{217.778}{4.444 \times 2 \times 10^5} = 2.45 \times 10^{-4} \text{ mm}$$

from equation 1, we have

$$2 \times 10^5 [18\delta_2 - 8\delta_3] = 445$$

$$2 \times 10^5 [18\delta_2 - 8 \times 2.45 \times 10^{-4}] = 445$$

$$18\delta_2 - 1.96 \times 10^{-3} = 2.225 \times 10^{-3}$$

$$\delta_2 = 2.325 \times 10^{-4} \text{ mm}$$

from the relation

$$\sigma = E[B]\{\delta\}_e \text{ we get,}$$

$$\sigma_1 = 2 \times 10^5 \frac{1}{250} [-1 \ 1] \begin{Bmatrix} 0 \\ 2.325 \times 10^{-4} \end{Bmatrix} = 0.186 \text{ N/mm}^2$$

$$\sigma_2 = 2 \times 10^5 \frac{1}{250} [-1 \ 1] \begin{Bmatrix} 2.325 \times 10^{-4} \\ 2.45 \times 10^{-4} \end{Bmatrix} = 0.01 \text{ N/mm}^2$$

### Reaction at Support:

$$R_1 = [k_{11} \ k_{12} \ k_{13}] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} - F_1 = 2 \times 10^5 [10 \ -10 \ 0] \begin{Bmatrix} 0 \\ 2.325 \times 10^{-4} \\ 2.45 \times 10^{-4} \end{Bmatrix} - 25$$

$$\therefore R_1 = 490 \text{ N}$$

[Obviously in this simple problem reaction = total load].

**Example 11.2:** Assemble the element properties for a bar with uniformly varying area subject to self weight only as shown in Fig. 11.6(a). Neglect the possibility of buckling. Treat it as a single element.

$$\xi = \frac{x - x_c}{l_e/2}, d\xi = \frac{2}{l_e} dx \text{ or } dx = \frac{l_e}{2} d\xi$$

$$\{u\} = u = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix}$$

$$\epsilon_x = \frac{du}{dx} = \frac{du}{d\xi} \cdot \frac{d\xi}{dx}$$



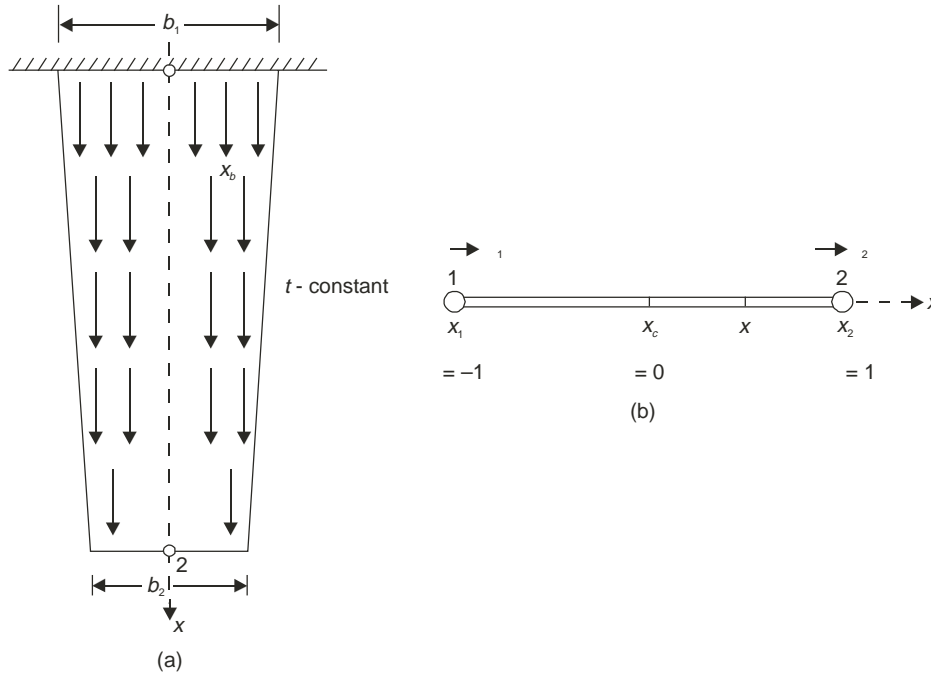


Fig. 11.6

$$= \left[ -\frac{1}{2} \quad \frac{1}{2} \right] \frac{2}{l_e} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \frac{1}{l_e} [-1 \quad 1] \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = [B] \{\delta\}_e$$

where

$$[B] = \frac{1}{l_e} [-1 \quad 1]$$

Area which varies linearly also can be represented in natural coordinate system as

$$A = [N] \{A\} = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}$$

[Check: when  $\xi = -1$ ,  $A = A_1$ ; when  $\xi = +1$ ,  $A = A_2$ ]

**Element stiffness:**

$$\begin{aligned} [k]_e &= \iiint_v [B]^T [D][B] dV = \int_0^{l_e} [B]^T [D][B] A dx \\ [k] &= \int_{-1}^1 [B]^T [D][B] A \frac{l_e}{2} d\xi \\ &= \int_{-1}^1 \frac{1}{l_e} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E \frac{1}{l_e} [-1 \quad 1] \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \frac{l_e}{2} d\xi \end{aligned}$$

$$\begin{aligned}
&= \frac{E}{4l_e} \int_{-1}^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 - \xi & 1 + \xi \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} d\xi \\
&= \frac{E}{4l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \xi - \frac{\xi^2}{2} & \xi + \frac{\xi^2}{2} \end{bmatrix}_{-1}^1 \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \\
&= \frac{E}{4l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \frac{E}{4l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} 2 \{A_1 + A_2\} \\
&= E \frac{A_1 + A_2}{2l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{E \bar{A}}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \dots(11.19)
\end{aligned}$$

where  $\bar{A}$  = Average area =  $\frac{A_1 + A_2}{2}$

### Consistant Load

Only body force is acting and it is in the  $x$ -direction

$$\begin{aligned}
\{X_b\} &= X = \rho A dx = \rho A \frac{l_e}{2} d\xi \\
\{F\}_e &= \iiint_v [N]^T \{X_b\} dV = \int_{-1}^1 \begin{Bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{Bmatrix} \rho A \frac{l_e}{2} d\xi \\
&= \rho \frac{l_e}{2} \int_{-1}^1 \begin{Bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{Bmatrix} \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} d\xi = \frac{\rho l_e}{8} \int_{-1}^1 \begin{bmatrix} (1-\xi)^2 & 1-\xi^2 \\ 1-\xi^2 & (1+\xi)^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} d\xi
\end{aligned}$$

$$\begin{aligned}
\text{Now,} \quad \int_{-1}^1 (1-\xi)^2 d\xi &= \int_{-1}^1 (1-2\xi+\xi^2) d\xi \\
&= \left[ \xi - \xi^2 + \frac{\xi^3}{3} \right]_{-1}^1 \\
&= 1 - 1 + \frac{1}{3} - \left( -1 - 1 - \frac{1}{3} \right) \\
&= \frac{8}{3}
\end{aligned}$$

$$\text{Similarly} \quad \int_{-1}^1 (1+\xi)^2 d\xi = \left[ \xi + \frac{\xi^2}{2} + \frac{\xi^3}{3} \right]_{-1}^1 = 1 + \frac{1}{3} - \left( -1 + \frac{1}{3} \right) = \frac{8}{3}$$

and

$$\int_{-1}^1 (1 + \xi)^2 d\xi = \frac{8}{3}$$

$$\begin{aligned} \therefore \{F\}_e &= \frac{\rho l}{8} \begin{bmatrix} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \\ &= \frac{\rho l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \\ &= \frac{\rho l}{6} \begin{Bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{Bmatrix} \end{aligned}$$

...(11.20)

Hence the element equilibrium equation is

$$\frac{E \bar{A}}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \frac{\rho l}{6} \begin{Bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{Bmatrix} \quad \text{Answer}$$

...(11.21)

**Example 11.3:** Determine the extension of the bar shown in Fig. 11.7 due to self weight and a concentrated load of 400N applied at its end. Given  $b_1 = 150 \text{ mm}$   $b_2 = 75 \text{ mm}$   $t = 20 \text{ mm}$

$$E = 2 \times 10^5 \text{ N/mm}^2 \quad \rho = 0.8 \times 10^{-4} \text{ N/mm}^3$$

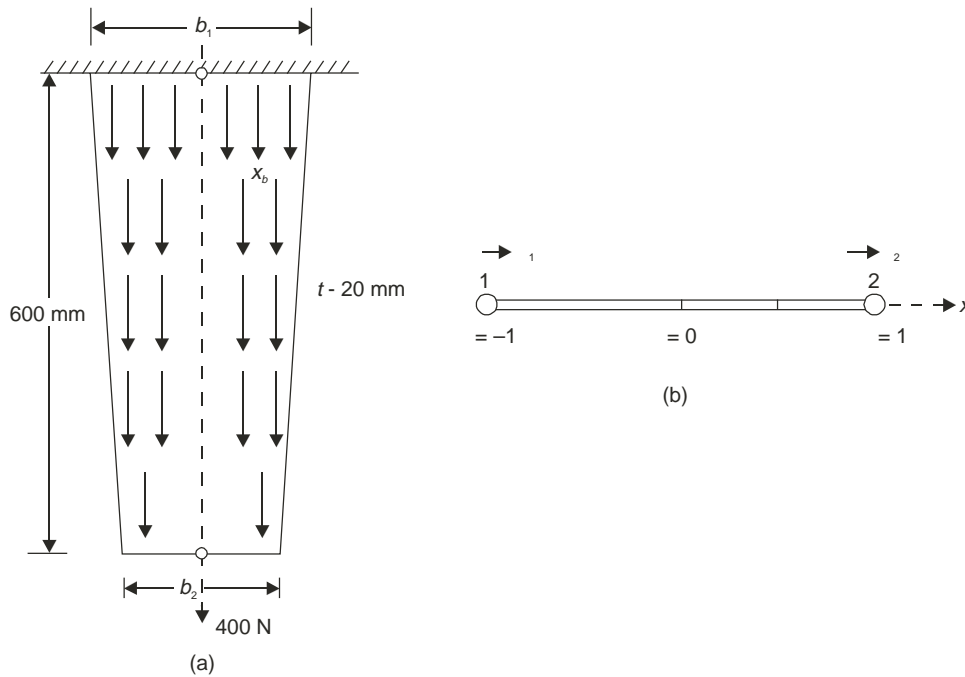


Fig. 11.7

**Solution:**

$$A_1 = 150 \times 20 = 3000 \text{ mm}^2; \quad A_2 = 75 \times 20 = 1500 \text{ mm}^2;$$

$$\therefore \text{Average area } \bar{A} = \frac{3000 + 1500}{2} = 2250 \text{ mm}^2$$

$$\therefore [k] = \frac{E \bar{A}}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2 \times 10^5 \times 2250}{600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

since it is the only element, above expression is global stiffness matrix also. Due to **self weight**

$$\begin{aligned} \{F\}_e &= \frac{\rho l_e}{6} \begin{Bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{Bmatrix} \\ &= \frac{0.8 \times 10^{-4} \times 600}{6} \begin{Bmatrix} 2 \times 3000 + 1500 \\ 3000 + 2 \times 1500 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 48 \end{Bmatrix} \end{aligned}$$

Due to concentrated load

$$\begin{aligned} \{F\}_e &= \begin{Bmatrix} 0 \\ 400 \end{Bmatrix} \\ \therefore \{F\}_e &= \begin{Bmatrix} 60 \\ 448 \end{Bmatrix} \end{aligned}$$

since there is only one element,

$$\{F\} = \{F\}_e = \begin{Bmatrix} 60 \\ 448 \end{Bmatrix}$$

The equation is

$$= \frac{2 \times 10^5 \times 2250}{600} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 448 \end{Bmatrix}$$

The boundary condition is  $\delta_1 = 0$ . Hence the equation reduces to

$$= \frac{2 \times 10^5 \times 2250}{600} \delta_2 = 448$$

or

$$\delta_2 = 5.973 \times 10^{-4} \text{ mm}$$

$$\begin{aligned} \therefore \text{Extension of the bar} &= \delta_2 - \delta_1 \\ &= 5.973 \times 10^{-4} \text{ mm} \end{aligned}$$

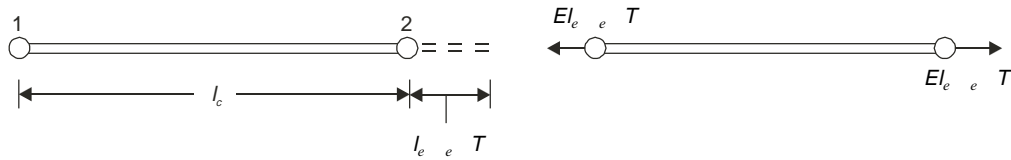
## Temperature Stresses

Change in the temperature in a member causes stresses, if its free expansion is prevented. Let  $\Delta T$  be the rise in temperature and  $\alpha$  be the coefficient of thermal expansion. To find the stresses developed due to change in temperature we can use any one of the following two methods:

- (i) Direct Approach
- (ii) Variational Approach

**(i) Direct Approach**

The free expansion of the element shown in Fig. 11.8(a) will be  $l_e \alpha_e \Delta T$ . This is as good



**Fig. 11.8**

as applying tensile forces  $E_e A_e \alpha_e \Delta T$  at the ends of the element as shown in Fig. 11.8(b). Hence the nodal load vector due to rise in temperature in the element is

$$\{F_{eT}\} = E A \alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \quad \dots(11.23)$$

This may be added to nodal load vector  $\{F\}$  due to the other loads.

If  $u$  is the final displacement of any point in the element, displacement due to elastic strain

$$\begin{aligned} &= \text{Total displacement} - \text{Free expansion} \\ &= u - l_e \alpha_e \Delta T \end{aligned}$$

$\therefore$  Elastic Strain

$$\begin{aligned} &= \varepsilon = [B]\{\delta\}_e - \frac{l_e \alpha_e \Delta T}{l_e} \\ &= [B]\{\delta\}_e - \alpha_e \Delta T \end{aligned} \quad \dots(11.24)$$

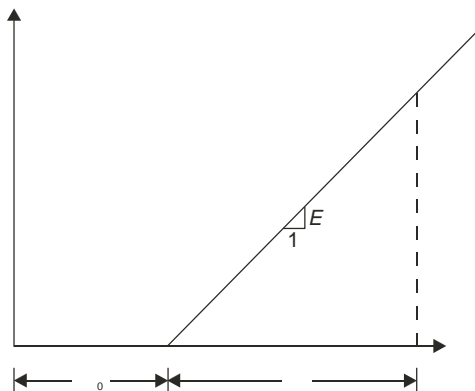
Elastic Stress  $\sigma = E_e \varepsilon$

$$= E_e [B]\{\delta\}_e - E_e \alpha_e \Delta T \quad \dots(11.25)$$

[Note: Free expansion will not cause stresses in a member]

**(ii) Variation Approach**

Strain due to change in temperature may be treated as initial strain as shown in Fig. 11.9



**Fig. 11.9**

Thus,  $\varepsilon_0 = \alpha \Delta T$  ... (11.25)

Stress strain relation is

$$\sigma = E(\varepsilon - \varepsilon_0) \quad \dots(11.26)$$

If our interest is to find the stresses due to temperature rise only, then equation 9.16(b) for total potential energy reduces to

$$\begin{aligned} \Pi &= \frac{1}{2} \iiint_v \{\varepsilon\}^T [D] \{\varepsilon\} dV = \sum_e \frac{1}{2} \int_{-1}^1 \{\varepsilon\}^T E_e \{\varepsilon\} \frac{A_e l_e}{2} d\xi \\ &= \sum_e \frac{1}{4} E_e A_e l_e \int_{-1}^1 (\varepsilon - \varepsilon_0)^T (\varepsilon - \varepsilon_0) d\xi \\ &= \sum_e \frac{1}{4} E_e A_e l_e \int_{-1}^1 ([B]\{\delta\}_e - \varepsilon_0)^T ([B]\{\delta\}_e - \varepsilon_0) d\xi \\ &= \sum_e \frac{1}{4} E_e A_e l_e \int_{-1}^1 (\{\delta\}_e^T [B]^T [B] \{\delta\}_e - 2\{\delta\}_e^T [B]^T \varepsilon_0 + \varepsilon_0^2) d\xi \end{aligned}$$

Minimization of potential energy  $\frac{d\Pi}{d\{\delta\}} = 0$ , gives

$$= \frac{1}{4} E_e A_e l_e \int_{-1}^1 (2[B]^T [B] \{\delta\}_e - 2[B]^T \varepsilon_0 + 0) d\xi = 0$$

The first term correspond to element stiffness matrix as found earlier. The second term corresponds to load vector due to temperature changes. Thus the load vector due to temperature effect is given by

$$\{F_{eT}\} = \frac{E_e A_e l_e}{2} \frac{1}{l_e} \int_{-1}^1 [B]^T \varepsilon_0 d\xi$$

substituting  $[B] = \frac{1}{l_e} [-1 \ 1]$

and

$\varepsilon_0 = \alpha \Delta T$ , we get

$$\begin{aligned} \{F_{eT}\} &= \frac{E_e A_e l_e}{2} \frac{1}{l_e} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \alpha \Delta T \{\varepsilon\}_{-1}^1 \\ &= E_e A_e \alpha \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \quad \dots(11.27) \end{aligned}$$

This load vector may be added to load vector due to body forces, surface forces or the applied load and the problem solved to get displacements due to all the causes. After nodal displacements  $\{\delta\}$  are found, member stress may be found by

$$\begin{aligned} \sigma &= E_e (\varepsilon - \varepsilon_0) = E_e \varepsilon - E_e \varepsilon_0 \\ &= E_e [B] \{\delta\}_e - E_e \alpha_e \Delta T \end{aligned} \quad \dots(11.28)$$

**Example 11.4:** Determine the nodal displacements at node 2, stresses in each material and support reactions in the bar shown in Fig. 11.10, due to applied force  $P = 400 \times 10^3 \text{N}$  and temperature rise of  $30^\circ\text{C}$ . Given:

$$\begin{aligned} A_1 &= 2400 \text{ mm}^2 & A_2 &= 1200 \text{ mm}^2 \\ l_1 &= 300 \text{ mm} & l_2 &= 400 \text{ mm} \\ E_1 &= 0.7 \times 10^5 \text{ N/mm}^2 & E_2 &= 2 \times 10^5 \text{ N/mm}^2 \end{aligned}$$

and  $\alpha_1 = 22 \times 10^{-6}/^\circ\text{C}$        $\alpha_2 = 12 \times 10^{-6}/^\circ\text{C}$

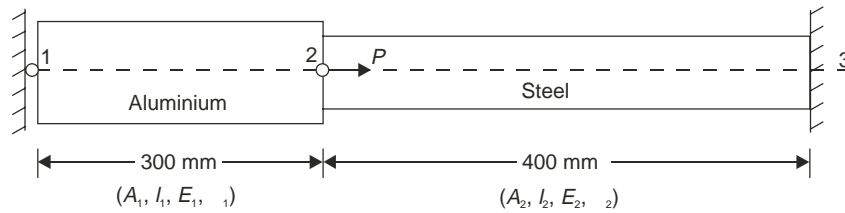


Fig. 11.10

**Solution:**

$$\begin{aligned} [k]_e &= \frac{E_1 A_1}{l_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{0.7 \times 10^5 \times 2400}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{bmatrix} 560 & -560 \\ -560 & 560 \end{bmatrix} \begin{matrix} 1 & 2 \leftarrow \text{Global} \\ \downarrow & \\ & 2 \end{matrix} \\ [k]_2 &= \frac{2 \times 10^5 \times 1200}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^3 \begin{bmatrix} 600 & -600 \\ -600 & 600 \end{bmatrix} \begin{matrix} 2 & 3 \leftarrow \text{Global} \\ \downarrow & \\ & 3 \end{matrix} \\ \therefore [k] &= 10^3 \begin{bmatrix} 560 & -560 & 0 \\ -560 & 560 + 600 & -600 \\ & -600 & 600 \end{bmatrix} = 10^3 \begin{bmatrix} 560 & -560 & 0 \\ -560 & 1160 & -600 \\ 0 & -600 & 600 \end{bmatrix} \end{aligned}$$

**Nodal Force Vector**

Due to temperature changes

$$\{F_{eT}\}_1 = 0.7 \times 10^5 \times 2400 \times 22 \times 10^{-6} \times 30 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -110880 \\ 110880 \end{Bmatrix} \begin{matrix} \text{Global} \\ \downarrow \\ 1 \\ 2 \end{matrix}$$

$$\{F_{eT}\}_2 = 2 \times 10^5 \times 1200 \times 12 \times 10^{-6} \times 30 \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -86400 \\ 86400 \end{Bmatrix} \begin{matrix} \text{Global} \\ 2 \\ 3 \end{matrix}$$

$$\therefore \{F_{eT}\} = \begin{Bmatrix} -110880 \\ 110880 - 86400 \\ 86400 \end{Bmatrix} = \begin{Bmatrix} -110880 \\ 24480 \\ 86400 \end{Bmatrix}$$

Due to applied forces

$$\{F\} = \begin{Bmatrix} 0 \\ 400000 \\ 0 \end{Bmatrix}$$

$\therefore$  Load vector due to applied loads and temperature effect is

$$\{F\} = \begin{Bmatrix} -110880 + 0 \\ 24480 + 400000 \\ 86400 + 0 \end{Bmatrix} = 10^3 \begin{Bmatrix} -110.88 \\ 424.48 \\ 86.40 \end{Bmatrix}$$

The equilibrium equation is

$$10^3 \begin{bmatrix} 560 & -560 & 0 \\ -560 & 1160 & -600 \\ 0 & -600 & 600 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} -110.88 \\ 424.48 \\ 86.40 \end{Bmatrix}$$

The boundary conditions are  $\delta_1 = \delta_3 = 0$

$\therefore$  The equation reduces to

$$1160\delta_2 = 424.48$$

i.e.  $\delta_2 = 0.36593$  **Answer**

$$\sigma = E_e [B] \{\delta\}_e - E_e \alpha_e \Delta T$$

$$\begin{aligned} \therefore \sigma_1 &= 0.7 \times 10^5 \times \frac{1}{300} [-1 \ 1] \begin{Bmatrix} 0 \\ 0.36593 \end{Bmatrix} - 0.7 \times 10^5 \times 22 \times 10^{-6} \times 30 \\ &= 39.18 \text{ N/mm}^2 \quad \text{Answer} \end{aligned}$$



$$\sigma_2 = 2 \times 10^5 \times \frac{1}{400} [-1 \ 1] \begin{Bmatrix} 0.36593 \\ 0 \end{Bmatrix} - 2 \times 10^5 \times 12 \times 10^{-6} \times 30$$

$$= 50.965 \text{ N/mm}^2 \quad \text{Answer}$$

$$R_1 = [k_{11} \ k_{12} \ k_{13}] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} - F_1$$

$$= 10^3 [560 \ -560 \ 0] \begin{Bmatrix} 0 \\ 0.36593 \\ 0 \end{Bmatrix} + 110080 = -94041 \text{ N} \quad \text{Answer}$$

$$R_3 = 10^3 [0 \ -600 \ 600] \begin{Bmatrix} 0 \\ 0.36593 \\ 0 \end{Bmatrix} - 86400 = -305959 \text{ N} \quad \text{Answer}$$

[Check:  $\sum H = 0 \rightarrow -94040 + 400000 - 305959 = 0$ ].

### 11.3 TWO DIMENSIONAL TRUSSES (PLANE TRUSSES)

Fig. 11.11 shows a typical plane truss. The truss may be statically determinate or indeterminate. In the analysis all joints are assumed pin connected and all loads act at joints only. These assumptions result into no bending of any member. All members are subjected to only direct stresses—tensile or compressive. Now we are interested to see the finite element analysis procedure for such trusses.

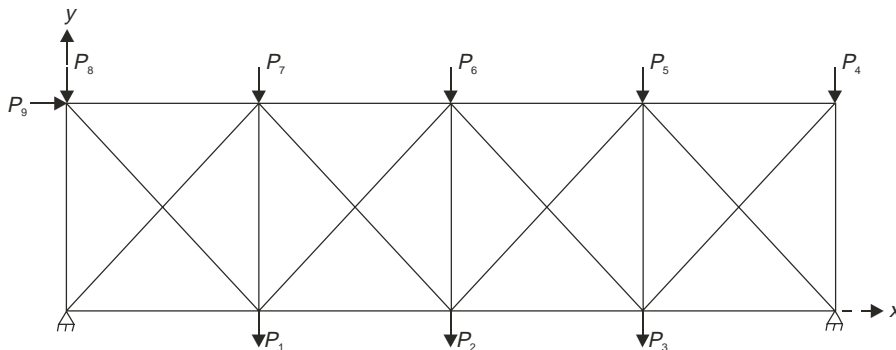


Fig. 11.11 A typical plane truss

#### Step 1: Field Variables and Elements

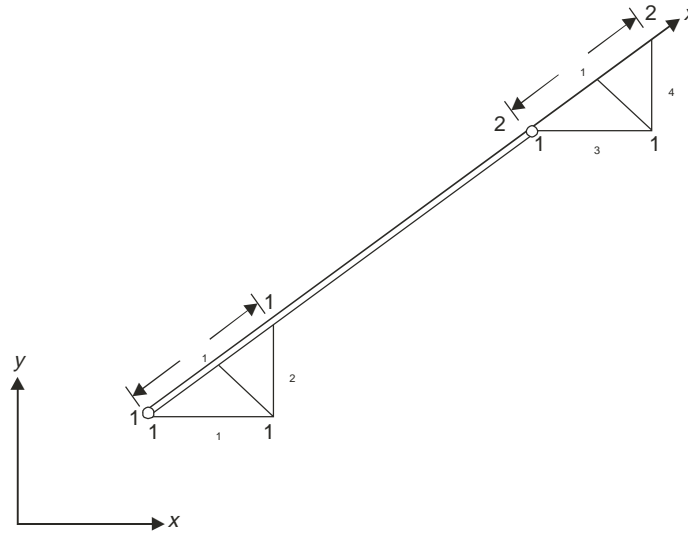
Joint displacements are selected as basic field variables. Since there is no bending of the members, we have to ensure only displacement continuity ( $C^0$ -continuity) and there is no need to worry about slope continuity ( $C^1$ -continuity). Hence we select two noded bar elements for the analysis of trusses. Since the members are subjected to only axial forces, the displacements are only in the axial directions of the members. Therefore the

nodal variable vector for the typical bar element shown in Fig. 11.12 is

$$\{\delta'\} = \begin{Bmatrix} \delta'_1 \\ \delta'_2 \end{Bmatrix} \quad \dots(11.29)$$

where  $\delta'_1$  and  $\delta'_2$  are in the axial directions of the element. But the axial direction is not same for all members. If we select  $x$ - $y$  as global coordinate system, there are two displacement components at every node. Hence the nodal variable vector for a typical element is,

$$\{\delta\}^T = [\delta_1 \ \delta_2 \ \delta_3 \ \delta_4] \quad \dots(11.30)$$



**Fig. 11.12** Typical element and its nodal displacements

as shown in Fig. 11.12

From the Figure it is clear that

$$\delta'_1 = \delta_1 \cos \theta + \delta_2 \sin \theta$$

$$\delta'_2 = \delta_3 \cos \theta + \delta_4 \sin \theta$$

If  $l$  and  $m$  are the direction cosines,

$$l = \cos \theta, \quad m = \sin \theta,$$

$$\therefore \delta'_1 = l \delta_1 + m \delta_2$$

$$\delta'_2 = l \delta_3 + m \delta_4$$

i.e.

$$\{\delta'\} = \begin{Bmatrix} \delta'_1 \\ \delta'_2 \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}$$

i.e.

$$\{\delta'\} = [L]\{\delta\} \quad \dots(11.31)$$

where  $[L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$

and  $[L]$  is called transformation (or rotation) matrix. If the coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  of node 1 and 2 of the elements are known, we can find

$$l = \frac{x_2 - x_1}{l_e}, m = \frac{y_2 - y_1}{l_e}$$

where  $l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$  ... (11.32)

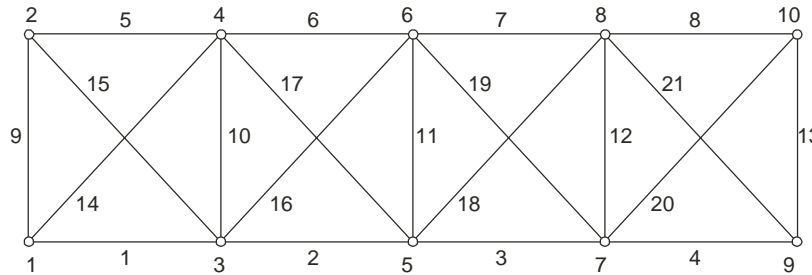
**Step 2:** Discretising

A member may be taken as an element conveniently. Hence in the typical truss considered. There are

- (a) 4 – top chord members
- (b) 4 – bottom chord members
- (c) 5 – vertical members and
- (d) 8 – diagonal members

∴ Total elements selected are –21

There are 10 nodal points and they are numbered as shown in Fig. 11.13.



**Fig. 11.13** Numbering nodes and members

The numbering is such that the band width is minimum. In this case maximum difference in the node numbers of an element is in diagonal members and is equal to 3. The degree of freedom of each node is 2, one in x-direction and another in y-direction. Hence the maximum band width

$$= (3 + 1) \times 2 = 8$$

Total degrees of freedom is

$$\begin{aligned} &= \text{Total number of nodes} \times \text{degree of freedom of each node} \\ &= 10 \times 2 = 20 \end{aligned}$$

$$\therefore \{\delta\}^T = [\delta_1 \ \delta_2 \ \delta_3 \ \delta_4 \ \dots \ \delta_{19} \ \delta_{20}]$$

The nodal connectivity details is as given below:

Element No.	Element Node 1	Element Node 2
1	1	3
2	3	5
3	5	7
:	:	:
:	:	:
20	7	10
21	8	9

Global  
numbers

**Step 3:** Interpolation Functions

Since bar element is used,

$$\{u\} = [N]\{\delta'\}$$

where

$$[N] = [N_1 \ N_2] = \left[ \frac{x'_2 - x'}{l} \quad \frac{x' - x'_1}{l} \right] = \left[ \frac{1 - \xi'}{2} \quad \frac{1 + \xi'}{2} \right]$$

**Step 4:** Element Properties

**(a) Stiffness Matrix:** In the analysis of bars and columns, we have seen the element stiffness matrix is

$$[k]_e = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

when viewed in local coordinate system, the truss is also a one dimensional two noded bar element. Hence the element stiffness matrix of truss element in local coordinate system,  $[k']_e$  is given by

$$\begin{aligned} [k']_e &= \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \therefore U_e &= \frac{1}{2} \{\delta'\}^T [k'] \{\delta'\} \\ \{\delta'\} &= [L] \{\delta\} \\ \therefore U_e &= \frac{1}{2} ([L]\{\delta\})^T [k'] [L]\{\delta\} \\ &= \frac{1}{2} \{\delta\}^T [L]^T [k'] [L] \{\delta\} = \frac{1}{2} \{\delta\}^T [k_e] \{\delta\} \end{aligned}$$

where

$$[k]_e = [L]^T [k'] [L] \quad \dots(11.33a)$$

and it may be called as element stiffness matrix in global coordinate system.

$$\therefore [k]_e = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{E_e A_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$= \frac{E_e A_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \begin{bmatrix} l & m & -l & -m \\ -l & -m & l & m \end{bmatrix} = \frac{E_e A_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \quad \dots(11.33b)$$

[Note: The above expression is same as equation 3.6, which was obtained by direct approach]

### (b) Consistant Loads:

Since all loads are acting at joints which are node points also, the load vector can be assembled straight way. Normally load vector  $\{F\}$  of size equal to degrees of freedom is generated as null vector and then load vector values are supplied. In the problem shown in Fig. 11.11,

$$\begin{aligned} F_6 &= -P_1, & F_{10} &= -P_2, & F_{14} &= -P_3, & F_{16} &= -P_5 \text{ and } F_{20} &= -P_4 \\ F_3 &= -P_9, & F_8 &= -P_7, & F_{12} &= -P_6, \end{aligned}$$

### Step 5: Global Properties

It may be noted that for member  $ij$ ,

$$\begin{aligned} \delta_1 &= \delta_{2i-1}, \quad \delta_2 = \delta_{2i}, \\ \delta_3 &= \delta_{2j-1} \text{ and } \delta_4 = \delta_{2j-1} \end{aligned}$$

Hence the position of elements of  $[k]_e$  in global stiffness matrix for member  $ij$  are as shown below:

$$[k]_e = \frac{E_e A_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \begin{matrix} 2i-1 \\ 2i \\ 2j-1 \\ 2j \end{matrix}$$

The global stiffness matrix of required size is initially developed as null matrix. Then for element number 1, element matrix is generated and they are added to the existing values of elements of global matrix at approximate places. Then next element stiffness matrix is generated and placed in appropriate positions in global matrix. The process is continued till all elements are handled.

Assembling of load vector is already explained in step 4.

### Step 6: Boundary Conditions

If hand calculations are made usually elimination approach is used and if computers are used penalty approach is used for imposing boundary conditions. The method is exactly same as explained in the analysis of columns and tension members.

### Step 7: Solution of Simultaneous Equations

This step is also same as explained in the analysis of tension bars and columns.

### Step 8: Additional Calculations

Analysts are interested in finding stresses and forces in the members of the truss. We know

$$\sigma = E_e \varepsilon$$

But

$$\varepsilon = \frac{\text{Change in length}}{\text{Original length}} = \frac{\delta'_2 - \delta''_1}{l_e}$$

$$\begin{aligned}\therefore \sigma &= E_e = \frac{\delta'_2 - \delta'_1}{l_e} \\ &= \frac{E_e}{l_e} [-1 \quad 1] \begin{Bmatrix} \delta'_1 \\ \delta'_2 \end{Bmatrix}\end{aligned}$$

But

$$\begin{aligned}\{\delta'\} &= [L]\{\delta\} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \\ \therefore \sigma &= \frac{E_e}{l_e} [-1 \quad 1] \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \{\delta\} \\ &= \frac{E_e}{l_e} [-l \quad -m \quad l \quad m] \{\delta\} \quad \dots(11.34)\end{aligned}$$

Using equation 11.34 stresses are calculated in all the elements / members. If forces are required the stresses may be multiplied by the cross sectional areas. Positive value indicates tension and the negative compression. The reactions at supports may be calculated on the lines explained in the analysis of tension bars and columns. To make the analysis procedure clear, a small problem is solved below with hand calculations.

**Example 11.4:** For the three – bar truss shown in Fig. 11.14, determine the nodal displacements and the stress in each member. Find the support reactions also. Take modulus of elasticity as 200 GPa.

**Solution:** Element numbers, node numbers and displacement numbers are as shown in Fig. 11.15. Taking node 1 as the origin, the coordinates of various nodes are 1 (0,0), 2 (800, 0), 3 (400, 400).

$$\therefore l_{e1} = \sqrt{(800 - 0)^2 + (0 - 0)^2} = 800 \text{ mm}$$

$$l_{e2} = \sqrt{(400 - 800)^2 + (400 - 0)^2} = 400\sqrt{2} \text{ mm}$$

$$l_{e3} = \sqrt{(400 - 0)^2 + (400 - 0)^2} = 400\sqrt{2} \text{ mm}$$

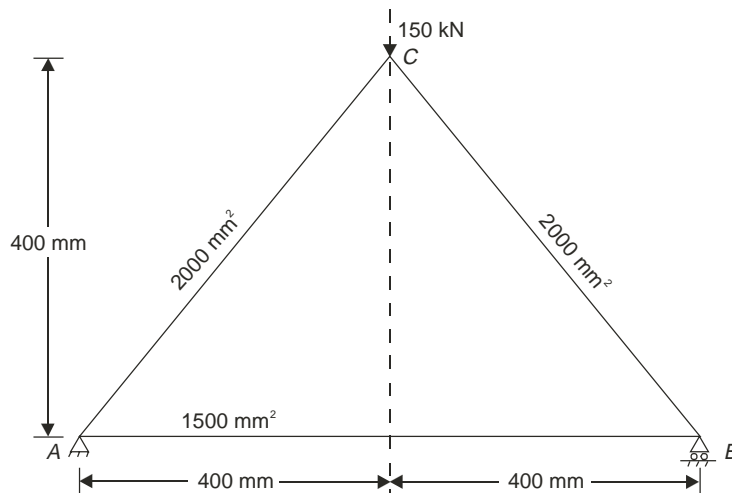


Fig. 11.14

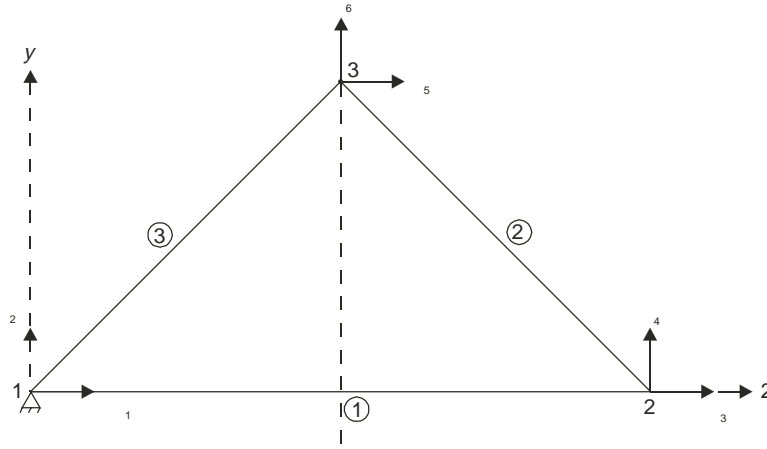


Fig. 11.15

Element No.	Node 1	Node 2	Element
1	1	2	Global
2	2	3	
3	1	3	

$$\begin{aligned}
 \theta_1 &= 0 & \theta_2 &= 135^\circ & \theta_3 &= 45^\circ \\
 l_1 &= 1.0 & l_2 &= -0.707 & l_3 &= 0.707 \\
 m_1 &= 0 & m_2 &= 0.707 & m_3 &= 0.707 \\
 E_1 &= E_2 = E_3 = 200\text{GPa} = 200\text{ kN/mm}^2
 \end{aligned}$$

$$[k]_{e1} = \frac{E_1 A_1}{l_{e1}} \begin{bmatrix} l_1^2 & l_1 m_1 & -l_1^2 & -l_1 m_1 \\ l_1 m_1 & m_1^2 & -l_1 m_1 & -m_1^2 \\ -l_1^2 & -l_1 m_1 & l_1^2 & l_1 m_1 \\ -l_1 m_1 & -m_1^2 & l_1 m_1 & m_1^2 \end{bmatrix} = \frac{200 \times 1500}{800} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 375 & 0 & -375 & 0 \\ 0 & 0 & 0 & 0 \\ -375 & 0 & 375 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Global Numbers} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$[k]_{e2} = \frac{200 \times 2000}{400\sqrt{2}} \begin{bmatrix} 3 & 4 & 5 & 6 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{matrix} \text{Global numbers} \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} 3 & 4 & 5 & 6 \\ 353.55 & -353.55 & -353.55 & 353.55 \\ -353.55 & 353.55 & 353.55 & -353.55 \\ -353.55 & 353.55 & 353.55 & -353.55 \\ 353.55 & -353.55 & -353.55 & 353.55 \end{bmatrix} \begin{matrix} \text{Global numbers} \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \\
 [k]_{e3} &= \frac{200 \times 2000}{400\sqrt{2}} \begin{bmatrix} 1 & 2 & 5 & 6 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{matrix} \text{Global numbers} \\ 1 \\ 2 \\ 5 \\ 6 \end{matrix} \\
 &= \begin{bmatrix} 1 & 2 & 5 & 6 \\ 353.55 & 353.55 & -353.55 & -353.55 \\ 353.55 & 353.55 & -353.55 & -353.55 \\ -353.55 & -353.55 & 353.55 & 353.55 \\ -353.55 & -353.55 & 353.55 & 353.55 \end{bmatrix} \begin{matrix} \text{Global numbers} \\ 1 \\ 2 \\ 5 \\ 6 \end{matrix} \\
 [k] &= \begin{bmatrix} 375.0+ & 0+ & -375.0+ & 0+ & 0+ & 0+ \\ 353.55 & 353.55 & 0 & 0 & -353.55 & -353.55 \\ 0+ & 0+ & 0+ & 0+ & 0+ & 0+ \\ 353.55 & 353.55 & 0 & 0 & -353.55 & -353.55 \\ -375.0+ & 0+ & 375.0+ & 0+ & 0+ & 0+ \\ 0 & 0 & 353.55 & -353.55 & -353.55 & 353.55 \\ 0+ & 0+ & 0+ & 0+ & 0+ & 0+ \\ 0 & 0 & -353.55 & 353.55 & 353.55 & -353.55 \\ 0+ & 0+ & -353.55+ & 353.55+ & 353.55+ & -353.55+ \\ -353.55 & -353.55 & 0 & 0 & 353.55 & 353.55 \\ 0+ & 0+ & 353.55+ & -353.55+ & -353.55+ & 353.55+ \\ -353.55 & -353.55 & 0 & 0 & 353.55 & 353.55 \end{bmatrix} \\
 &F_1 = F_2 = F_3 = F_4 = F_5 = 0 \text{ and } F_6 = -150 \\
 \therefore &\begin{bmatrix} 728.55 & 353.55 & -375.0 & 0 & -353.55 & -353.55 \\ 353.55 & 353.55 & 0 & 0 & -353.55 & -353.55 \\ -375.0 & 0 & 728.55 & -353.75 & -353.55 & 353.55 \\ 0 & 0 & -353.55 & 353.55 & 353.55 & -353.55 \\ -353.55 & -353.55 & -353.55 & 353.55 & 707.1 & 0 \\ -353.55 & -353.55 & 353.55 & -353.55 & 0 & 707.1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -150 \end{Bmatrix}
 \end{aligned}$$



The boundary conditions are,

$$\delta_1 = \delta_2 = \delta_4 = 0$$

Hence the equation reduces to

$$\begin{bmatrix} 728.55 & -353.55 & 353.55 \\ -353.55 & 707.10 & 0 \\ 353.55 & 0 & 707.1 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -150 \end{Bmatrix}$$

$$\begin{bmatrix} 728.55 & -353.55 & 353.55 \\ 0 & 535.53 & 171.57 \\ 0 & 171.57 & 535.53 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -150 \end{Bmatrix}$$

$$\therefore \begin{bmatrix} 728.55 & -353.55 & 353.55 \\ 0 & 556.98 & 171.57 \\ 0 & 0 & 480.56 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -150 \end{Bmatrix}$$

from equation 3;  $480.56\delta_6 = -150$  i.e.  $\delta_6 = -0.312$  mm

from equation 2;  $556.98\delta_5 + 171.57(-0.312) = 0$

$$\delta_5 = 0.1 \text{ mm}$$

from equation 1;  $728.55\delta_3 - 353.55(0.1) + 353.55(-0.312) = 0$

$$\therefore \delta_3 = \frac{353.55 \times 0.1 + 353.55 \times 0.312}{728.55} = 0.2 \text{ mm}$$

$$\therefore \sigma_1 = \frac{E_1}{l_1} \begin{bmatrix} -l_1 & -m_1 & l_1 & m_1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix}$$

$$= \frac{200}{800} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.2 \\ 0 \end{Bmatrix} = -0.05 \text{ Answer}$$

$$\therefore P_1 = \sigma_1 A_1 = -0.05 \times 1500 = -75 \text{ kN Answer}$$

$$\sigma_2 = \frac{200}{400\sqrt{2}} \begin{bmatrix} 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix} \begin{Bmatrix} 0.2 \\ 0.0 \\ 0.1 \\ -0.312 \end{Bmatrix}$$

$$= \frac{1}{2\sqrt{2}} [0.707(0.2 - 0 - 0.1 - 0.312)] = 0.053 \quad \text{Answer}$$

$$\therefore P_2 = \sigma_2 A_2 = -0.053 \times 2000 = 106 \text{ kN} \quad \text{Answer}$$

$$\sigma_3 = \frac{200}{400\sqrt{2}} [-0.707 \quad -0.707 \quad 0.707 \quad 0.707] \begin{Bmatrix} 0.2 \\ 0.0 \\ 0.1 \\ -0.312 \end{Bmatrix}$$

$$= \frac{1}{2\sqrt{2}} [0.707(-0 - 0 + 0.1 - 0.312)] = 0.053 \text{ kN/mm}^2 \quad \text{Answer}$$

$$\therefore P_3 = \sigma_3 A_3 = 0.053 \times 2000 = 106 \text{ kN} \quad \text{Answer}$$

$$R_1 + 0 = [728.55 \quad 353.55 \quad -375.0 \quad 0 \quad -353.55 \quad -353.55] \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.2 \\ 0.0 \\ 0.1 \\ -0.312 \end{Bmatrix}$$

$$\therefore R_1 = 0 \quad \text{Answer}$$

$$R_2 + 0 = [353.55 \quad 353.55 \quad 0 \quad 0 \quad -353.55 \quad -353.55] \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.2 \\ 0.0 \\ 0.1 \\ -0.312 \end{Bmatrix}$$

$$\therefore R_2 = 75 \text{ kN} \quad \text{Answer}$$

$$\text{Similarly } R_3 = 75 \text{ kN} \quad \text{Answer}$$

**Example 11.5:** If the support B of truss shown in Fig. 11.15 yields by 0.1 mm, determine the member forces due to applied load and yielding of the support.

**Solution:** The equation of equilibrium remains as in the previous problem. Only boundary condition changes. In this case  $\delta_4 = 0.1 \text{ mm}$ . The other two boundary conditions are same as in the previous case i.e.,  $\delta_1 = \delta_2 = 0$ .

Imposing the boundary conditions  $\delta_1 = \delta_2 = 0$ , the equation reduces to

$$\begin{bmatrix} 728.55 & -353.55 & -353.55 & 353.55 \\ -353.55 & 353.55 & 353.55 & -353.55 \\ -353.55 & 353.55 & 707.1 & 0 \\ 353.55 & -353.55 & 0 & 707.1 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -150 \end{Bmatrix}$$

Introducing  $\delta_4 = -0.1$  we get

$$\begin{bmatrix} 728.55 & -353.55 & 353.55 \\ -353.55 & 707.1 & 0 \\ 353.55 & 0 & 707.1 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 - 353.55 \times 0.1 \\ 0 + 353.55 \times 0.1 \\ -150 - 353.55 \times 0.1 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} 728.55 & -353.55 & 353.55 \\ -353.55 & 707.1 & 0 \\ 353.55 & 0 & 707.1 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} -35.355 \\ +35.355 \\ -185.355 \end{Bmatrix}$$

$$\begin{bmatrix} 728.55 & -353.55 & 353.55 \\ 0 & 535.53 & 171.57 \\ 0 & 171.57 & 535.53 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 35.355 \\ -18.198 \\ -168.198 \end{Bmatrix}$$

$$\therefore \begin{bmatrix} 728.55 & -353.55 & 353.55 \\ 0 & 535.53 & 171.57 \\ 0 & 0 & 480.56 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 35.355 \\ -18.198 \\ -111.957 \end{Bmatrix}$$

$$\delta_6 = -\frac{111.318}{480.56} = -0.2318 \text{ mm}$$

$$535.53 \delta_5 + 171.57 \times (0.2318) = -18.198$$

$$\therefore \delta_5 = 0.04028 \text{ mm}$$

$$728.55 \delta_3 - 353.55(0.04028) + 353.55(-0.2318) = 35.355$$

$$\delta_3 = 0.18056 \text{ mm}$$

$$\begin{aligned} P_1 &= \frac{E_1 A_1}{l_{e1}} \begin{bmatrix} -l_1 & -m_1 & l_1 & m_1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \\ &= \frac{200 \times 1500}{800} \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.18056 \\ -0.1 \end{Bmatrix} = 67.71 \text{ kN} \end{aligned}$$

$$P_2 = \frac{200 \times 2000}{400\sqrt{2}} [0.707 \quad -0.707 \quad -0.707 \quad 0.707] \begin{Bmatrix} 0.18056 \\ -0.1 \\ 0.04028 \\ -0.2318 \end{Bmatrix} = 4.24 \text{ kN}$$

$$P_3 = \frac{200 \times 2000}{400\sqrt{2}} [-0.707 \quad -0.707 \quad 0.707 \quad 0.707] \begin{Bmatrix} 0 \\ 0 \\ 0.04028 \\ -0.2318 \end{Bmatrix} = -95.76 \text{ k}$$

$$\{P\} = \begin{Bmatrix} 67.71 \\ 4.24 \\ -95.76 \end{Bmatrix} \quad \text{Answer}$$

### Temperature Stresses

(i) **Direct Approach:** When viewed in local coordinate system, this element is one dimensional and for such element thermal forces due to  $\Delta T$  rise in temperature are  $E_e A_e \alpha_e \Delta T$  as shown in Fig. 11.16

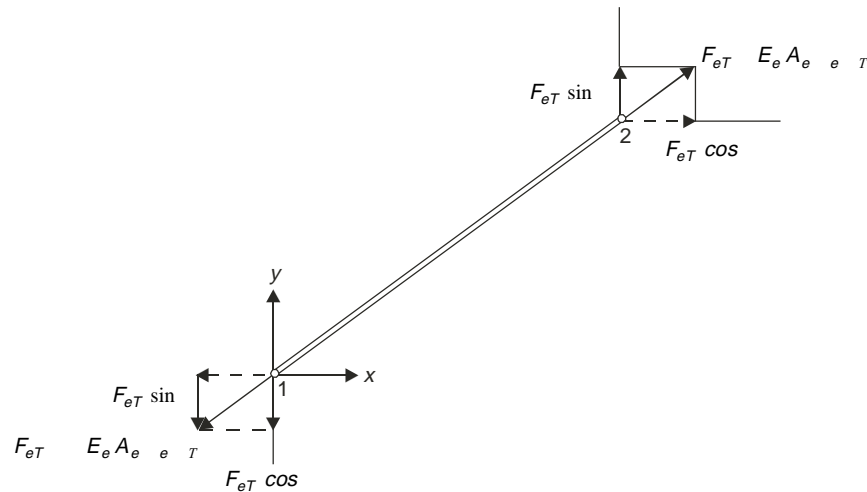


Fig. 11.16

$$\therefore \langle F'_{eT} \rangle = E_e A_e \alpha_e \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

where  $\{F'_{eT}\}$  is load vector due to temperature effect in local coordinate system.

In global  $x$ - $y$  system, it is seen easily that the load vector is,

$$\{F'_{eT}\} = E_e A_e \alpha_e \Delta T \begin{Bmatrix} -1 \times \cos \alpha \\ -1 \times \sin \alpha \\ 1 \times \cos \alpha \\ 1 \times \sin \alpha \end{Bmatrix} = E_e A_e \alpha_e \Delta T \begin{Bmatrix} -l \\ -m \\ l \\ m \end{Bmatrix}$$

**(ii) Variational Approach:** If we represent the load vector due to temperature effect by  $\{F'_{eT}\}$  in local coordinate system, then from equation 11.27, we get,

$$\therefore \langle F'_{eT} \rangle = E_e A_e \alpha_e \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

Let  $\{F'_{eT}\}$  be the corresponding load vector in global system. Then potential energy due to this load is obviously same whether expressed in local or in global coordinates system. Thus,

$$\{\delta'\}_e^T \{F'_{eT}\} = \{\delta\}_e^T \{F_{eT}\}$$

but from equation 11.31,

$$\{\delta'\}_e = [L]\{\delta\}_e$$

$$\therefore \{\delta\}_e^T [L]^T \{F'_{eT}\} = \{\delta\}_e^T \{F_{eT}\}$$

i.e.,

$$\{F_{eT}\} = [L]^T \{F'_{eT}\}$$

$$= \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} E_e A_e \alpha_e \Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = E_e A_e \alpha_e \Delta T \begin{Bmatrix} -l \\ -m \\ l \\ m \end{Bmatrix} \quad \dots(11.35)$$

The stress and force in the element can be found as was done in case of tension bars /columns

i.e.,

$$\begin{aligned} \sigma &= E(\varepsilon - \varepsilon_0) \\ &= \frac{E_e}{l_e} ([B]\{\delta\}_e - l_e \alpha \Delta T) = \frac{E_e}{l_e} [-l \quad -m \quad l \quad m] \{\delta\}_e - E_e \alpha \Delta T \end{aligned} \quad \dots(11.36)$$

and

$$P = A\sigma = \frac{E_e A_e}{l_e} [-l \quad -m \quad l \quad m] \{\delta\}_e - E_e A_e \alpha \Delta T \quad \dots(11.37)$$

### Stresses Due to Lack of Fit

At the time of fabricating the statically indeterminate frames, if a member is found to be slightly shorter or longer (lacks in Fit), the member is forced in position. This is possible by introducing initial forces in the member. If the member is longer by  $\delta l$ , the initial force applied on joint in outward direction is,

$$P_0 = \frac{E_e A_e}{l_e} \delta l$$

i.e. initial stress =  $E_e \frac{\delta l}{l_e}$

or initial strain  $\epsilon_0 = \frac{\delta l}{l_e}$

This problem of initial stress may be handled on the lines similar to stresses due to temperature effect.

The term  $\alpha \Delta T$  in case of temperature effect is to be replaced by  $\frac{\delta l}{l_e}$  in this case.

**Example 11.5:** Fig. 11.17 shows an indeterminate pin connected plane stress with cross sectional area of diagonal members equal to 2000 mm<sup>2</sup> and all other members with cross sectional area of 1000 mm<sup>2</sup>. If Young's modulus  $E = 200\text{kN/mm}^2$

- (i) Assemble global stiffness matrix
- (ii) Determine load vector if temperature of member 1–3 increases by 25°C. Given  $\alpha = 12 \times 10^{-6}/^\circ\text{C}$
- (iii) Determine load vector if member 1–3 is longer by 0.2 mm.
- (iv) Introduce Boundary Conditions

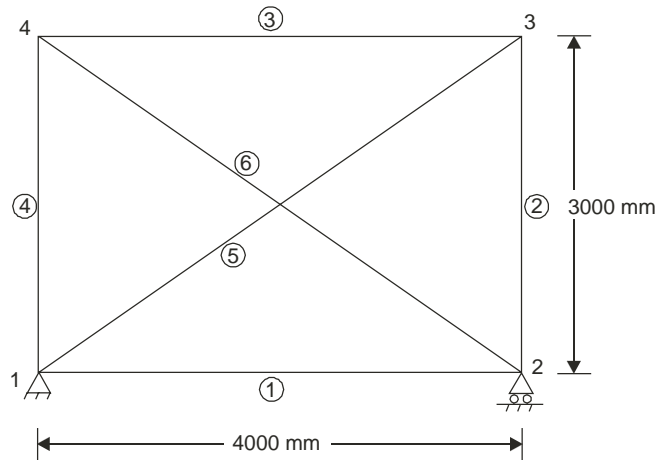


Fig. 11.17

**Solution:** Joints and members are numbered as shown in Fig. 11.17.

Member No.	End 1	End 2	L	m	$l_e$ in mm	$A_e$ in mm <sup>2</sup>
1	1	2	1.0	0.0	4000	1000
2	2	3	0.0	1.0	3000	1000
3	3	4	-1.0	0.0	4000	1000
4	1	4	0.0	1.0	3000	1000
5	1	3	0.8	0.6	5000.0	2000
6	2	4	-0.8	0.6	5000.0	2000

We know

$$[k_e] = \frac{E_e A_e}{l_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$[k]_1 = \frac{200 \times 1000}{4000} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Global} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} = \begin{bmatrix} 50 & 0 & -50 & 0 \\ 0 & 0 & 0 & 0 \\ -50 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \text{Global} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$[k]_2 = \frac{200 \times 1000}{3000} \begin{bmatrix} 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{Global} \\ \downarrow \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 66.67 & 0 & -66.67 \\ 0 & 0 & 0 & 0 \\ 0 & -66.67 & 0 & 66.67 \end{bmatrix} \begin{matrix} \leftarrow \text{Global} \\ \downarrow \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$[k]_3 = \begin{bmatrix} 50 & 0 & -50 & 0 \\ 0 & 0 & 0 & 0 \\ -50 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \text{Global} \\ \downarrow \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

$$k_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 66.67 & 0 & -66.67 \\ 0 & 0 & 0 & 0 \\ 0 & -66.67 & 0 & 66.67 \end{bmatrix} \begin{matrix} \leftarrow \text{Global} \\ \downarrow \\ 1 \\ 2 \\ 7 \\ 8 \end{matrix}$$

$$k_5 = \frac{200 \times 2000}{5000} \begin{bmatrix} 1 & 2 & 5 & 6 \\ 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix} \begin{matrix} \leftarrow \text{Global} \\ \downarrow \\ 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

$$k_6 = \begin{bmatrix} 1 & 2 & 5 & 6 & \leftarrow \text{Global} \\ \downarrow \\ 51.2 & 38.4 & -51.2 & -38.4 & 1 \\ 38.4 & 28.8 & -38.4 & -28.8 & 2 \\ -51.2 & -38.4 & 51.2 & 38.4 & 5 \\ -38.4 & -28.8 & 38.4 & 28.8 & 6 \\ 3 & 4 & 7 & 8 & \leftarrow \text{Global} \\ \downarrow \\ 51.2 & -38.4 & -51.2 & 38.4 & 3 \\ -38.4 & 28.8 & 38.4 & -28.8 & 4 \\ -51.2 & 38.4 & 51.2 & -38.4 & 7 \\ 38.4 & -28.8 & -38.4 & 28.8 & 8 \end{bmatrix}$$

Global Stiffness Matrix  $k$

1	2	3	4	5	6	7	8	
50.0	0	-50.0	0	0	0	0	0	1
0	0			0	0			
51.2	38.4			51.2	-38.4			
0	0	0	0					2
0	66.67					0	-66.67	
38.4	28.87			-38.4	-28.8			
-50	0	50.0	0	0	0			3
		0	0	0	0			
		51.2	-38.4			-51.2	38.4	
0	0	0	0	0	-66.67			4
		0	66.67	0				
		-38.4	28.80			38.4	-28.8	
		0	0	0	0			5
				50	0	-50	0	
-51.2	-38.4			51.2	38.4			
		0	-66.67	0	66.67			6
				0	0	0	0	
-38.4	-28.8			38.4	28.8			
0	0			-50	0	50	0	7
						0	0	
		-51.2	38.4			51.2	-38.4	
0	-66.67			0	0	0	0	8
		38.4	-28.8			-38.4	28.80	



$$k = \begin{bmatrix} 101.2 & 38.4 & -50 & 0 & -51.2 & -38.4 & 0 & 0 \\ 38.4 & 95.47 & 0 & 0 & -38.4 & -28.8 & 0 & -66.67 \\ -50 & 0 & 101.2 & -38.4 & 0 & 0 & -51.2 & 38.4 \\ 0 & 0 & -38.4 & 95.47 & 0 & -66.67 & 38.4 & -28.8 \\ -51.2 & -38.40 & 0 & 0 & 101.2 & 38.4 & -50 & 0 \\ -38.4 & -28.8 & 0 & -66.67 & 38.4 & 95.47 & 0 & 0 \\ 0 & 0 & -51.3 & 38.4 & -50.0 & 0 & 101.2 & -38.4 \\ 0 & -66.67 & 38.4 & -28.8 & 0 & 0 & -38.4 & 95.47 \end{bmatrix} \quad \text{Answer}$$

(ii) Load vector for temperature forces:

$$F_{5T} = E_5 A_5 \alpha_5 \Delta T \begin{Bmatrix} -l \\ -m \\ l \\ m \end{Bmatrix}$$

$$= 200 \times 2000 \times 12 \times 10^{-6} \times 25 \begin{Bmatrix} -0.8 \\ -0.6 \\ 0.8 \\ 0.6 \end{Bmatrix} = \begin{Bmatrix} -96.0 \\ -72.0 \\ 96.0 \\ 72.0 \end{Bmatrix}$$

Global  
↓  
1  
2  
5  
6

$$\{F_T\}^T = [-96.0 \quad -72.0 \quad 0 \quad 0 \quad 96.0 \quad 72.0]$$

(ii) Load vector if the member 5(1-3) is longer by 0.2mm

$$\{F_5\} = E_5 A_5 \frac{\delta l}{l_e} \begin{Bmatrix} -l \\ -m \\ l \\ m \end{Bmatrix} = 200 \times 2000 \times \frac{0.2}{500} \begin{Bmatrix} -0.8 \\ -0.6 \\ 0.8 \\ 0.6 \end{Bmatrix} = \begin{Bmatrix} -128.0 \\ -96.0 \\ 128.0 \\ 96.0 \end{Bmatrix}$$

Global  
↓  
1  
2  
5  
6

Equations after introducing boundary conditions: Since it is too big problem for hand calculations penalty method may be used and solution may be obtained using standard programs. Now  $\Delta_1 = \Delta_2 = \Delta_4 = 0$

For temperature forces the final equation is:

$$k = \begin{bmatrix} 101.2+C & 38.4 & -50 & 0 & -51.2 & -38.4 & 0 & 0 \\ 38.4 & 95.47+C & 0 & 0 & -38.4 & -28.8 & 0 & -66.67 \\ -50.0 & 0 & 101.2 & -38.4 & 0 & 0 & -51.2 & 38.4 \\ 0 & 0 & -38.4 & 95.47+C & 0 & -66.67 & 38.4 & -28.8 \\ -51.2 & -38.40 & 0 & 0 & 101.2 & 38.4 & -50 & 0 \\ -38.4 & -28.8 & 0 & -66.67 & 38.4 & 95.47 & 0 & 0 \\ 0 & 0 & -51.3 & 38.4 & -50.0 & 0 & 101.2 & -38.4 \\ 0 & -66.67 & 38.4 & -28.8 & 0 & 0 & -38.4 & 95.47 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 96.0 \\ 72.0 \\ 0 \\ 0 \end{Bmatrix}$$

In case of stresses due to lack of fit only, only right hand side changes,

$$\{F\}^T = [0 \ 0 \ 0 \ 0 \ 128.0 \ 96.0 \ 0 \ 0]$$

### 11.4 THREE DIMENSIONAL TRUSSES (SPACE TRUSSES)

Typical two noded truss element is shown in Fig. 11.18(a). In this  $x'$  is the local coordinate system. While  $x, y, z$  are the global coordinate system. Fig. 11.18(b) shows the

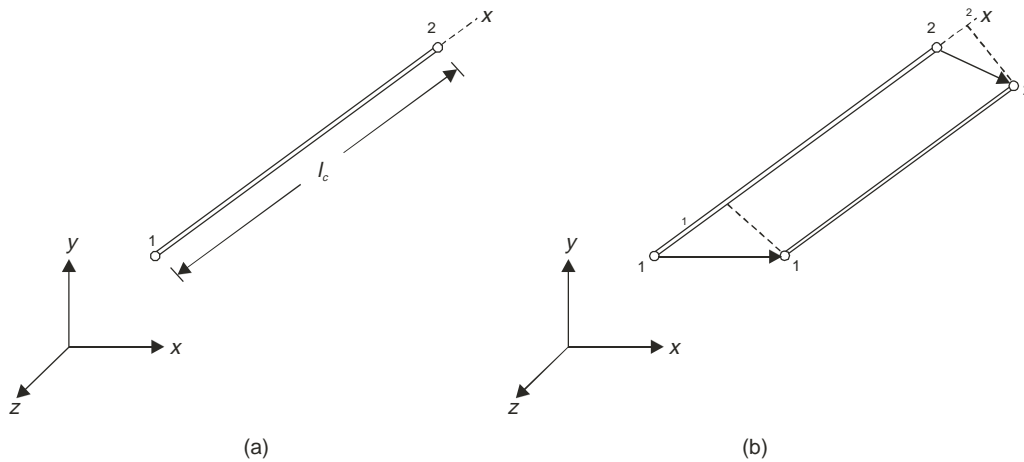


Fig. 11.18

displaced position of the element. In local system, the displacement vector =  $\begin{Bmatrix} \delta'_1 \\ \delta'_2 \end{Bmatrix}$

In global system, the displacement vector is

$$\{\delta\}^T = [\delta_1 \ \delta_2 \ \delta_3 \ \delta_4 \ \delta_5 \ \delta_6]$$

If  $l$ ,  $m$  and  $n$  are the direction cosines, we know,

$$l = \frac{x_2 - x_1}{l_e}, m = \frac{y_2 - y_1}{l_e} \text{ and } n = \frac{z_2 - z_1}{l_e}$$

The length of the member  $l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ . We know,

$$\delta'_1 = l\delta_1 + m\delta_2 + n\delta_3$$

and

$$\delta'_2 = l\delta_4 + m\delta_5 + n\delta_6$$

In matrix form,

$$\begin{Bmatrix} \delta'_1 \\ \delta'_2 \end{Bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_6 \end{Bmatrix}$$

$$\{\delta'\} = [L]\{\delta\} \quad \dots(11.38)$$

where

$$L = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix}$$

is called transformation matrix. From equation 11.33(a), we know the relationship between global and local stiffness matrices is,

$$\begin{aligned} [k]_e &= [L]^T \{k'\}_e [L] = \begin{bmatrix} l & 0 \\ m & 0 \\ n & 0 \\ 0 & l \\ 0 & m \\ 0 & n \end{bmatrix} \frac{E_e A_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \\ &= \frac{E_e A_e}{l_e} \begin{bmatrix} l & 0 \\ m & 0 \\ n & 0 \\ 0 & l \\ 0 & m \\ 0 & n \end{bmatrix} \begin{bmatrix} l & m & n & -l & -m & -n \\ -l & -m & -n & l & m & n \end{bmatrix} \\ &= \frac{E_e A_e}{l_e} \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -ln & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & -m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix} \quad \dots(11.39) \end{aligned}$$

**Example 11.6:** The tripod shown in Fig. 11.19 carries a vertically downward load of 10kN at joint 4. If Young's modulus of the material of tripod stand is 200kN/mm<sup>2</sup> and the cross sectional area of each leg is 2000mm<sup>2</sup>, determine the forces developed in the legs of the tripod.

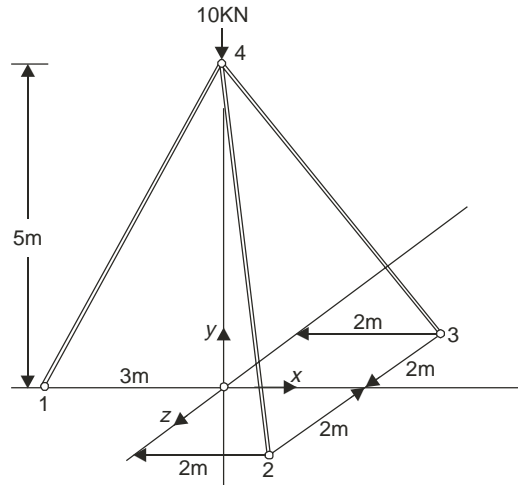


Fig. 11.19

**Solution:** Each member is taken as a bar element in space. The coordinates of various joints are 1(-3, 0, 0); 2(2, 0, 2), 3(2, 0, -2) and 4(0, 5, 0)

Length of each element is given by

$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and the direction cosines are given by

$$l = \frac{x_2 - x_1}{l_e}, m = \frac{y_2 - y_1}{l_e} \text{ and } n = \frac{z_2 - z_1}{l_e}$$

The details of the three elements are given below in the tabular form:

Element No.	Node 1	Node 2	$l_e$ in mm	$l$	$m$	$n$
1	1	4	5831	0.514	0.857	0
2	2	4	5745	-0.348	0.870	-0.348
3	3	4	5745	-0.348	0.870	-0.348

From equation 11.39,

$$k_1 = \begin{bmatrix} 18.11 & 30.184 & 0 & -18.11 & -30.184 & 0 \\ 30.184 & 50.350 & 0 & -30.184 & -50.350 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -18.11 & -30.184 & 0 & 18.11 & 30.184 & 0 \\ -30.184 & -50.350 & 0 & 30.184 & 50.350 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 10 \\ 11 \\ 12 \end{matrix} \leftarrow \begin{matrix} \text{Global} \\ \downarrow \end{matrix}$$

$$[k_2] = \begin{matrix} & \begin{matrix} 4 & 5 & 6 & 10 & 11 & 12 \end{matrix} & \begin{matrix} \leftarrow \text{Global} \\ \downarrow \end{matrix} \\ \begin{matrix} 4 \\ 5 \\ 6 \\ 10 \\ 11 \\ 12 \end{matrix} & \begin{bmatrix} 8.425 & -21.097 & 8.425 & -8.425 & 21.097 & -8.425 \\ -21.097 & 52.707 & -21.037 & 21.09 & -52.707 & 21.097 \\ 8.425 & -21.097 & 8.425 & -8.425 & 21.097 & -8.425 \\ -8.425 & 21.097 & -8.425 & 8.425 & -21.097 & 8.425 \\ 21.097 & -52.707 & 21.097 & -21.097 & 52.707 & -21.097 \\ -8.425 & 21.097 & -8.425 & 8.425 & -21.097 & 8.425 \end{bmatrix} \end{matrix}$$

$$[k_3] = \begin{matrix} & \begin{matrix} 7 & 8 & 9 & 10 & 11 & 12 \end{matrix} & \begin{matrix} \leftarrow \text{Global} \\ \downarrow \end{matrix} \\ \begin{matrix} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix} & \begin{bmatrix} 8.425 & -21.097 & -8.425 & -8.425 & 21.097 & 8.425 \\ -21.097 & 52.707 & 21.097 & 21.097 & -52.707 & -21.097 \\ -8.425 & 21.097 & 8.425 & 8.425 & -21.097 & -8.425 \\ -8.425 & 21.097 & 8.425 & 8.425 & -21.097 & -8.425 \\ 21.097 & -52.707 & -21.097 & -21.097 & 52.707 & 21.097 \\ 8.245 & -21.097 & -8.425 & -8.425 & 21.097 & 8.425 \end{bmatrix} \end{matrix}$$

Global matrix of size  $12 \times 12$  can be assembled.  
The load vector is

$$[F]^T = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -10 \ 0]$$

Hence the equilibrium equation can be assembled. Noting that  $\delta_1 = \delta_2 = \dots = \delta_9 = 0$ , the reduced matrix equation will be of size  $3 \times 3$ , the elements being from rows and columns of 10, 11, 12. The reduced equation of equilibrium is i.e.,

18.110	30.148	0
8.425	-21.097	8.425
8.425	-21.097	-8.425
30.148	50.35	0
-21.097	52.707	-21.097
-21.097	52.707	21.097
0	0	0
8.425	-21.097	8.425
-8.425	21.097	8.425

$$\begin{Bmatrix} \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix}$$

i.e. 
$$\begin{bmatrix} 34.96 & -12.046 & 0 \\ -12.046 & 155.764 & 0 \\ 0 & 0 & 16.950 \end{bmatrix} \begin{Bmatrix} \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix}$$

$$\text{i.e.} \quad \begin{bmatrix} 34.96 & -12.046 & 0 \\ 0 & 151.589 & 0 \\ 0 & 0 & 16.950 \end{bmatrix} \begin{Bmatrix} \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix}$$

$$\therefore \Delta_{12} = 0$$

From equation 2,

$$151.584 \delta_{12} = -10 \text{ or } \delta_{11} = -0.065697$$

$$34.96 \delta_{10} - 12.046(-0.06597) = 0$$

$$\therefore \delta_{10} = -0.02273$$

$$F_1 = \frac{200 \times 2000}{5831} [-0.514 \quad -0.857 \quad 0 \quad 0.514 \quad 0.857 \quad 0] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.02273 \\ 0.06597 \\ 0 \end{Bmatrix}$$

$$= -4.680 \text{ kN}$$

$$F_2 = \frac{200 \times 2000}{5745} [0 - 0 + 0 - 0.348(-0.02273) + 0.87(-0.06597) + 0]$$

$$= -3.445 \text{ kN}$$

$$F_3 = \frac{200 \times 2000}{5745} [0 - 0 + 0 - 0.348(-0.02273) + 0.87(-0.06597) + 0]$$

$$= -3.445 \text{ kN}$$

$$\{F\}^T = [-4.680 \quad -3.445 \quad -3.445] \text{ Answer}$$

**Note:** As the degrees of freedom is high three dimensional problems are not suited for hand calculations. The computer program with penalty method of imposing boundary conditions is ideally suited for such problems.

## QUESTIONS

1. Differentiate between a bar element and a truss element
2. Using variational approach derive element stiffness matrix of
  - (a) bar element
  - (b) plane truss element
3. Derive the expressions for nodal load vector in a two noded bar element due to
  - (a) Body force
  - (b) Surface load

Use variational approach

4. Explain the elimination method and penalty method of imposing boundary conditions. Comment on the two methods.
5. In axially loaded cases, how do you find the support reactions after getting required displacements? Explain.
6. Using variational approach determine the expression for constant load, due to rise in temperature  $\Delta T$  in an element.
7. Determine the nodal displacement, element stresses and support reactions of the axially loaded bar as shown in Fig. 11.20. Take  $E = 200 \text{ GPa}$  and  $P = 30 \text{ kN}$

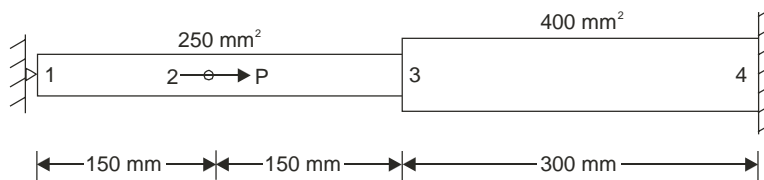


Fig. 11.20

[Answer:  $\Delta_2 = 0.062307 \text{ mm}$   $\Delta_3 = 0.034615 \text{ mm}$

$\sigma_1 = 83.08 \text{ N/mm}^2$   $\sigma_2 = 36.92 \text{ N/mm}^2$

$\sigma_3 = 23.08 \text{ N/mm}^2$   $R_1 = -20.77 \text{ kN}$   $R_4 = -9.23 \text{ kN}$ ]

8. Obtain the forces in the plane truss shown in Fig. 11.21 and determine the support reactions also. Use finite element method. Take  $E = 200 \text{ GPa}$  and  $A = 2000 \text{ mm}^2$ .

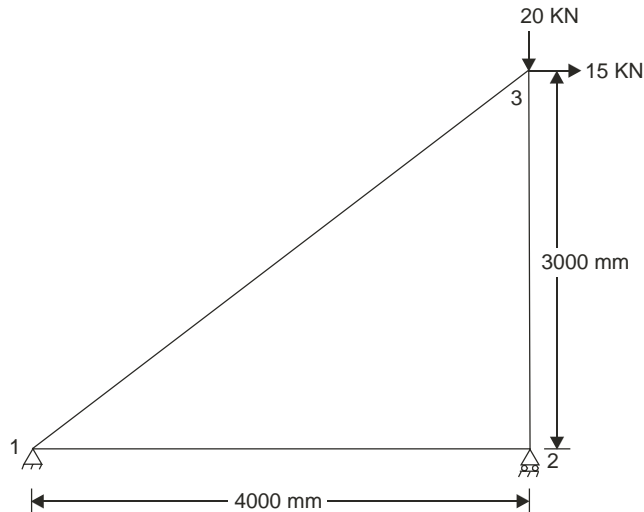


Fig. 11.21

[Answer:  $\{F\}^T = [0 \quad -37.25 \quad 18.75]$  in kN

$R_1 = -11.25 \text{ kN}$   $R_2 = -15 \text{ kN}$   $R_4 = 31.25 \text{ kN}$ ]

9. Determine the forces in the members of the truss shown in Fig. 11.22 Take  $E = 200\text{GPa}$ ,  $A = 2000\text{ mm}^2$ .

[Answer:  $\{F\}^T = [20 \quad -26.25 \quad 4.961 \quad 0 \quad -6.2752]$  in kN]

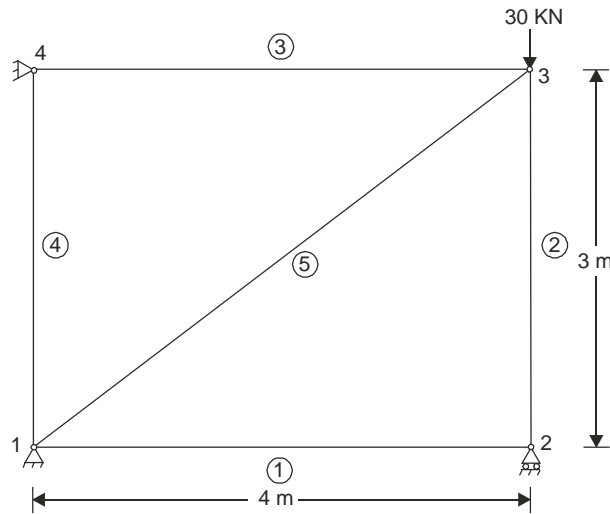


Fig. 11.22



# 12

## Finite Element Analysis—Plane Stress and Plane Strain Problems

### 12.1 INTRODUCTION

In this chapter, first we see the general method for the analysis of plane stress/plane strain problems using CST elements. Then a small problem is taken up and hand calculations are made. The idea of such approach is to explain the steps involved in computer programming and make it clear by giving physical feel of the calculations involved. The lengthy calculations involved are pointed out when higher order elements are used.

### 12.2 GENERAL PROCEDURE WHEN CST ELEMENTS ARE USED

The general procedure is explained referring to tension bar problem shown in Fig. 1.2 and the dam section analysis problem shown in Fig. 10.5.

*Step 1:* Field Variable and Element:

Since plane stress and plane strain problems are two dimensional problems, we need two dimensional elements. Any one from the family of triangular elements (CST/LST/QST) are ideally suited for these problems. Any one element from the family of two dimensional isoparametric elements (to be explained in next chapter) also may be used. In these elements there are two degree of freedom at each node i.e. the displacement in  $x$ -direction and displacement in  $y$ -direction. Hence total degree of freedom in

- (i) each element =  $2 \times$  No. of nodes per element
- (ii) structure =  $2 \times$  No. of nodes in entire structure.

For a CST element shown in Fig. 12.1, the displacement vector may be taken as

$$\begin{aligned}\{\delta\}_e^T &= \{\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6\} \quad \dots(12.1a) \\ &= \{u_1 \quad u_2 \quad u_3 \quad v_1 \quad v_2 \quad v_3\}_e\end{aligned}$$

or as

$$\{\delta\}^T = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3\} \quad \dots(12.1b)$$

In most of the programs the order shown in equation 12.1 (b) is selected. Hence in this chapter the displacement vector  $\{\delta\}$  is used in the form of equation 12.1 (b). Then the  $x$  and  $y$  displacements of the node in global system are referred as  $2n - 1$ th and  $2n$ th displacements.

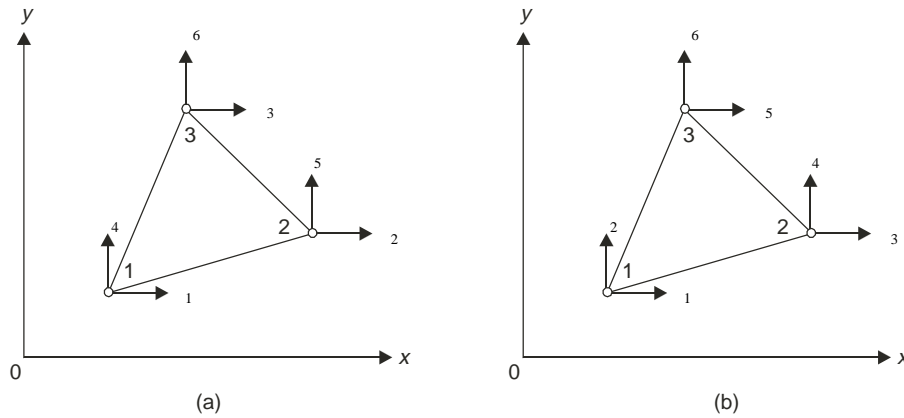


Fig. 12.1 Nodal variables in CST elements

**Step 2: Discretization**

Discretization of the structure should be made keeping in mind all the points listed in Chapter 10. For all nodes  $x$  and  $y$  coordinates are to be supplied/generated. Then nodal connectivity detail is to be supplied. For the dam analysis problem shown in Fig. 10.5, the nodal connectivity detail is of the form shown in Table 12.1.

Table 12.1 Nodal connectivity

Element No.	1	2	3	Local Numbers
				Global Numbers
1	1	2	7	
2	2	7	8	
:				
7	4	11	10	
8	4	5	11	
:				
10	6	12	11	
:				

**Step 3: Shape/Interpolation Functions**

As shown in equation 5.15, the shape function terms are

$$N_1 = \frac{a_1 + b_1x + c_1y}{2A}, \quad N_2 = \frac{a_2 + b_2x + c_2y}{2A} \quad \text{and} \quad N_3 = \frac{a_3 + b_3x + c_3y}{2A}$$

where  $a_1 = x_2y_3 - x_3y_2$      $a_2 = x_3y_1 - x_1y_3$      $a_3 = x_1y_2 - x_2y_1$   
 $b_1 = y_2 - y_3$      $b_2 = y_3 - y_1$      $b_3 = y_1 - y_2$   
 $c_1 = x_3 - x_2$      $c_2 = x_1 - x_3$      $c_3 = x_2 - x_1$

and 
$$2A = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

When we select nodal displacement vector as shown in Fig. 12.1 (b),

$$u(x, y) = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \{\delta\}_e \quad \dots(12.3)$$

**Step 4:** Element Properties

Since strain vector

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_2 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

and nodal displacement vector is in the form 12.3, the strain displacement vector ( $\{\varepsilon\} = [B]\{\delta\}$ ),  $[B]$  is given by

$$[B] = \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \quad \dots(12.4)$$

According to variational principal (equation 9.26)

$$[k]_e = \iiint_v (B)^T [D][B] dV$$

Since  $[B]$ ,  $[D]$  are constant matrices we get

$$[k]_e = [B]^T [D][B]V \quad \dots(12.5)$$

where  $V = A t$

This is exactly same as equation 7.4 which was obtained by Turner by the direct approach. In equation 12.5,  $[D]$  is the elasticity matrix, which is as presented in Chapter 2 (equation 2.14 and 2.15). In case of isotropic materials, for plane stress case,

$$[D] = \frac{E}{1 - \mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1 - \mu}{2} \end{bmatrix} \quad \dots(12.6)$$

and for plane strain case,

$$[D] = \frac{E}{(1 + \mu)(1 - 2\mu)} \begin{bmatrix} 1 - \mu & \mu & 0 \\ \mu & 1 - \mu & 0 \\ 0 & 0 & \frac{1 - 2\mu}{2} \end{bmatrix} \quad \dots(12.7)$$

Using equation 12.5, the element stiffness matrix can be found.

### Consistent Loads

Consistent loads can be derived using the equation

$$\{F\}_e = \oint [N]^T \{X_b\} dV + \oint [N]^T \{T\} ds \quad (\text{equation 9.26})$$

If there are nodal forces, they are to be added directly to the vector  $\{F\}_e$ .

#### Step 5: Global Properties

Using nodal connectivity details the exact position of every term of stiffness matrix and nodal vector must be identified and placed in global stiffness matrix.

#### Step 6: Boundary Conditions

Since in most of the problems in plane stress and plane strain degree of freedom is quite high, the computers are to be used. These problems are not suitable for hand calculations. When computer programs are to be developed, imposition of boundary condition is conveniently done by penalty method.

#### Step 7: Solution of Simultaneous Equations

Gauss elimination method or Cholesky's decompositions method may be used. In elasticity problems, there exists symmetry and banded nature of stiffness matrix. Hence the programs are developed to store only half the band width of stiffness matrix and solve simultaneous equations using Choleski's decomposition method.

#### Step 8: Additional Calculations

After getting nodal displacements stresses and strains in each element is assembled using the relations

$$\{\varepsilon\} = [B] \{\delta\}_e$$

and

$$\{\sigma\} = [D][B] \{\delta\}_e$$

The calculated value of stress for an element is constant. It is assumed to represent the value at the centroid of the element. As a designer is normally interested in the principal stresses, for each element these values also may be calculated.

**Example 12.1:** Find the nodal displacements and element stresses in the propped beam shown in Fig. 12.2. Idealize the beam into two CST elements as shown in the figure. Assume plane stress condition. Take  $\mu = 0.25$ ,  $E = 2 \times 10^5 \text{ N/mm}^2$ , Thickness = 15mm.

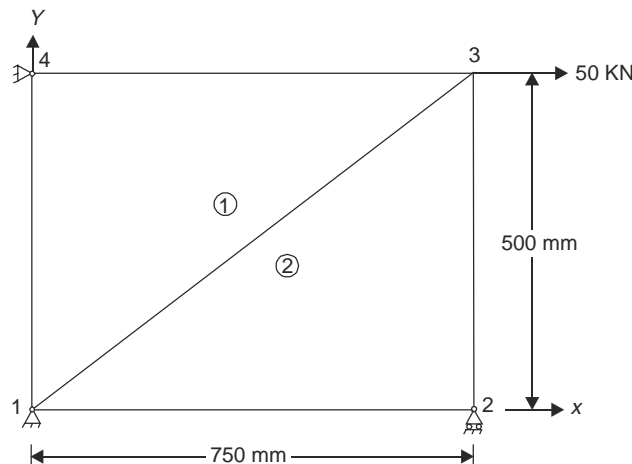
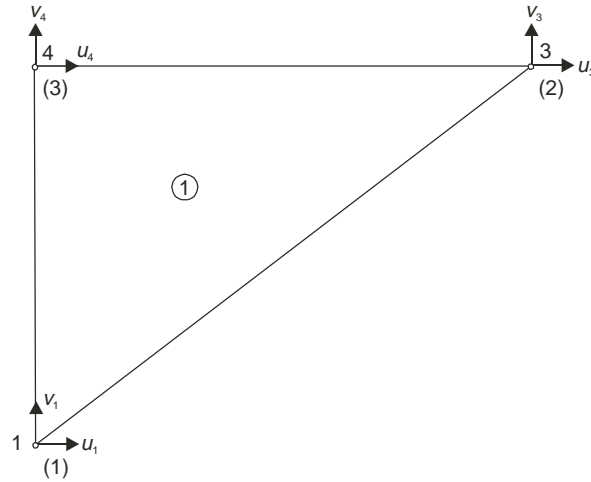


Fig. 12.2

**Solution:** For element (1), global nodal numbers are 1, 3, 4. Local numbers 1, 2, 3 selected are indicated in Fig. 12.3. Selecting node 4 as the origin of global coordinate system.



**Fig. 12.3**

1(0, 0), 2(750, 500) and 3(0, 500)

$$2A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 750 & 500 \\ 1 & 0 & 500 \end{vmatrix} = 750 \times 500 - 0 = 750 \times 500$$

$$\therefore [B] = \frac{1}{750 \times 750} \begin{bmatrix} 0 & 0 & 500 & 0 & -500 & 0 \\ 0 & -750 & 0 & 0 & 0 & 750 \\ -750 & 0 & 0 & 500 & 750 & -500 \end{bmatrix}$$

$$= \frac{1}{750} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 1.5 \\ -1.5 & 0 & 0 & 1 & 1.5 & -1 \end{bmatrix}$$

$$[D] = \frac{E}{(1 + \mu)(1 - 2\mu)} \begin{bmatrix} 1 - \mu & \mu & 0 \\ \mu & 1 - \mu & 0 \\ 0 & 0 & \frac{1 - 2\mu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{1.25 \times 0.5} \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} = 0.2 \times 10^5 \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
 [D][B] &= \frac{1}{750} \times 0.2 \times 10^5 \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 1.5 \\ -1.5 & 0 & 0 & 1 & 1.5 & -1 \end{bmatrix} \\
 &= \frac{0.2 \times 10^5}{750} \begin{bmatrix} 0 & -1.5 & 3 & 0 & -3 & 1.5 \\ 0 & -4.5 & 1 & 0 & -1 & 4.5 \\ -1.5 & 0 & 0 & 1 & 1.5 & -1 \end{bmatrix} \\
 [k]_1 &= t A [B]^T [D][B] \\
 &= \frac{15 \times 750 \times 500}{2} \times \frac{1}{750} \begin{bmatrix} 0 & 0 & -1.5 \\ 0 & -1.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1.5 \\ 0 & 1.5 & -1 \end{bmatrix} \frac{0.2 \times 10^5}{750} \begin{bmatrix} 0 & -1.5 & 3 & 0 & -3 & 1.5 \\ 0 & -4.5 & 1 & 0 & -1 & 4.5 \\ -1.5 & 0 & 0 & 1 & 1.5 & -1 \end{bmatrix} \\
 &= 100000 \begin{bmatrix} u_1 & v_1 & u_3 & v_3 & u_4 & v_4 & \text{Global} \\ 2.25 & 0 & 0 & -1.5 & -2.25 & 1.50 & \downarrow \hat{u}_1 \\ 0 & 6.75 & -1.5 & 0 & 1.5 & 6.75 & v_1 \\ 0 & -1.5 & 3.0 & 0 & -3.0 & 1.5 & u_3 \\ -1.5 & 0 & 0 & 1 & 1.5 & -1.0 & v_3 \\ -2.25 & 1.5 & -3 & 1.5 & 5.25 & -3.0 & u_4 \\ 1.5 & 6.75 & 1.5 & -1 & -3.0 & 7.75 & v_4 \end{bmatrix}
 \end{aligned}$$

For element (2),  
Local and global node numbers are as shown in Fig. 12.4.

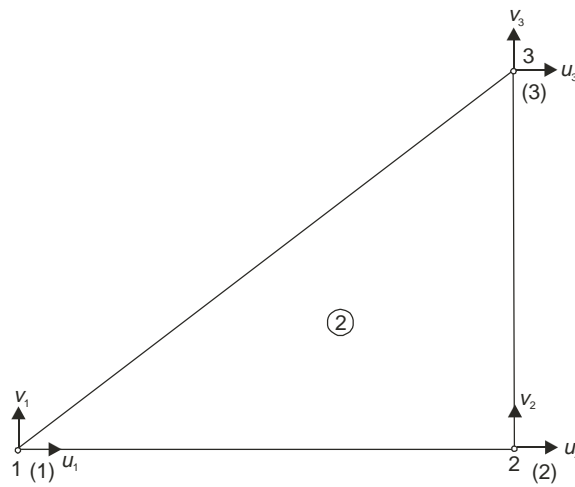


Fig. 12.4

The coordinates of nodes are

$$1(0, 0), 2(750, 0), 3(750, 500)$$

$$b_1 = y_2 - y_3 = -500$$

$$b_2 = y_3 - y_1 = -500$$

$$b_3 = y_1 - y_2 = -0$$

$$c_1 = x_3 - x_2 = 0$$

$$c_2 = x_1 - x_3 = -750$$

$$c_3 = x_2 - x_1 = 750$$

$$2A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 750 & 0 \\ 1 & 750 & 500 \end{vmatrix} = 1(750 \times 500) = 750 \times 500$$

$$[B] = \frac{1}{750 \times 500} \begin{bmatrix} -500 & 0 & 500 & 0 & 0 & 0 \\ 0 & 0 & 0 & -750 & 0 & 750 \\ 0 & -500 & -750 & 500 & 750 & 0 \end{bmatrix}$$

$$= \frac{1}{750} \begin{bmatrix} -1.0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.5 & 0 & 1.5 \\ 0 & -1.0 & -1.5 & 1.0 & 1.5 & 0 \end{bmatrix}$$

$$[D] = 0.2 \times 10^5 \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ same as for element 1.}$$

$$\therefore [D][B] = \frac{1}{750} \times 0.2 \times 10^5 \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.0 & 0 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.5 & 0 & 1.5 \\ 0 & -1.0 & -1.5 & 1.0 & 1.5 & 0 \end{bmatrix}$$

$$= \frac{0.2 \times 10^5}{750} \begin{bmatrix} -3.0 & 0 & 3.0 & -1.5 & 0 & 1.5 \\ -1.0 & 0 & 1.0 & -4.5 & 0 & 4.5 \\ 0 & -1.0 & -1.5 & 1.0 & 1.5 & 0 \end{bmatrix}$$

$$[k]_2 = t A [B]^T [D][B]$$

$$[k]_2 = 15 \times \frac{750 \times 500}{2} \times \frac{1}{750} \begin{bmatrix} -1.0 & 0 & 0 \\ 0 & 0 & -1.0 \\ 1.0 & 0 & -1.5 \\ 0 & -1.5 & 1.0 \\ 0 & 0 & 1.5 \\ 0 & 1.5 & 0 \end{bmatrix} \frac{0.2 \times 10^5}{750} \begin{bmatrix} -3.0 & 0 & 3.0 & -1.5 & 0 & 1.5 \\ -1.0 & 0 & 1.0 & -4.5 & 0 & 4.5 \\ 0 & -1.0 & -1.5 & 1.0 & 1.5 & 0 \end{bmatrix}$$

$$= 10000 \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \\ 3.0 & 0 & -3.0 & 1.5 & 0 & -1.5 \\ 0 & 1.0 & 1.5 & -1.0 & -1.5 & 0 \\ -3.0 & 1.5 & 5.25 & -3.0 & -2.25 & 1.5 \\ 1.5 & -1.0 & -3.0 & 7.75 & 1.5 & -6.75 \\ 0 & -1.5 & -2.25 & 1.1 & 2.25 & 0 \\ -1.5 & 0 & 1.5 & -6.75 & 0 & 6.25 \end{bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix}$$

$[k] = 100000$

$u_1$	$v_1$	$u_2$	$v_2$	$u_3$	$v_3$	$u_4$	$v_4$	
2.25	0			0	-1.5	-2.25	1.50	$u_1$
3.0	0	-3.0	1.5	0	-1.5			
0	6.75			-1.5	0	1.5	6.75	$v_1$
0	1.0	1.5	-1.0	-1.5	0			
-3.0	1.5	5.25	-3.0	-2.25	1.5			$u_2$
1.5	-1.0	-3.0	7.75	1.5	-6.75			$v_2$
0	-1.5	-2.25	1.5	3.0	0	-3.0	1.5	$u_3$
0	-1.5	-2.25	1.5	2.25	0			
-1.5	0			0	1.0	1.5	-1.0	$v_3$
-1.5	0	1.5	-6.75	0	6.75			
-2.25	1.5			-3.0	1.5	5.25	-3.0	$u_4$
1.5	6.75			1.5	-1.0	-3.0	7.75	$v_4$

$$\{F\}^T = [0 \ 0 \ 0 \ 0 \ 50000 \ 0 \ 0 \ 0]$$

∴ The equation is

$$[k][\delta] = \{F\}.$$



$$\text{i.e.,} \quad 100000 \begin{bmatrix} 5.5 & 0 & -3.0 & 1.5 & 0 & -3.0 & -2.25 & 1.50 \\ 0 & 7.25 & 1.5 & -1.0 & -3.0 & 0 & 1.5 & 6.25 \\ -3.0 & 1.5 & 5.25 & -3.0 & -2.25 & 1.5 & 0 & 0 \\ 1.5 & -1.0 & -3.0 & 7.25 & 1.5 & -6.75 & 0 & 0 \\ 0 & -3.0 & -2.25 & 1.5 & 5.25 & 0 & -3.0 & 1.5 \\ -3.0 & 0 & 1.5 & -6.75 & 0 & 7.75 & 1.5 & -1.0 \\ -2.25 & 1.5 & 0 & 0 & -3.0 & 1.5 & 5.25 & -3.0 \\ 1.5 & 6.75 & 0 & 0 & 1.5 & -1.0 & -3.0 & 7.75 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 50000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The boundary conditions are

$$\delta_1 = \delta_2 = \delta_4 = \delta_7 = \delta_8 = 0$$

$\therefore$  Reduced equation is,

$$100000 \begin{bmatrix} 5.25 & -2.25 & 1.5 \\ -2.25 & 5.25 & 0 \\ 1.5 & 0 & 7.75 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 50000 \\ 0 \end{Bmatrix}$$

$$\therefore \begin{bmatrix} 5.25 & -2.25 & 1.5 \\ -2.25 & 5.25 & 0 \\ 1.5 & 0 & 7.75 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5 \\ 0 \end{Bmatrix}$$

$$\therefore \begin{bmatrix} 5.25 & -2.25 & 1.5 \\ 0 & 4.2857 & 0.6429 \\ 0 & 0.6429 & 7.3214 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5 \\ 0 \end{Bmatrix}$$

$$\therefore \begin{bmatrix} 5.25 & -2.25 & 1.5 \\ 0 & 4.2857 & 0.6429 \\ 0 & 0 & 7.17139 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.5 \\ -0.075 \end{Bmatrix}$$

$$\delta_6 = -0.010459$$

$$4.2857 \delta_5 + 0.6429 (-0.010459) = 0.5$$

$$\delta_5 = 0.118236$$

$$5.25 \delta_3 - 2.25 (0.118236) + 1.5 (-0.010459) = 0$$

$$\delta_3 = 0.053661$$

$$\{\delta\}^T = [0 \ 0 \ 0.053661 \ 0 \ 0.118236 \ -0.010459 \ 0 \ 0]$$

$$\{\sigma_1\} = [D][B]\{\delta\}.$$

$$= \frac{0.2 \times 10^5}{750} \begin{bmatrix} 0 & -1.5 & 3 & 0 & -3 & 1.5 \\ 0 & -4.5 & 1 & 0 & -1 & 4.5 \\ -1.5 & 0 & 0 & 1 & -1.5 & -1.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.053661 \\ 0 \\ 0.118236 \\ -0.010459 \end{Bmatrix} = \begin{Bmatrix} -5.584 \\ -2.977 \\ 5.000 \end{Bmatrix}$$

$$\{\sigma_2\} = \frac{0.2 \times 10^5}{750} \begin{bmatrix} -3.0 & 0 & 3.0 & -1.5 & 0 & 1.5 \\ -1.0 & 0 & 1.0 & -4.5 & 0 & 4.5 \\ 0 & -1.0 & -1.5 & 1.0 & 1.5 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.118236 \\ -0.010459 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 9.877 \\ 4.408 \\ 5.008 \end{Bmatrix}$$

**Example 12.2:** Derive the expression for consistent load vector due to self weight in a CST element.

**Solution:** The general expression for the consistent load in any element due to the body force is

$$\{F\}_e = \iiint_V [N]^T \{X_b\} dV$$

$$\text{For self weight } \{X_b\} = \begin{Bmatrix} 0 \\ -\rho \end{Bmatrix}$$

Where  $\rho$  is unit weight of the material

It is advantageous to take interpolation functions in the natural coordinate system, since closed form integration formulae can be used.

We know for CST element,

$$[N] = \begin{bmatrix} L_1 & 0 & L_2 & 0 & L_3 & 0 \\ 0 & L_1 & 0 & L_2 & 0 & L_3 \end{bmatrix}$$

when nodal vector selected is in the order  $\begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$

$$\therefore [F]_e = \iint_A \begin{bmatrix} L_1 & 0 \\ 0 & L_1 \\ L_2 & 0 \\ 0 & L_2 \\ L_3 & 0 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} 0 \\ -\rho \end{Bmatrix} hdA = \iint_A \begin{bmatrix} 0 \\ -L_1\rho \\ 0 \\ -L_2\rho \\ 0 \\ -L_3\rho \end{bmatrix} hdA$$

Noting that the standard integration formula is

$$\iint_A L_1^p L_2^q L_3^r dA = \frac{p! q! r!}{(p + q + r + 2)} 2A$$

we get 
$$\iint_A -L_1 \rho h dA = -\rho \frac{1! 0! 0!}{(1 + 0 + 0 + 2)} h 2A = -\rho \frac{1}{2 \times 3} 2A = -\frac{\rho h A}{3}$$

Similarly 
$$\iint_A -L_2 \rho dA = -\frac{\rho h A}{3}$$

and 
$$\iint_A -L_3 \rho dA = -\frac{\rho h A}{3}$$

$$\therefore \{F\}_e = -\frac{\rho h A}{3} \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{Bmatrix} \text{ Answer}$$

**Example 12.3:** Find the expression for nodal vector in a CST element subject to pressures  $P_{x1}, P_{y1}$  on side 1,  $P_{x2}, P_{y2}$  on side 2 and  $P_{x3}, P_{y3}$  on side 3 as shown in Fig. 12.5.

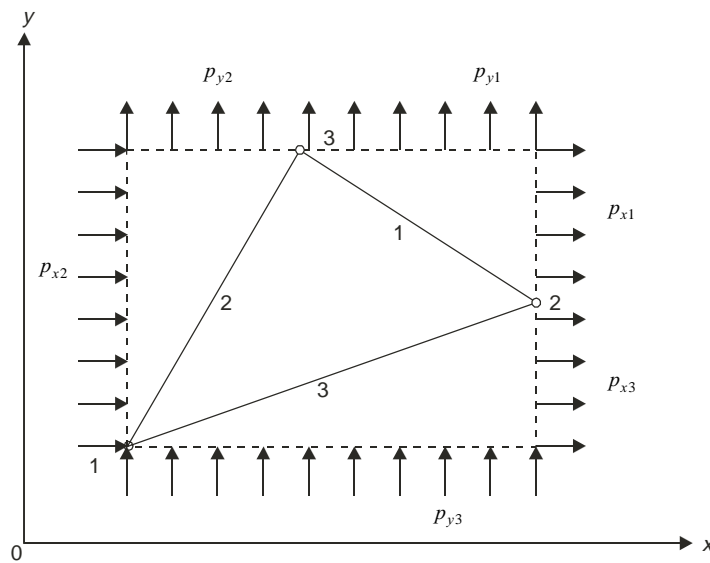


Fig. 12.5

**Solution:** Let us first consider nodal vector due to pressure  $P_{x1}$  and  $P_{y1}$  only. [Refer Fig. 12.6]

We know in CST element

$$[N] = [L]$$

Along side 1,  $L_1 = 0 \quad \therefore L_2 + L_3 = 1$

i.e.,  $L_3 = 1 - L_2$

$ds_1 = l_1 dL_2$ , when  $s$  is measured from node 3 towards 2.

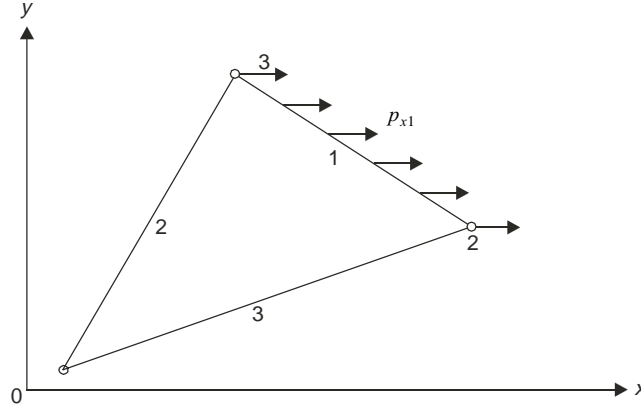


Fig. 12.6

The surface forces are

$$\{X_s\} = \begin{Bmatrix} P_{x1} \\ P_{y1} \end{Bmatrix}$$

Hence the line integral form exists for nodal force vector as given below:

$$\{F\}_e = \int [N]^T \{X_s\} ds_1 t$$

$$= t \int_0^l \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_2 & 0 \\ 0 & L_2 \\ L_3 & 0 \\ 0 & L_3 \end{bmatrix} \begin{Bmatrix} P_{x1} \\ P_{y1} \end{Bmatrix} ds = t \int_0^l \begin{bmatrix} 0 \\ 0 \\ L_2 P_{x1} \\ L_2 P_{y1} \\ L_3 P_{x1} \\ L_3 P_{y1} \end{bmatrix} ds$$

Noting that the standard integration form for natural coordinate system is

$$\oint L_1^p L_2^q ds = \frac{p! q!}{(p + q + 1)!} l$$

we get

$$\int L_2 P_{x1} ds = \frac{1! 0!}{(1 + 0 + 1)!} l_1 P_{x1} = \frac{l_1}{2} P_{x1}$$

$$\int L_2 P_{y1} ds = \frac{l_1}{2} p_{y1}$$

$$\int L_3 P_{x1} ds = \frac{0! 1!}{(0 + 1 + 1)!} l_1 P_{x1} = \frac{l_1}{2} p_{x1}$$

and

$$\int L_3 P_{y1} ds = \frac{l_1}{2} p_{y1}$$

$$\therefore \{F\}_e^T = \frac{tl_1}{2} [0 \quad 0 \quad p_{x1} \quad p_{y1} \quad p_{x1} \quad p_{y1}]$$

Similarly due to forces on side 2 we get

$$\{F\}_e^T = tl_2 [p_{x2} \quad p_{y2} \quad 0 \quad 0 \quad p_{x2} \quad p_{y2}]$$

and due to forces on side 3,

$$\{F\}_e^T = tl_3 [p_{x3} \quad p_{y3} \quad p_{x3} \quad p_{y3} \quad 0 \quad 0]$$

$\therefore$  Nodal vector due to forces on all the three sides is,

$$\{F_e\} = \begin{Bmatrix} l_2 p_{x2} + l_3 p_{x3} \\ l_2 p_{y1} + l_3 p_{y3} \\ l_1 p_{x1} + l_3 p_{x3} \\ l_1 p_{y1} + l_3 p_{y3} \\ l_1 p_{x1} + l_2 p_{x2} \\ l_1 p_{y1} + l_2 p_{y2} \end{Bmatrix}$$

### 12.3 USE OF HIGHER ORDER ELEMENTS

The procedure explained for CST element may be extended to LST (6 noded) QST (10 noded) and rectangular family of elements also. However the procedure becomes lengthy. The shape functions to be used for these elements are already presented in Chapter 5. For LST elements the shape functions are

$$N_1 = L_1 (2L_1 - 1), \quad N_2 = L_2 (2L_2 - 1), \quad N_3 = L_3 (2L_3 - 1)$$

$$N_4 = 4 L_1 L_2, \quad N_5 = 4 L_2 L_3 \text{ and } N_6 = 4 L_3 L_1$$

We know,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4 + N_5 u_5 + N_6 u_6$$

$$\therefore \epsilon_x = \frac{du}{dx} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} & \frac{dN_4}{dx} & \frac{dN_5}{dx} & \frac{dN_6}{dx} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{Bmatrix}$$

Now

$$\frac{dN_1}{dx} = \frac{dN_1}{dL_1} \frac{dL_1}{dx} + \frac{dN_1}{dL_2} \frac{dL_2}{dx} + \frac{dN_1}{dL_3} \frac{dL_3}{dx}$$

$$\begin{aligned}
 &= b_1 \frac{dN_1}{dL_1} + b_2 \frac{dN_1}{dL_2} + b_3 \frac{dN_1}{dL_3} \\
 &= b(4L_1 - 1)
 \end{aligned}$$

Similarly the expressions for  $\frac{dN_2}{dx} \dots \frac{dN_6}{dx}$  can be assembled. It gives first row of  $[B]$  matrix knowing

that  $\epsilon_y = \frac{dv}{dx}$  and  $v = N_1 v_1 + N_2 v_2 + \dots + N_6 v_6$  the second row of  $[B]$  matrix can be assembled. The third row of  $[B]$  matrix correspond to

$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx}.$$

This also can be assembled.

After finding  $[B]$  matrix our interest is to assemble stiffness matrix

$$[k]_e = \iiint_v [B]^T [D] [B] dV$$

Noting that  $[B]$  matrix is not a constant matrix, direct integration using closed form expressions become lengthy process, though not impossible. Similarly the assembly of consistent loads using the expression

$$\{F\}_e = \iiint_v N^T X_b dV + \iint_s N^T X_s ds$$
 is lengthy process.

The isoparametric concept and numerical integration techniques to be explained in Chapter 13, have not only simplified and standardized the FEA analysis, but have made such elements adoptable for curved boundaries also.

The procedure in using rectangular family of elements is same as explained above and the observations are also similar.

## QUESTIONS

1. For the CST element shown in Fig. 12.7, assemble strain–displacement matrix. Take,  $t = 20$  mm,  $E = 2 \times 10^5$  N/mm<sup>2</sup>.

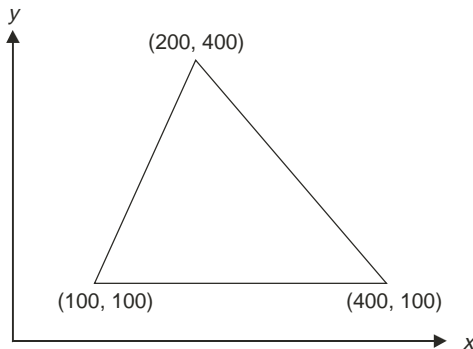


Fig. 12.7

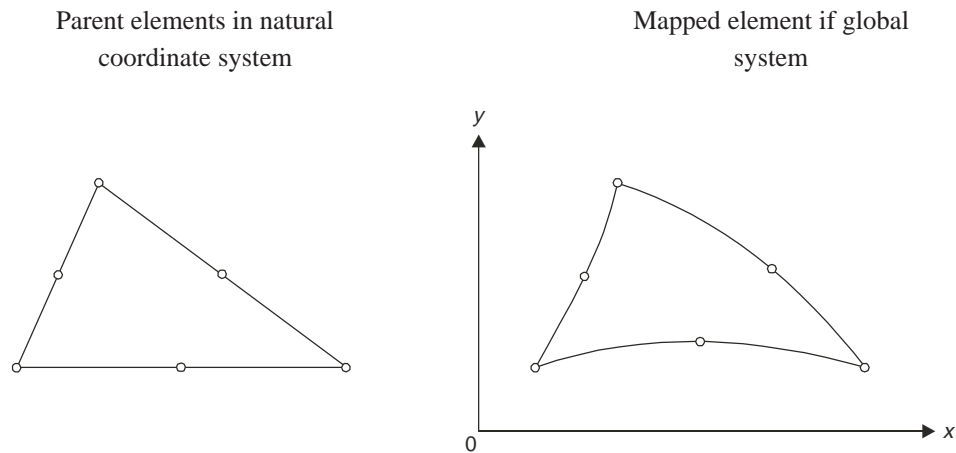
2. Derive the constant load vector in a CST element due to
  - (i) Self weight
  - (ii) Uniform pressures  $P_{x1}$ ,  $P_{y1}$  acting on the side 1.

# 13

## Isoparametric Formulation

### 13.1 INTRODUCTION

The various elements so far we have seen are having straight edges. To take care of curved boundaries refined meshes are to be used when straight edged elements are employed. Even with refined meshes analysts were not happy with the results since unnecessary stress concentrations are introduced. Higher order elements also do not overcome the problem of suitably approximating curved boundaries. The isoparametric concept brought out by Taig [1] and latter on generalized by B.M. Irons [2] revolutionized the finite elements analysis and it also helped in properly mapping the curved boundaries. They brought out the concept of mapping regular triangular and rectangular elements in natural coordinate system, to arbitrary shapes in global system as shown in Fig. 13.1. In this chapter method of coordinate transformation of natural coordinates to global coordinate system is presented. The terms isoparametric, super parametric and subparametrics are defined. The basic theorems on which isoparametric concept is based are explained and need for satisfying uniqueness theorem of mapping is presented. Assembling of stiffness matrix is illustrated. For assembling stiffness matrix integration is to be carried out numerically. The Gaussian integration technique which is commonly employed is explained briefly. To make the procedure clear few small numerical problems are illustrated and lastly application to structural engineering problems is presented.



**Fig. 13.1** *Concept of mapping in isoparametric elements*



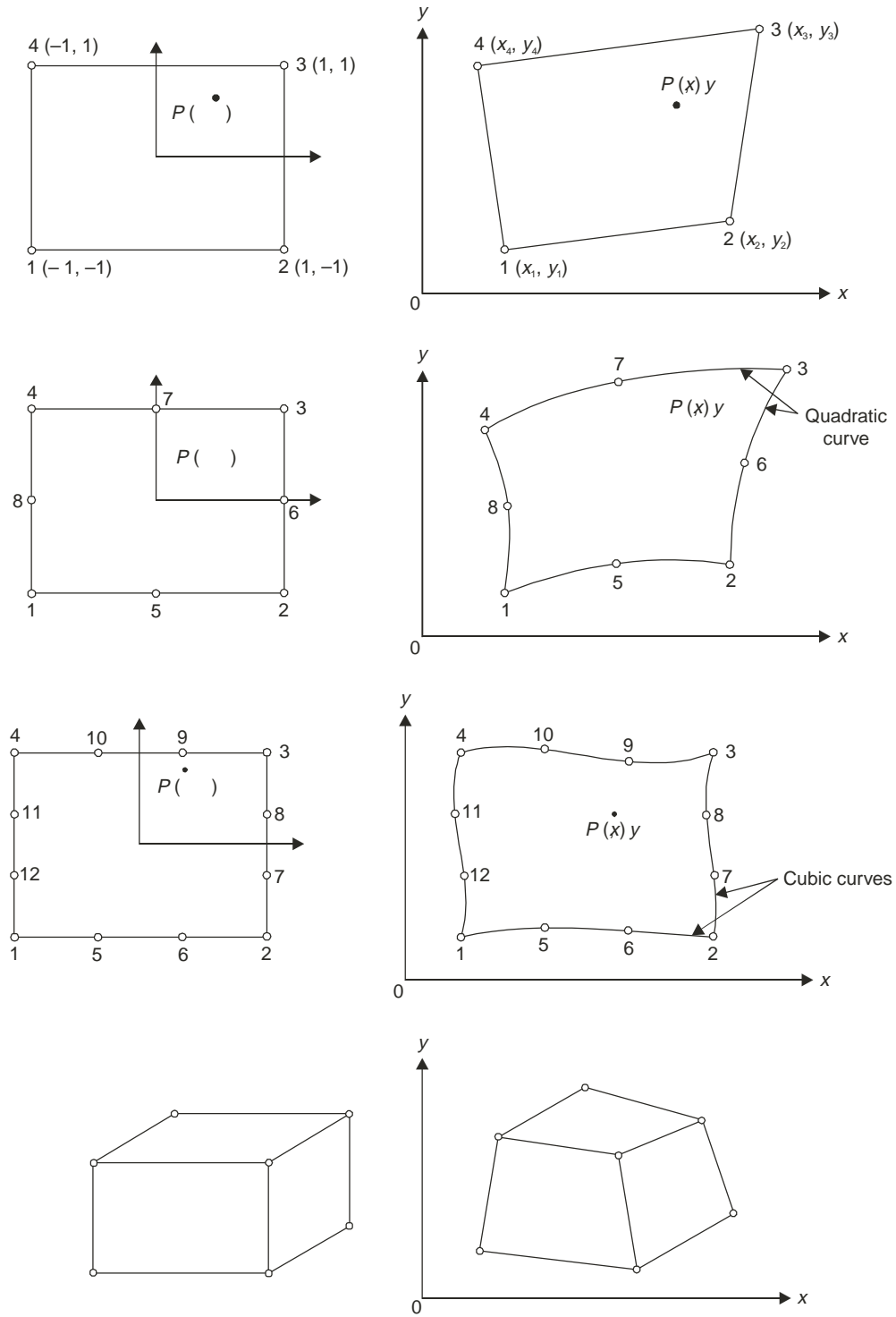


Fig. 13.1 (contd)

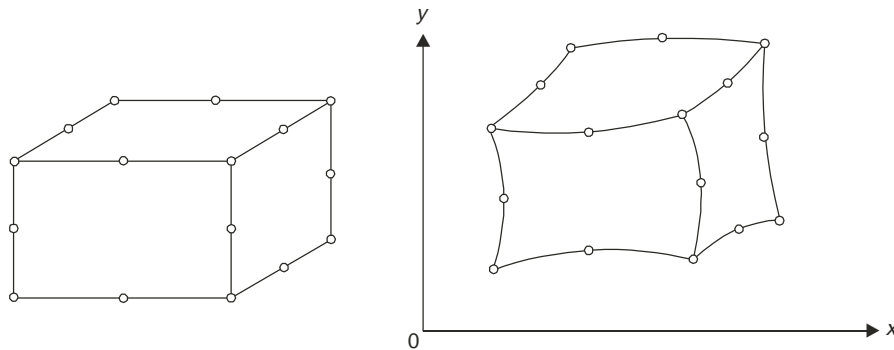


Fig. 13.1 (contd)

### 13.2 COORDINATE TRANSFORMATION

So far we have used the shape functions for defining deflection at any point in terms of the nodal displacement. Taig [1] suggested use of shape function for coordinate transformation from natural local coordinate system to global Cartesian system and successfully achieved in mapping parent element to required shape in global system. Thus the Cartesian coordinate of a point in an element may be expressed as

$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + \dots + N_n x_n \\ y &= N_1 y_1 + N_2 y_2 + \dots + N_n y_n \\ z &= N_1 z_1 + N_2 z_2 + \dots + N_n z_n \end{aligned}$$

or in matrix form

$$\{x\} = [N] \{x\}_e$$

where  $N$  are shape functions and  $(x)_e$  are the coordinates of nodal points of the element. The shape functions are to be expressed in natural coordinate system.

For example consider mapping of a rectangular parent element into a quadrilateral element:

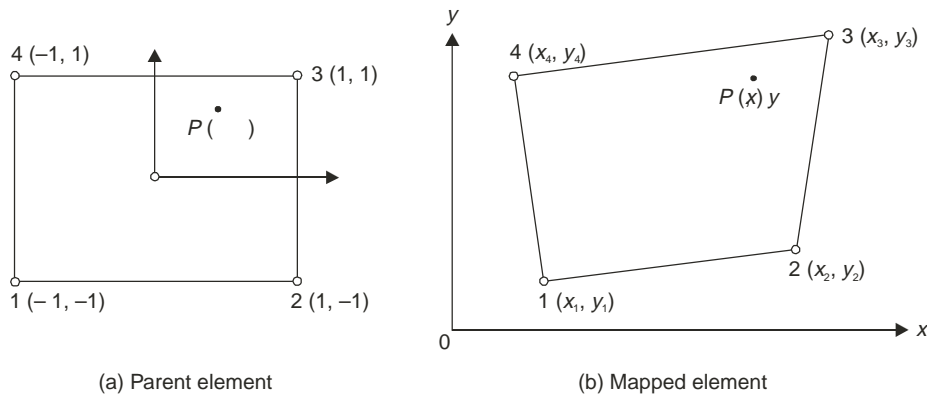


Fig. 13.2 Mapping of rectangular element in natural local coordinate system to global cartesian coordinate system

The parent rectangular element shown in Fig. 13.2 (a) has nodes 1, 2, 3 and 4 and their coordinates are  $(-1, -1)$ ,  $(-1, 1)$ ,  $(1, 1)$  and  $(1, -1)$ . The shape functions of this element are

$$N_1 = \frac{(1 - \xi)(1 - \eta)}{4}, \quad N_2 = \frac{(1 + \xi)(1 - \eta)}{4}$$

$$N_3 = \frac{(1 + \xi)(1 + \eta)}{4} \quad \text{and} \quad N_4 = \frac{(1 - \xi)(1 + \eta)}{4}$$

$P$  is a point with coordinate  $(\xi, \eta)$ . In global system the coordinates of the nodal points are

$$(x_1, y_1), (x_2, y_2), (x_3, y_3) \text{ and } (x_4, y_4)$$

To get this mapping we define the coordinate of point  $P$  as

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

and

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$

Noting that shape functions are such that at node  $i$ ,  $N_i = 1$  and all others are zero, it satisfy the coordinate value at all the nodes. Thus any point in the quadrilateral is defined in terms of nodal coordinates.

Similarly other parent elements are mapped suitably when we do coordinate transformation.

### 13.3 BASIC THEOREMS OF ISOPARAMETRIC CONCEPT

Isoparametric concept is developed based on the following three basic theorems:

**Theorem I:** If two adjacent elements are generated using shape functions, then there is continuity at the common edge.

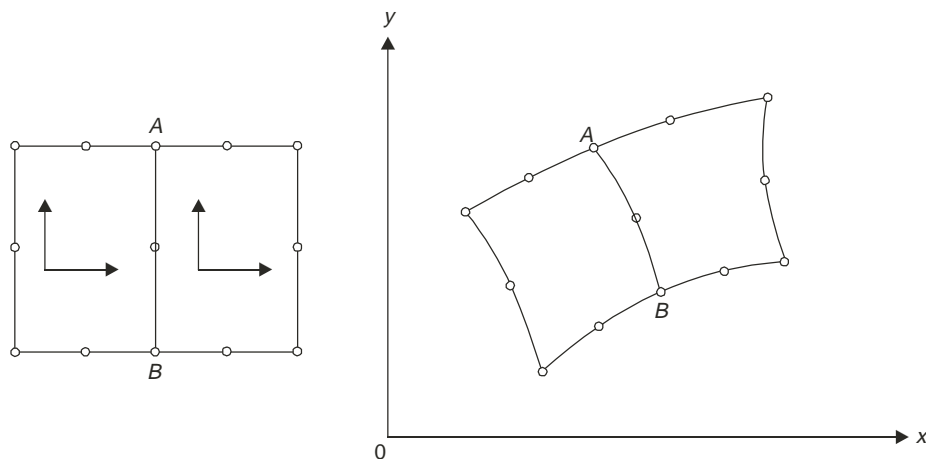


Fig. 13.3

It may be observed that in the parent element, for any point on edge  $AB$ , shape functions  $N_i = 0$  for nodes not on the edge and  $N_i$  exists for nodes on the edge. Hence the final function is the same for the common edge  $AB$  in any two adjacent elements, when we give the same coordinate values for the nodes on common edge. Hence edge  $AB$  is contiguous in the adjacent elements.

**Theorem II:** It states, if the shape functions used are such that continuity of displacement is represented in the parent coordinates, then the continuity requirement, will be satisfied in the isoparametric elements also.

The proof is same as for theorem 1.

**Theorem III:** The constant derivative conditions and condition for rigid body are satisfied for all isoparametric elements if,

$$\sum N_i = 1$$

**Proof:** Let the displacement function be

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z \quad \dots(13.1)$$

$\therefore$  Nodal displacement at 'i'th node is given by

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i$$

In finite element analysis we define nodal displacement at any point in the element in terms of nodal displacement as

$$\begin{aligned} u &= \sum N_i u_i \\ \therefore u &= \sum N_i (\alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i) \\ &= \alpha_1 \sum N_i + \alpha_2 \sum N_i x_i + \alpha_3 \sum N_i y_i + \alpha_4 \sum N_i z_i \end{aligned}$$

From the isoparametric concept, we know

$$\sum N_i x_i = x$$

$$\sum N_i y_i = y$$

$$\sum N_i z_i = z$$

$$\therefore u = \alpha_1 \sum N_i + \alpha_2 x + \alpha_3 y + \alpha_4 z \quad \dots(13.2)$$

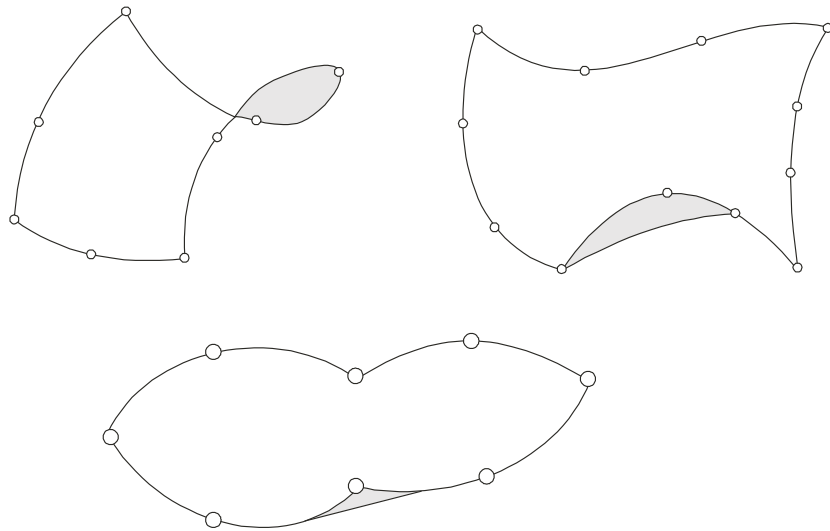
Hence if equation 13.2 has to represent equation 13.1 uniquely, then

$$\sum N_i = 1$$

The shape functions developed in natural coordinate systems satisfy this requirement. Hence they can be safely used for isoparametric representation. This theorem is known as **convergence criteria for isoparametric elements**.

### 13.4 UNIQUENESS OF MAPPING

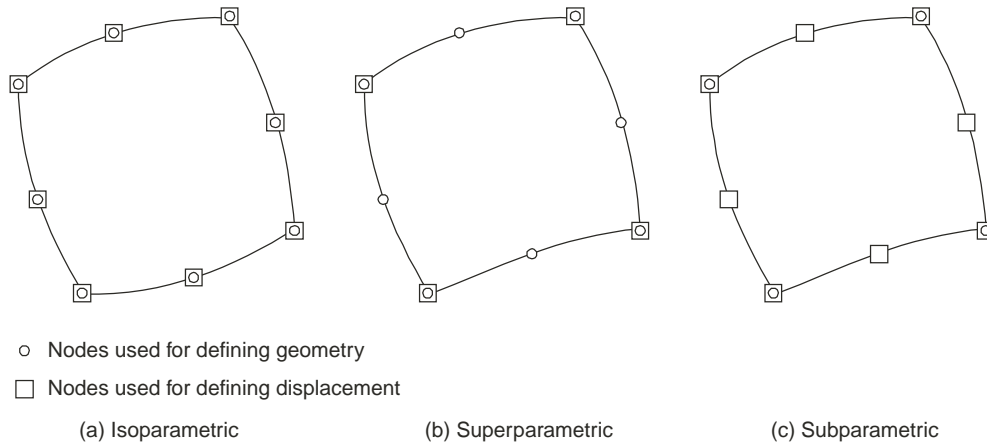
It is absolutely necessary that a point in parent element represents only one point in the isoparametric element. Some times, due to violent distortion it is possible to obtain undesirable situation of nonuniqueness. Some of such situations are shown in Fig. 13.4. If this requirement is violated determinant of Jacobian matrix (to be explained latter) becomes negative. If this happens coordinate transformation fails and hence the program is to be terminated and mapping is corrected.



**Fig. 13.4** *Non-uniqueness of mapping*

### 13.5 ISOPARAMETRIC, SUPERPARAMETRIC AND SUBPARAMETRIC ELEMENTS

We have seen that in the finite element analysis with isoparametric elements, shape functions are used for defining the geometry as well as displacements. If the shape functions defining the boundary and displacements are the same, the element is called as **isoparametric element**. For example, in Fig. 13.5 (a) all the eight nodes are used in defining the geometry and displacement. Thus, in this case



**Fig. 13.5** *Isoparametric, superparametric and subparametric elements*

$u = [N] \{\delta\}_e$ ,  $x = [N] \{x\}_e$  and  $y = [N] \{y\}_e$  where  $[N]$  is quadratic shape function of serendipity family. The element in which more number of nodes are used to define geometry compared to the number of nodes used to define displacement are known as **superparametric element**. One such element is shown in Fig. 13.5 (b) in which 8 nodes are used to define the geometry and displacement is defined using only 4 nodes. In the stress analysis where boundary is highly curved but stress gradient is not high, one can use these elements advantageously.

Figure 13.5 (c) shows a **subparametric element** in which less number of nodes are used to define geometry compared to the number of nodes used for defining the displacements. Such elements can be used advantageously in case of geometry being simple but stress gradient high.

### 13.6 ASSEMBLING STIFFNESS MATRIX

Assembling element stiffness matrix is a major part in finite element analysis. Since it involves coordinate transformation from natural local coordinate system to Cartesian global system, isoparametric elements need special treatment. In this article assembling of element stiffness matrix for 4 noded quadrilateral element is explained in detail. The procedure can be easily extended to higher order elements by using suitable functions and noting the increased number of nodes.

Figure 13.6 shows the typical parent element and isoparametric quadrilateral element.

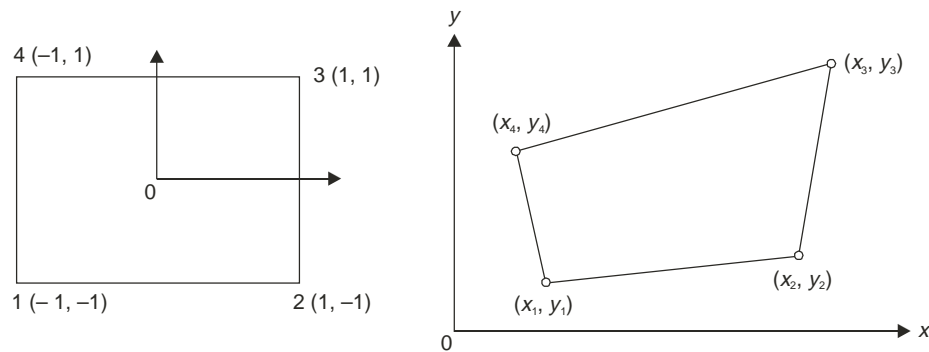


Fig. 13.6 Typical isoparametric quadrilateral element

For parent element, the shape functions are

$$N_i = \frac{(1 + \xi\xi_i)(1 + \eta\eta_i)}{4} \quad \dots(13.3)$$

i.e.,

$$N_1 = \frac{(1 - \xi)(1 - \eta)}{4}, \quad N_2 = \frac{(1 + \xi)(1 - \eta)}{4}$$

$$N_3 = \frac{(1 + \xi)(1 + \eta)}{4} \quad \text{and} \quad N_4 = \frac{(1 - \xi)(1 + \eta)}{4}$$

We use the above functions for defining the displacement as well as for defining the geometry of any point within the element in terms of nodal values.

When we use shape functions for the geometry,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_4 \\ y_4 \end{Bmatrix} \quad \dots(13.4)$$

The above relation helps to determine the  $(x, y)$  coordinates of any point in the element when the corresponding natural coordinates  $\xi$  and  $\eta$  are given.

We are also using the same functions for defining the displacement at any point in the element

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix} \quad \dots(13.5)$$

In assembling the stiffness matrix we need the derivatives of displacements with respect to global  $x, y$  system. It is easy to find derivatives with respect to local coordinates  $\xi$  and  $\eta$  but it needs suitable assembly to get the derivatives w.r.t. to global Cartesian system.

The relationship between the coordinates can be computed using chain rule of partial differentiation. Thus,

$$\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi} \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta} \frac{\partial}{\partial y}$$

i.e.,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix} \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} \quad \dots(13.6)$$

where

$$[J] = \begin{Bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{Bmatrix} \quad \dots(13.7)$$

The matrix  $[J]$  shown above is called Jacobian matrix. It relates derivative of the function in local coordinate system to derivative in global coordinate system. In case of three dimensional problem it is given by

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

Now going back to isoparametric quadrilateral element,

Let

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Where

$$J_{11} = \frac{\partial x}{\partial \xi} \quad J_{12} = \frac{\partial y}{\partial \xi}$$

$$J_{21} = \frac{\partial x}{\partial \eta} \quad J_{22} = \frac{\partial y}{\partial \eta}$$

we know,

$$x = \sum_{i=1}^4 N_i x_i = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$\therefore J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} x_4$$

Similarly  $J_{12}$ ,  $J_{21}$  and  $J_{22}$  can be assembled.

Then we get

$$J = \begin{bmatrix} \sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} y_i \\ \sum_{i=1}^4 \frac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^4 \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \quad \dots(13.8)$$

For any specified point the above matrix can be assembled. Now,

$$\begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix}$$



$$\begin{aligned} \therefore \begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{Bmatrix} &= [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \\ &= \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{Bmatrix} \end{aligned} \quad \dots(13.9)$$

where  $J_{11}^*$ ,  $J_{12}^*$ ,  $J_{21}^*$  and  $J_{22}^*$  are the elements of Jacobian inverse matrix. Since for a given point Jacobian matrix is known its inverse can be calculated and Jacobian inverse matrix is assembled. With this transformation relation known, we can express derivatives of the displacements as shown below:

$$= \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* \\ 0 & 0 & J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} \quad \dots(13.10)$$

The strain displacement relation is given by

$$\begin{aligned} \{\varepsilon\} &= \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} \\ &= \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} \end{aligned} \quad \dots(13.11)$$

$$\text{But } u = \sum_{i=1}^4 N_i u_i \text{ and } v = \sum_{i=1}^4 N_i v_i$$

$$\therefore \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Substituting it in equation 13.11 strain displacement matrix  $[B]$  is obtained as,

$$\{B\} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix} \dots(13.12)$$

Then element stiffness matrix is given by

$$[k] = \oint [B]^T [D][B] dV$$

In this case,

$$[k] = t \iint [B]^T [D][B] dx dy \dots(13.13)$$

where  $t$  is the thickness.

It can be shown that elemental area in Cartesian coordinates  $(x, y)$  can be expressed in terms of the area in local coordinates  $(\xi, \eta)$  as

$$\partial x \partial y = |J| \partial \xi \partial \eta \dots(13.14)$$

Where  $|J|$  is the determinant of the Jacobian.

$$\therefore [k] = t \iint [B]^T [D][B] |J| \partial \xi \partial \eta \dots(13.15)$$

Integration is to be performed so as to cover entire area i.e. the limit of integration is from  $\xi$  is from  $-1$  to  $1$  and  $\eta$  is also from  $-1$  to  $1$ . It is difficult to carryout all the multiplications in equation 13.15 and then the integration. It is convenient to go for numerical integration.

### 13.7 NUMERICAL INTEGRATION

In mathematics numerical integration techniques like trapezoidal rule, Simpson's  $\frac{1}{3}$  rd rule, Simpson's  $\frac{3}{8}$  th rule and Gauss quadrature formula are available. In trapezoidal rule, the variation of the function between two sampling point is assumed linear. In Simpson's  $\frac{1}{3}$  rd rule 3 sampling points are used and second degree curve is fitted. In Simpson's  $\frac{3}{8}$  th rule 4 sampling points are selected and 3rd degree (cubic) curve is fitted. All these methods are based on Newton Cotes formula in which values at  $n$  equally spaced sampling points are required to fit in  $n - 1$  degree variation curve.

In finite element analysis Gauss quadrature formula is preferred since in this values at  $n$  sampling points can be used to fit in  $2n - 1$  degree variation, as the evaluation of functions like  $B^TDBdV$  is a time consuming process. In this method, the numerical integration is achieved by the following expression.

$$\int_{-1}^1 f(\xi) d\xi = \sum_{i=1}^n w_i f(\xi_i) \quad \dots(13.16)$$

Where  $w_i$  – weight function and  $f(\xi_i)$  is values of the function at pre determined sampling points.

In Gauss quadrature formula sampling points are cleverly placed. In this, both  $n$  sampling points and  $n$  weights are treated as variables to make exact  $2n - 1$  degree polynomial. This is an open quadrature formula, the function values need not be known at end points but they must be known at predetermined sampling points.

The location of sampling points  $\xi_i$  and weight function  $w_i$  are determined using Legendre polynomials. Hence this method is some times called as Gauss Legendre quadrature formula. Table 13.1 shows location of Gauss sampling points ( $\xi_i$ ) and corresponding weight function ( $w_i$ ) for different number ( $n$ ) of Gauss integration scheme.

**Table 13.1** Location of sampling points and weight functions

in Gauss Integration  $\int_{-1}^1 f(\xi) d\xi = \sum_{i=1}^n W_i f(\xi_i)$

$n$	$\xi$	$W_i$
1	$\xi_1 = 0.00000000$	$W_1 = 2.00000000$
2	$-\xi_1 = \xi_2 = 0.57735027$	$W_1 = W_2 = 1.00000000$
3	$-\xi_1 = \xi_3 = 0.77459667$ $\xi_2 = 0.00000000$	$W_1 = W_3 = 0.55555556$ $W_2 = 0.88888889$
4	$-\xi_1 = \xi_4 = 0.86113631$ $-\xi_2 = \xi_3 = 0.33998104$	$W_1 = W_4 = 0.34785485$ $W_2 = W_3 = 0.65214515$

Fig. 13.7 shows the integration scheme for 3 point Gauss integration. It may be noted that the sampling points are symmetrically placed, all weight are positive and the weights of symmetrically placed points are same. Depending upon the degree of variation, the number of Gauss points  $n$  can be chosen so that integrations are exact. Since in finite element analysis, the exact degree of variation of the functions like element stiffness matrix are not known, preliminary investigations may be made by changing  $n$  to get stable result. Many investigators have reported that two point house integration is more than sufficient. The above scheme may be extended to 2 and 3 dimensional problems also.

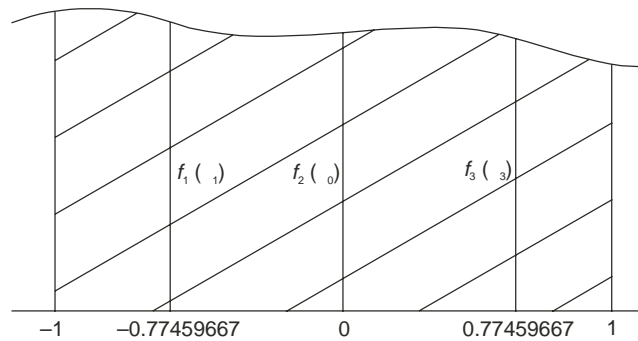


Fig. 13.7 Scheme of 3 point Gauss integration

For two dimensional problem  $n = 2$  means  $2 \times 2 = 4$  Gaussian points and for three dimensional problems it works out to be  $2 \times 2 \times 2 = 8$ . Thus,

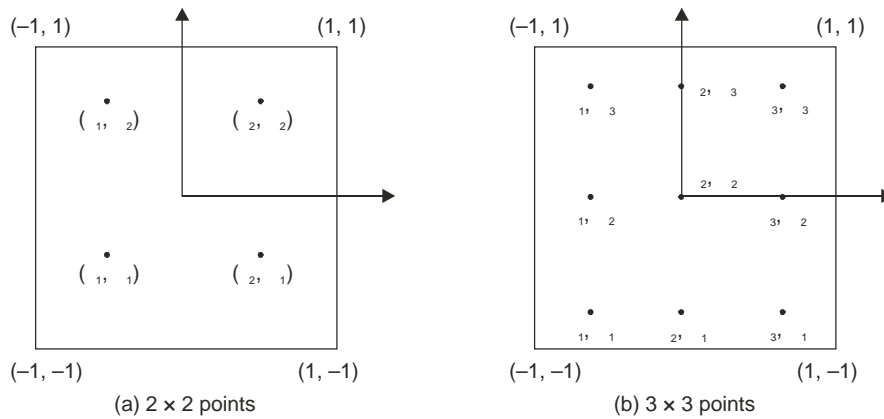


Fig. 13.8 Gauss sampling points in two dimensional elements

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta = \int_{-1}^1 \sum_{i=1}^n W_i f(\xi, \eta) d\eta$$

$$\begin{aligned}
 &= \sum_{j=1}^n W_j \left\{ \sum_{i=1}^n W_i f(\xi, \eta) \right\} \\
 &= \sum_{j=1}^n \sum_{i=1}^n W_j W_i f(\xi, \eta)
 \end{aligned}$$

For a two dimensional problem, Gauss points for  $n = 2$  and  $n = 3$  are shown in Fig. 13.8.

### 13.8 NUMERICAL EXAMPLES

Finite element analysis using isoparametric element involves too many calculations and hence not suitable for hand calculations. For such analysis one has to go for computer analysis by developing programs. With slight additional statements various elements can be easily incorporated in a single analysis package. However to make the procedure clear to learners of finite element analysis a small numerical problem is taken up here.

**Example 13.1:** Assemble Jacobian matrix and strain displacement matrix corresponding to the Gauss point (0.57735, 0.57735) for the element shown in Fig. 13.9. Then indicate how do you proceed to assemble element stiffness matrix.

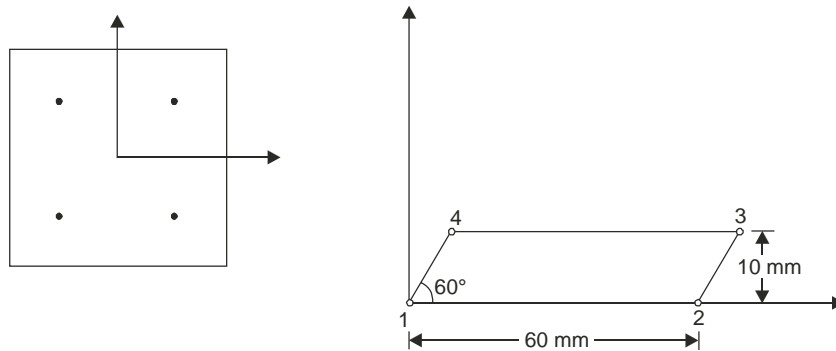


Fig. 13.9

**Solution:** The coordinates of node points in Cartesian system are (0, 0), (60, 0), (65.7735, 10) and (5.7735, 10).

The shape functions are

$$N_i = \frac{1}{4} (1 + \xi \xi_i)(1 + \eta \eta_i)$$

$$\therefore \frac{\partial N_i}{\partial \xi} = \frac{1}{4} \xi_i (1 + \eta \eta_i)$$

and

$$\frac{\partial N_i}{\partial \eta} = \frac{1}{4} \eta_i (1 + \xi \xi_i)$$

$$\therefore \frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1 - \eta) = -\frac{1}{4}(1 - 0.57735) = -0.10566$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1 - \eta) = \frac{1}{4}(1 - 0.57735) = 0.10566$$

$$\frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1 + \eta) = \frac{1}{4}(1 + 0.57735) = 0.39438$$

$$\frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1 + \eta) = -\frac{1}{4}(1 + 0.57735) = -0.39438$$

Similarly,

$$\frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1 - \xi) = -\frac{1}{4}(1 - 0.57735) = -0.10566$$

$$\frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1 + \xi) = -\frac{1}{4}(1 + 0.57735) = -0.39438$$

$$\frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1 + \xi) = \frac{1}{4}(1 + 0.57735) = 0.39438$$

$$\frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1 - \xi) = \frac{1}{4}(1 - 0.57735) = 0.10566$$

The Jacobian Matrix is given by

$$[J] = \begin{bmatrix} \sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i \\ \sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$\therefore J_{11} = \sum \frac{\partial N_i}{\partial \xi} x_i$$

$$= -0.10566 \times 0 + 0.10566 \times 60 + 0.39438 \times 65.7735 - 0.39438 \times 5.7735 = 30.0000$$

$$J_{12} = \sum \frac{\partial N_i}{\partial \xi} y_i = 0 + 0 + 0.39438 \times 10 - 0.39438 \times 10 = 0$$

$$J_{21} = \sum \frac{\partial N_i}{\partial \eta} x_i = 0 - 0.39438 \times 60 + 0.39438 \times 65.7735 + 0.10566 \times 5.7735 = 2.88698$$

$$J_{22} = \sum \frac{\partial N_i}{\partial \eta} y_i = 0 + 0 + 0.39438 \times 10 + 0.10566 \times 10 = 5.0000$$

$$\therefore [J] = \begin{bmatrix} 30.0000 & 0 \\ 2.88698 & 5.0000 \end{bmatrix}$$

$$[J^*] = [J]^{-1} \frac{1}{30.0000 \times 5.0000} \begin{bmatrix} 5.0000 & -2.88698 \\ 0 & 30.0000 \end{bmatrix} = \begin{bmatrix} 0.033333 & -0.019246 \\ 0 & 0.166667 \end{bmatrix}$$

The strain displacement matrix is given by

$$\{B\} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} & 0 \\ \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \frac{\partial N_3}{\partial \xi} & 0 & \frac{\partial N_4}{\partial \xi} \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \frac{\partial N_3}{\partial \eta} & 0 & \frac{\partial N_4}{\partial \eta} \end{bmatrix}$$

Where  $J_{ij}^*$  are the elements of Jacobian inverse matrix,

$$\therefore [B] = \begin{bmatrix} 0.033333 & -0.019246 & 0 & 0 \\ 0 & 0 & 0 & 0.166667 \\ 0 & 0.166667 & 0.033333 & -0.019246 \end{bmatrix} \times$$

$$\begin{bmatrix} -0.10566 & 0 & 0.10566 & 0 & 0.39438 & 0 & -0.39438 & 0 \\ -0.10566 & 0 & -0.39438 & 0 & 0.39438 & 0 & 0.10566 & 0 \\ 0 & -0.10566 & 0 & 0.10566 & 0 & 0.39438 & 0 & -0.39438 \\ 0 & -0.10566 & 0 & -0.39438 & 0 & 0.39438 & 0 & -0.10566 \end{bmatrix}$$

$$= \begin{bmatrix} 1.48843 \times 10^{-3} & 0 & 0.011112 & 0 & 5.55563 \times 10^{-3} & 0 & -0.022915 & 0 \\ 0 & -0.01761 & 0 & -0.06573 & 0 & 0.06573 & 0 & 0.01761 \\ -0.01761 & 1.48843 \times 10^{-3} & -0.06573 & 0.011112 & 0.06573 & 5.55563 \times 10^{-3} & 0.01761 & 0.015179 \end{bmatrix}$$

*Answer*

### Assembling Element Stiffness Matrix

$D$  matrix can be assembled for the plane stress or plane strain as the case may be, by using material properties

of the structure. Then the following matrix multiplication may be carried out to assemble  $\frac{t}{2} [B]^T [D] [B]$  for the Gauss point (0.57735, 0.57735). This value when multiplied with weight function (in this case 1) gives the contribution of the Gauss point (0.57735, 0.57735) to the element stiffness matrix.

Assembling of the Jacobian matrix and  $[B]$  matrix may be carried out for the Gauss point  $(0.57735, -0.57735)$  and then its contribution to stiffness matrix may be found. On the same line assemble the contribution of remaining two Gauss points. Addition of all the four Gauss points contribution gives the element stiffness matrix of size  $3 \times 8$ .

**Example 13.2:** Determine the Cartesian coordinate of the point  $P(\xi = 0.5, \eta = 0.6)$  shown in Fig. 13.10.

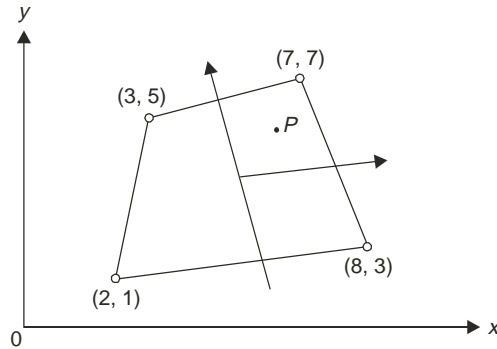


Fig. 13.10

**Solution:** It is given that

$$\xi = 0.5 \text{ and } \eta = 0.6$$

$$N_1 = \frac{(1 - \xi)(1 - \eta)}{4} = \frac{(1 - 0.5)(1 - 0.6)}{4} = 0.05$$

$$N_2 = \frac{(1 + \xi)(1 - \eta)}{4} = \frac{(1 + 0.5)(1 - 0.6)}{4} = 0.15$$

$$N_3 = \frac{(1 + \xi)(1 + \eta)}{4} = \frac{(1 + 0.5)(1 + 0.6)}{4} = 0.6$$

$$N_4 = \frac{(1 - \xi)(1 + \eta)}{4} = \frac{(1 - 0.5)(1 + 0.6)}{4} = 0.2$$

$$\therefore x = \sum N_i x_i = 0.05 \times 2 + 0.15 \times 8 + 0.6 \times 7 + 0.2 \times 3 = 6.1$$

$$y = \sum N_i y_i = 0.05 \times 1 + 0.15 \times 3 + 0.6 \times 7 + 0.2 \times 5 = 5.7$$

The Cartesian coordinates of point  $P$  are  $(6.1, 5.7)$

**Example 13.3:** In the element shown in Fig. 13.10,  $P$  is the point  $(6, 5)$ . On this point the load components in  $x$  and  $y$  directions are 8 kN and 12 kN respectively. Determine its nodal equivalent forces.

**Solution:** We have to first determine the local natural coordinates of point  $P$ . We know

$$x = \sum N_i x_i \text{ and } y = \sum N_i y_i$$



For the quadrilateral element

$$N_i = \frac{(1 + \xi\xi_i)(1 + \eta\eta_i)}{4}$$

i.e., 
$$N_1 = \frac{(1 - \xi)(1 - \eta)}{4} \quad N_2 = \frac{(1 + \xi)(1 - \eta)}{4}$$

$$N_3 = \frac{(1 + \xi)(1 + \eta)}{4} \quad N_4 = \frac{(1 - \xi)(1 + \eta)}{4}$$

$$\therefore x = \sum N_i x_i \text{ gives}$$

$$6 = \frac{1}{4} [(1 - \xi)(1 - \eta)2 + (1 + \xi)(1 - \eta)8 + (1 + \xi)(1 + \eta)7 + (1 - \xi)(1 + \eta)3]$$

$$\therefore 24 = 2(1 - \xi - \eta + \xi\eta) + 8(1 + \xi - \eta - \xi\eta) + 7(1 + \xi + \eta + \xi\eta) + 3(1 - \xi + \eta - \xi\eta)$$

$$= 20 + 10\xi + 0\eta - 2\xi\eta$$

$$4 = 10\xi - 2\xi\eta$$

...(1)

or  $2 = 5\xi - \xi\eta$

$$\therefore y = \sum N_i y_i \text{ gives}$$

$$5 = \frac{1}{4} [1(1 - \xi - \eta + \xi\eta) + 3(1 + \xi - \eta - \xi\eta) + 7(1 + \xi + \eta + \xi\eta) + 5(1 - \xi + \eta - \xi\eta)]$$

$$20 = 16 + 4\xi + 8\eta$$

or  $4 = 4\xi + 8\eta$

or  $1 = \xi + 2\eta$

...(2)

From equation (2),  $\eta = \frac{1 - \xi}{2}$

Substituting it in equation 1, we get

$$2 = 5\xi - \xi\left(\frac{1 - \xi}{2}\right)$$

or  $4 = 10\xi - \xi(1 - \xi) = 9\xi - \xi^2$

i.e.  $\xi^2 + 9\xi - 4 = 0$

$$\xi = \frac{-9 + \sqrt{9^2 + 4 \times 4}}{2} = 0.42443$$

$$\therefore \eta = \frac{1 - 0.42443}{2} = 0.28779$$

Now, the equivalent load is given by

$$\{F\} = [N]^T \begin{Bmatrix} X \\ Y \end{Bmatrix}$$

For Point  $P$ .

$$N_1 = \frac{(1 - 0.42443)(1 - 0.28779)}{4} = 0.10248$$

$$N_2 = \frac{(1 + 0.42443)(1 - 0.28779)}{4} = 0.25362$$

$$N_3 = \frac{(1 + 0.42443)(1 + 0.28779)}{4} = 0.45859$$

$$N_4 = \frac{(1 - 0.42443)(1 + 0.28779)}{4} = 0.18530$$

$$\{F_x\} = [N]^T \{X\}$$

$$\begin{Bmatrix} F_{x1} \\ F_{x2} \\ F_{x3} \\ F_{x4} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \{8\} = 8 \begin{Bmatrix} 0.10248 \\ 0.25362 \\ 0.45859 \\ 0.18530 \end{Bmatrix} = \begin{Bmatrix} 0.81984 \\ 2.02896 \\ 3.66872 \\ 1.48240 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{y1} \\ F_{y2} \\ F_{y3} \\ F_{y4} \end{Bmatrix} = \begin{Bmatrix} 0.10248 \\ 0.25362 \\ 0.45859 \\ 0.18530 \end{Bmatrix} \{12\} = \begin{Bmatrix} 1.22976 \\ 3.04344 \\ 5.50308 \\ 2.22360 \end{Bmatrix}$$

**Example 13.4:** The quadrilateral element shown in Fig. 13.11 is 20 mm thick and is subjected to surface forces  $T_x$  and  $T_y$ . Determine expressions for its equivalent nodal forces. If  $T_x = 10\text{N/mm}^2$  and  $T_y = 15\text{N/mm}^2$ , determine the numerical values of the nodal forces.

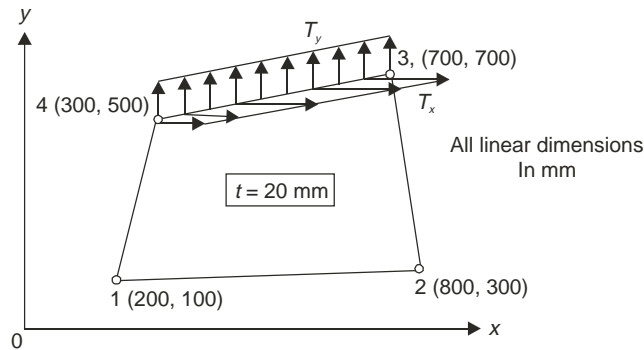


Fig. 13.11

**Solution:** The element is subjected to load along edge 3–4. We know along edge 3–4,  $\eta = 1$

$$\therefore N_1 = \frac{1}{4} (1 - \xi)(1 - \eta) = 0$$

$$N_2 = \frac{1}{4} (1 + \xi)(1 - \eta) = 0$$

$$N_3 = \frac{1}{4} (1 + \xi)(1 + \eta) = \frac{1}{2} (1 + \xi)$$

$$N_4 = \frac{1}{4} (1 - \xi)(1 + \eta) = \frac{1}{2} (1 - \xi)$$

Nodal forces are given by the expressions like

$$\begin{aligned} \{F_x\} &= \int [N]^T \{T_x\} ds \\ &= t \int [N]^T \{T_x\} dl \end{aligned}$$

We know,

$$\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2} \quad \text{and} \quad \frac{\Delta l}{\Delta \xi} = \sqrt{\left(\frac{\Delta x}{\Delta \xi}\right)^2 + \left(\frac{\Delta y}{\Delta \xi}\right)^2}$$

In isoparametric concept, we know

$$x = \sum N_i x_i \quad \text{and} \quad y = \sum N_i y_i$$

In this case, along line 3–4,

$$x = 0 + 0 + \frac{1}{2}(1 + \xi)x_3 + \frac{1}{2}(1 - \xi)x_4$$

$\therefore$  In limiting case,

$$\frac{dx}{d\xi} = \frac{\Delta x}{\Delta y} = \frac{1}{2}(x_3 - x_4)$$

Similarly,

$$y = 0 + 0 + \frac{1}{2}(1 + \xi)y_3 + \frac{1}{2}(1 - \xi)y_4$$

$\therefore$  In limiting case

$$\frac{dy}{d\xi} = \frac{1}{2}(y_3 - y_4)$$

$$\therefore \frac{dl}{d\xi} = \frac{\Delta l}{\Delta \xi} = \sqrt{\left\{\frac{1}{2}(x_3 - x_4)\right\}^2 + \left\{\frac{1}{2}(y_3 - y_4)\right\}^2} = \frac{1}{2}l_{34}$$

or

$$\begin{aligned}
 dl &= \frac{1}{2} l_{34} d\xi \\
 \therefore \{F_x\} &= t \int [N]^T \{T_x\} dl \\
 &= t \int_{-1}^1 \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}(1+\xi) \\ \frac{1}{2}(1-\xi) \end{bmatrix} \{T_x\} \frac{1}{2} l_{34} d\xi = t \frac{l_{34}}{4} \int_{-1}^1 \begin{bmatrix} 0 \\ 0 \\ (1+\xi)T_x \\ (1-\xi)T_x \end{bmatrix} d\xi
 \end{aligned}$$

For uniformly distributed load,  $T_x$  is constant,

$$\therefore \begin{Bmatrix} F_{x1} \\ F_{x2} \\ F_{x3} \\ F_{x4} \end{Bmatrix} = \frac{t l_{34}}{4} \begin{Bmatrix} 0 \\ 0 \\ 2T_x \\ 2T_y \end{Bmatrix} \text{ since } \int_{-1}^1 (1+\xi) d\xi = \int_{-1}^1 (1-\xi) d\xi = 2$$

$$\text{Similarly} = \frac{t l_{34}}{2} \begin{Bmatrix} 0 \\ 0 \\ T_x \\ T_y \end{Bmatrix}$$

$$\begin{Bmatrix} F_{y1} \\ F_{y2} \\ F_{y3} \\ F_{y4} \end{Bmatrix} = \int [N]^T \{T_y\} ds = \frac{t l_{34}}{2} \begin{Bmatrix} 0 \\ 0 \\ T_x \\ T_y \end{Bmatrix}$$

In this problem,

$$l_{34} = \sqrt{(700 - 300)^2 + (700 - 500)^2} = 447.21 \text{ mm}$$

$$t = 20 \text{ mm}$$

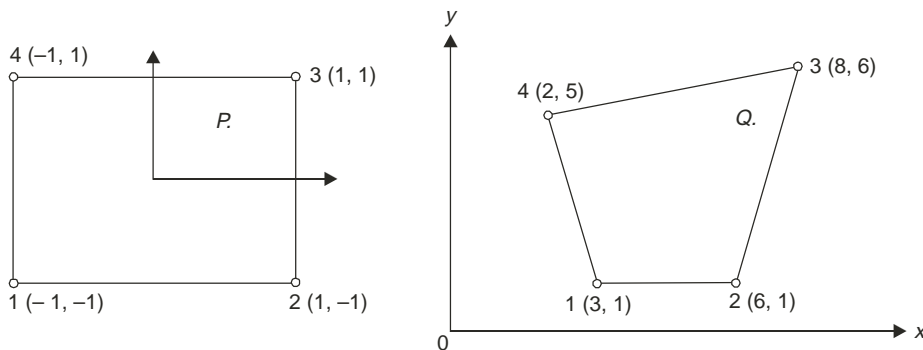
$$T_x = 10 \text{ N/mm}^2$$

$$\therefore \begin{Bmatrix} F_{x1} \\ F_{x2} \\ F_{x3} \\ F_{x4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 44721.36 \\ 44721.36 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{y1} \\ F_{y2} \\ F_{y3} \\ F_{y4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 67082.04 \\ 67082.04 \end{Bmatrix}$$

**QUESTIONS**

1. Explain the isoparametric concept in finite element analysis.
2. State and explain the three basic laws on which isoparametric concept is developed.
3. Discuss the convergence criteria for isoparametric elements.
4. Explain the terms isoparametric, subparametric and superparametric elements.
5. Write short notes on
  - (a) Uniqueness of mapping of isoparametric elements.
  - (b) Jacobian matrix
  - (c) Gaussian quadrature integration technique.
6. Explain the isoparametric elements and their advantages.
7. For the isoparametric quadrilateral elements shown in Fig. 13.12, determine
  - (a) Cartesian coordinates of the point  $P$  which has local coordinates  $\xi = 0.57735$  and  $\eta = 0.57735$  (Ans.  $x = 6.36603, y = 4.75088$ )
  - (b) Local coordinates of the point  $Q$  which has Cartesian coordinates  $(7, 4)$  (Ans.  $\xi = 0.91255, \eta = 0.21059$ )



**Fig. 13.12**

8. For the element shown in Fig. 13.13, assemble Jacobian matrix and strain displacement matrix for the Gaussian point  $(0.57735, 0.57735)$ .

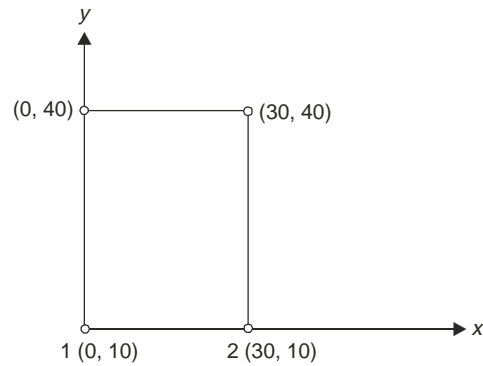


Fig. 13.13

$$[Ans. J = \begin{bmatrix} 15.0012 & 0 \\ 0 & 20.0016 \end{bmatrix}]$$

$$B = \begin{bmatrix} -7.044 \times 10^{-3} & 0 & 7.044 \times 10^{-3} & 0 & 0.026292 & 0 & -0.026292 & 0 \\ 0 & -5.283 \times 10^{-3} & 0 & -0.019719 & 0 & 0.019719 & 0 & 5.283 \times 10^{-3} \\ -0.019719 & -7.044 \times 10^{-3} & -0.019719 & 7.044 \times 10^{-3} & 0.019719 & 0.026292 & 5.283 \times 10^{-3} & 0.026292 \end{bmatrix}$$

**References**

1. Taig I.C., 'Structural Analysis by the Matrix Displacement Method', *Engl. Electric Aviation Report*, No. 5017, 1961
2. Irons B-14, 'Engineering Application of Numerical Integration in Stiffness Method', *Journal of AIAA*, 14 2035-7, 1966.

# 14

## Analysis of Beams and Rigid Frames

### 14.1 INTRODUCTION

In Chapter 3 direct stiffness formulation of beams was discussed, which is actually stiffness matrix method for the analysis. In Chapter 5 shape functions were derived for two noded beam element by considering polynomial interpolation function and also using Hermitian functions both ultimately result into same shape functions. Assembling of strain displacement matrix [B-matrix] was presented in Chapter 4. In this chapter assembling of stiffness matrix for a two noded beam element by variational approach is presented. Taking simple problems for hand calculation, formulation of system equations and solution procedure is explained. The members of rigid frames are similar to beam elements, but their orientation in global system are different. Transforming the stiffness equations of beam element to global system is required before assembling global system. This aspect is presented and illustrated with numerical problem taking simple frame analysis.

It was found difficult to extend the beam theory discussed above to plate bending. A new beam theory was developed in which lateral deflection  $w$  and rotation  $\theta_x$  were decoupled and treated as independent variables. However it needs inclusion of shear deformation and hence shear strain energy. Hence the beam theory becomes  $C^0$ -continuity problem. This is known as Timoshenko Beam theory and the element developed on this theory is also presented in this chapter.

### 14.2 BEAM ANALYSIS USING TWO NODED ELEMENTS

The typical beam element is shown in Fig. 14.1. Note the orientation of axes is as per the right hand thumb rule.

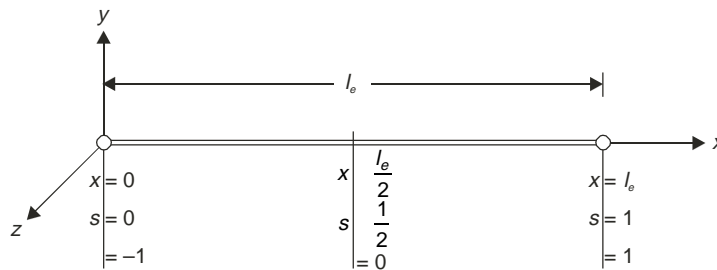


Fig. 14.1 A typical beam element

The nodal variable vector is

$$\{\delta\}^T = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4] = [w_1 \quad \theta_1 \quad w_2 \quad \theta_2]$$

where  $w_i$ —lateral displacement at node  $i$ .

$\theta_i$ —Rotation at node  $i$ .

In Chapter 5 the shape functions for such element have been determined as,

$$\begin{aligned} N_1 &= l - \frac{3x^2}{l_e^2} + \frac{2x^3}{l_e^3} \\ N_2 &= x - \frac{2x^2}{l_e} + \frac{x^3}{l_e^2} \\ N_3 &= \frac{3x^2}{l_e^2} - \frac{2x^3}{l_e^3} \quad \text{and} \quad N_4 = \frac{-x^2}{l_e} + \frac{x^3}{l_e^2} \end{aligned} \quad \dots(14.1)$$

If non-dimensiolising is done using  $s = \frac{x}{l_e}$ , the shape functions are (Chapter 4, Art 4).,

$$\begin{aligned} N_1 &= 1 - 3s^2 + 2s^3 \\ N_2 &= l_e s(s - 1)^2 \\ N_3 &= s(3 - 2s) \\ N_4 &= l_e s^2(s - 1) \end{aligned} \quad \dots(14.2)$$

If we use non-dimensiolising concept as used in isopatametric formulation i.e.  $\xi$  varying from  $-1$  to  $1$ , then the shape functions are

$$\begin{aligned} N_1 &= \frac{2 - 3\xi + \xi^3}{4} \\ N_2 &= \frac{l_e}{2} \frac{1 - \xi - \xi^2 + \xi^3}{4} \\ N_3 &= \frac{2 + 3\xi - \xi^3}{4} \quad \text{and} \quad N_4 = \frac{l_e}{2} \frac{-1 - \xi + \xi^2 + \xi^3}{4} \end{aligned} \quad \dots(14.3)$$

where  $\xi = \frac{2x}{l_e} - 1$ .

### Moment Curvature Relation

From basic solid mechanics we know

$$M = EI \frac{\partial^2 y}{\partial x^2}$$

Since

$$y = [N]\{\delta\}_e = [N_1 \quad N_2 \quad N_3 \quad N_4]\{\delta\}_e$$



we get

$$M = EI \begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2} \end{bmatrix} \{\delta_e\}_e$$

$$= [D] [B] \{\delta\}_e \quad \dots(14.4)$$

where

$$[D] = EI \quad \dots(14.5)$$

$$[B] = \begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2} \end{bmatrix} \quad \dots(14.6)$$

which is stress resultant curvature matrix.

### Strain Energy

From basic solid mechanics we know strain energy  $dU_e$  in an elemental length  $dx$  is given by

$$dU_e = \frac{1}{2} \frac{M^2}{EI} dV$$

$$U_e = \frac{1}{2} \int_0^{l_e} \left\{ EI \frac{\partial^2 y}{\partial x^2} \right\}^2 \frac{1}{EI} dx$$

$$= \frac{1}{2} \int_0^{l_e} EI \frac{\partial^2 y}{\partial x^2} dx = \frac{1}{2} \int_0^{l_e} EI ([B]\{\delta_e\})^T [B]\{\delta_e\} dx$$

$$= \frac{1}{2} \{\delta_e\}^T \int_0^{l_e} EI [B]^T [B] \{\delta_e\} \frac{l_e}{2} d\xi$$

since

$$\xi = \frac{2x}{l_e} - 1$$

$$\therefore U_e = EI \frac{l_e}{2} \{\delta_e\}^T \int_0^{l_e} [B]^T [B] \{\delta_e\} d\xi \quad \dots(14.7)$$

Now,

$$[B] = \begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2} \end{bmatrix}$$

since

$$\xi = \frac{2x}{l_e} - 1$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{2}{l_e}$$

$$\begin{aligned}
\therefore \frac{\partial^2 N_i}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial N_i}{\partial x} \right) = \frac{2}{l_e} \frac{\partial}{\partial \xi} \left( \frac{2}{l_e} \frac{\partial N_i}{\partial \xi} \right) \\
&= \frac{4}{l_e^2} \frac{\partial^2 N_i}{d\xi^2} \\
\therefore [B] &= \frac{4}{l_e^2} \left[ \frac{\partial^2 N_1}{\partial \xi^2} \quad \frac{\partial^2 N_2}{\partial \xi^2} \quad \frac{\partial^2 N_3}{\partial \xi^2} \quad \frac{\partial^2 N_4}{\partial \xi^2} \right] \\
&= \frac{4}{l_e^2} \left[ \frac{6\xi}{4} \quad -\frac{(1-3\xi)}{4} l_e \quad -\frac{6\xi}{4} \quad \frac{(1+3\xi)}{4} l_e \right] \\
&= \frac{1}{l_e^2} [6\xi \quad -(1-3\xi)l_e \quad -6\xi \quad (1+3\xi)l_e] \quad \dots(14.8)
\end{aligned}$$

$$\begin{aligned}
\therefore [B]^T [B] &= \frac{1}{l_e^2} \begin{bmatrix} 6\xi \\ -(1-3\xi)l_e \\ -6\xi \\ l_e(1+3\xi) \end{bmatrix} \frac{1}{l_e^2} [6\xi \quad -(1-3\xi)l_e \quad -6\xi \quad (1+3\xi)l_e] \\
&= \frac{1}{l_e^4} \begin{bmatrix} 36\xi^2 & (1-3\xi)l_e(6\xi) & -36\xi^2 & 6\xi(1+3\xi)l_e \\ & l_e^2(1-3\xi)^2 & 6\xi(1-3\xi)l_e & -(1+3\xi)(1-3\xi)l_e^2 \\ & SYM & 36\xi^2 & -6\xi(1+3\xi)l_e \\ & & & l_e^2(1+3\xi)^2 \end{bmatrix} \quad \dots(14.9)
\end{aligned}$$

Now noting that

$$\int_{-1}^1 \xi d\xi = 2, \quad \int_{-1}^1 \xi^2 d\xi = 0, \quad \text{and} \quad \int_{-1}^1 \xi^3 d\xi = \frac{2}{3}$$

We can write

$$\begin{aligned}
U_e &= \frac{1}{2} EI \{\delta\}_e^T \frac{l_e}{2} \frac{1}{l_e^4} \begin{bmatrix} 24 & 12l_e & -24 & 12l_e \\ & 8l_e^2 & -12l_e & 4l_e^2 \\ & SYM & 24 & -12l_e \\ & & & 8l_e^2 \end{bmatrix} \{\delta\}_e \\
&= \frac{1}{2} \{\delta\}_e^T \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ & 4l_e^2 & -6l_e & 2l_e^2 \\ & SYM & 12 & -6l_e \\ & & & 4l_e^2 \end{bmatrix} \{\delta\}_e = \frac{1}{2} \{\delta\}_e^T [k]_e \{\delta\}_e \quad \dots(14.10a)
\end{aligned}$$

Where  $[k]_e$  is element stiffness matrix and

$$\text{is equal to } \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 4l_e^2 & -6l_e & 2l_e^2 & \\ SYM & 12 & -6l_e & \\ & & & 4l_e^2 \end{bmatrix} \quad \dots(14.10b)$$

### Potential Energy

Potential energy of an element is equal to strain energy minus the work done by the external forces acting on the element. Thus

$$\Pi_e = \frac{1}{2} U_e - \int_0^{l_e} py dx - \sum P_m Y_m - \sum M_k \left( \frac{dy}{dx} \right)_k \quad \dots(14.11)$$

Where  $p$ —Distributed load per unit length

$P_m$ —Concentrated load at point  $m$

$M_k$ —External moment applied at  $k$ .

The strain energy term  $U_e$  has been already derived. The work done by external loads can be assembled as explained below:

Due to Uniformly Distributed load  $P$ /unit length:

$$\begin{aligned} \oint py dx &= \oint p [N_1 \quad N_2 \quad N_3 \quad N_4] \{\delta\}_e \frac{l_e}{2} d\xi \\ &= \frac{pl_e}{2} \int_{-1}^1 \left[ \frac{2-3\xi+\xi^3}{4} \quad \frac{l_e}{2} \frac{1-\xi-\xi^2+\xi^3}{4} \quad \frac{2+3\xi-\xi^3}{4} \quad \frac{l_e}{2} \frac{-1-\xi+\xi^2+\xi^3}{4} \right] \{\delta\}_e d\xi \end{aligned}$$

Noting that  $\int_{-1}^1 C d\xi = 2C$ ,  $\int_{-1}^1 \xi d\xi = \int_{-1}^1 \xi^3 d\xi = 0$  and

$$\int_{-1}^1 \xi^2 d\xi = \frac{2}{3}, \text{ we get}$$

$$\begin{aligned} \int_{-1}^1 py dx &= \frac{pl_e}{8} \left[ 2 \times 2 \quad \frac{l_e}{2} \left( 2 - \frac{2}{3} \right) \quad 2 \times 2 \quad \frac{l_e}{2} \left( -2 + \frac{2}{3} \right) \right] \{\delta\}_e \\ &= \left[ \frac{pl_e}{2} \quad \frac{pl_e^2}{12} \quad \frac{pl_e}{2} \quad -\frac{pl_e^2}{12} \right] \{\delta\}_e \quad \dots(14.12) \end{aligned}$$

This equivalent load on the element is shown in the Fig. 14.2. The point loads like  $P_m$  and  $M_k$  are readily taken care by introducing nodes at the points of application.

Thus work done by external load is assembled. Let it be represented by

$$\begin{aligned} \sum F_i \delta_i &= F_1 \delta_1 + F_2 \delta_2 + F_3 \delta_3 + F_4 \delta_4 \\ &= \{\delta\}_e^T \{F\}_e \end{aligned} \quad \dots(14.13)$$

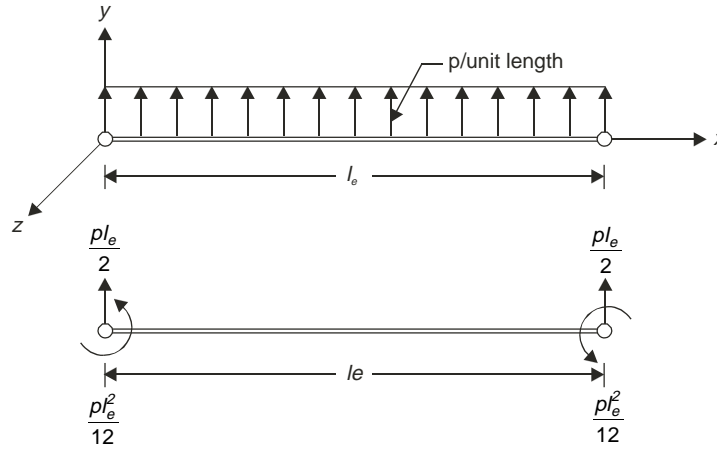


Fig. 14.2 Equivalent nodal loads of UDL

### Minimization of Potential Energy

It is to be noted that, in solid mechanics, minimization of potential energy of entire structure is to be assembled. In finite element analysis the total potential energy of the system is considered as the summation of total potential energy of the elements. Thus

$$\begin{aligned} \Pi &= \sum \Pi_e \\ &= \sum U_e - \sum \{\delta\}_e^T \{F\}_e = \frac{1}{2} \{F\}^T [k] \{\delta\} - \{\delta\}^T \{F\} \end{aligned}$$

where  $\{\delta\}$  and  $\{F\}$  are nodal unknown vector and load vector respectively. From the principle of minimization of potential energy we get,

$$\frac{d\Pi}{d\delta} = 0$$

$$[k] \{\delta\} - F = 0$$

or

$$[k] \{\delta\} = F \quad \dots(14.15)$$

In finite element analysis, element stiffness matrix  $[k]_e$  is assembled and placed in global matrix at appropriate place. When this process is completed for all the elements, we get global stiffness matrix  $[k]$ . Similarly global load vector  $\{F\}$  is assembled.

The necessary boundary conditions are imposed by (i) Elimination Method if the hand calculations are made or by (ii) Penalty Method if computers are used.

The solution of equation 14.14 gives the displacement vector  $\{\delta\}$ . The required stress resultants are determined for each element.

Moment at nodes:

$$\begin{aligned}
 M &= EI[B]\{\delta\}_e \\
 &= \frac{EI}{l_e^2} [6\xi \quad -(1-3\xi)l_e \quad -6\xi \quad (1+3\xi)l_e] \{\delta\}_e \quad \dots(14.15)
 \end{aligned}$$

Shear forces at nodes

$$\begin{aligned}
 V &= -\frac{dM}{dx} = -\frac{dM}{d\xi} \frac{d\xi}{dx} = -\frac{2}{l_e} \frac{dM}{d\xi} \\
 &= \frac{2EI}{l_e^3} [6 \quad 3l_e \quad -6 \quad 3l_e] \{\delta\}_e = \frac{EI}{l_e^3} [12 \quad 6l_e \quad -12 \quad 6l_e] \{\delta\}_e \quad \dots(14.16)
 \end{aligned}$$

The reactions at supports are nothing but end equilibrium forces. Hence

$$\{R\} = [k]_e [\delta]_e - \{F\}_e \quad \dots(14.17)$$

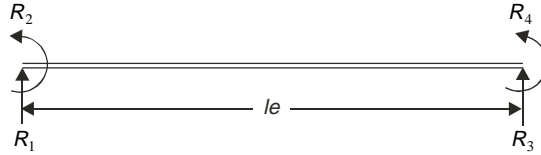


Fig. 14.3 Positive reactions

For uniformly distributed load the above equation will be

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} - \begin{Bmatrix} \frac{pl_e}{2} \\ \frac{pl_e^2}{12} \\ \frac{pl_e}{2} \\ \frac{pl_e^2}{12} \end{Bmatrix}$$

Note that the above reactions are to be interpreted as per sign convention i.e.  $R_1, R_3$  are positive in positive direction of z-axis (i.e. downward) and  $R_2, R_4$  (the end moments) are positive when they are clockwise (refer Fig. 14.3).

**Example 14.1:** Analyse the beam shown in Fig. 14.4 (a) by finite element method and determine the end reactions. Also determine the deflections at mid spans given

$$E = 2 \times 10^5 \text{ N/mm}^2 \text{ and } I = 5 \times 10^6 \text{ mm}^4$$

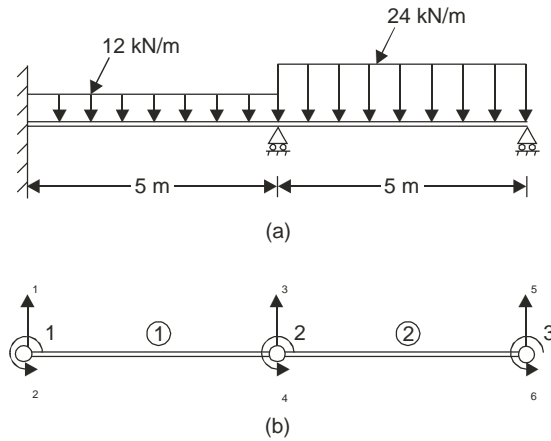


Fig. 14.4

**Solution:** Using kN and m units throughout,

$$E = 2 \times 10^5 \text{ N/mm}^2 = 2 \times 10^5 \times \frac{10^6}{10^3} = 2 \times 10^8 \text{ kN/m}^2$$

$$I = 5 \times 10^6 \text{ mm}^4 = 5 \times 10^{-6} \text{ m}^4$$

$$\therefore EI = 2 \times 10^8 \times 5 \times 10^{-6} = 1000 \text{ kN-m}^2$$

Let the two elements be numbered as shown in Fig. 14.4 (b). The nodal displacement vector is

$$\{\delta\}^T = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6]$$

$$k_1 = \frac{E_1 I_1}{l_1^3} \begin{bmatrix} 12 & 6l_1 & -12 & 6l_1 \\ 6l_1 & 4l_1^2 & -6l_1 & 2l_1^2 \\ -12 & -6l_1 & 12 & -6l_1 \\ 6l_1 & 2l_1^2 & -6l_1 & 4l_1^2 \end{bmatrix} = \frac{1000}{5^3} \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix}$$

$$= 8 \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \leftarrow \delta \text{ Global Numbers} \\ \downarrow \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \end{matrix} \end{matrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Similarly,

$$k_2 = 8 \begin{matrix} \begin{matrix} 3 & 4 & 5 & 6 \leftarrow \delta \text{ Global Numbers} \\ \downarrow \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \end{matrix} \end{matrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\therefore [k] = 8 \begin{bmatrix} 12 & 30 & -12 & 30 & & & \\ 30 & 100 & -30 & 50 & & & \\ -12 & 30 & \frac{12}{+12} & \frac{-30}{+30} & -12 & 30 & \\ 30 & 50 & \frac{-30}{+30} & \frac{100}{+100} & -30 & 50 & \\ & & -12 & -30 & 12 & -30 & \\ & & 30 & 50 & -30 & 100 & \end{bmatrix} \begin{matrix} \leftarrow \delta \text{ global numbers} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\therefore [k] = 8 \begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix} \begin{matrix} \leftarrow \delta \text{ global numbers} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

Consistent load vector is given by

$$F = \begin{Bmatrix} \frac{pl_e}{2} \\ \frac{pl_e^2}{12} \\ \frac{pl_e}{2} \\ \frac{pl_e^2}{12} \end{Bmatrix}$$

$$\therefore F^{(1)} = \begin{Bmatrix} \frac{(-12) \times 5}{2} \\ \frac{(-12) \times 5^2}{12} \\ \frac{(-12) \times 5}{2} \\ \frac{-(-12) \times 5^2}{12} \end{Bmatrix} = \begin{Bmatrix} -30 \\ -25 \\ -30 \\ 25 \end{Bmatrix} \begin{matrix} \delta \text{ Global Numbers} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\therefore F^{(2)} = \begin{Bmatrix} \frac{(-24) \times 5}{2} \\ \frac{(-24) \times 5^2}{12} \\ \frac{(-24) \times 5}{2} \\ \frac{-(-24) \times 5^2}{12} \end{Bmatrix} = \begin{Bmatrix} -60 \\ -50 \\ -60 \\ 50 \end{Bmatrix} \begin{matrix} \delta \text{ Global Numbers} \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\therefore \{F\} = \begin{Bmatrix} -30 \\ -25 \\ -90 \\ -25 \\ -60 \\ 50 \end{Bmatrix}$$

$\therefore$  The stiffness equation is

$$8 \begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -25 \\ -90 \\ -25 \\ -60 \\ 50 \end{Bmatrix}$$

Boundary conditions:

In the given problem the boundary conditions are

$$\delta_1 = \delta_2 = \delta_3 = \delta_5 = 0$$

Imposing them by elimination method, we get

$$8 \begin{bmatrix} 200 & 50 \\ 50 & 100 \end{bmatrix} \begin{Bmatrix} \delta_4 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} -25 \\ 50 \end{Bmatrix}$$

i.e.,

$$400 \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \delta_4 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} -25 \\ 50 \end{Bmatrix}$$

i.e.,

$$\begin{Bmatrix} \delta_4 \\ \delta_6 \end{Bmatrix} = \frac{1}{400} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} -25 \\ 50 \end{Bmatrix}$$



$$= \frac{1}{400} \times \frac{1}{8-1} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} -25 \\ 50 \end{Bmatrix} = \frac{1}{2800} \begin{bmatrix} -100 \\ 225 \end{bmatrix} \quad \text{Answer}$$

End reactions

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = 8 \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{-100}{2800} \end{Bmatrix} - \begin{Bmatrix} -30 \\ -25 \\ -30 \\ 25 \end{Bmatrix}$$

since  $\delta_1 = \delta_2 = \delta_3 = 0$  and  $\delta_4 = -\frac{100}{2800} = \begin{Bmatrix} 21.429 \\ 10.714 \\ -38.571 \\ -53.571 \end{Bmatrix}$  Answer

For element 2

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \end{Bmatrix} = 8 \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{-100}{2800} \\ 0 \\ \frac{225}{2800} \end{Bmatrix} - \begin{Bmatrix} -60 \\ -50 \\ -60 \\ 50 \end{Bmatrix} = \begin{Bmatrix} 70.714 \\ 53.571 \\ 49.286 \\ 0 \end{Bmatrix} \quad \text{Answer}$$

Deflection at mid span

$$y = [N_1 \quad N_2 \quad N_3 \quad N_4] \{\delta\}_e$$

$$= \left[ \frac{2 - 3\xi + \xi^3}{4} \quad \frac{l_e}{2} \frac{1 - \xi - \xi^2 + \xi^3}{4} \quad \frac{2 + 3\xi - \xi^3}{4} \quad \frac{l_e}{2} \frac{-1 - \xi + \xi^2 - \xi^3}{4} \right] \{\delta\}$$

For mid span  $\xi = 0$

$$\therefore y_{\text{centre}} = [0.5 \quad 0.125l_e \quad 0.5 \quad -0.125l_e] \{\delta\}$$

For element 1,

$$\therefore y_1 \text{ centre} = [0.5 \quad 0.125 \times 5 \quad 0.5 \quad -0.125 \times 5] \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -\frac{100}{2800} \end{Bmatrix}$$

$$= 0.02232 \text{ m} = 22.32 \text{ mm}$$

For element 2,

$$\therefore y_2 \text{ centre} = [0.5 \quad 0.125 \times 5 \quad 0.5 \quad -0.125 \times 5] \left\{ \begin{array}{c} 0 \\ 100 \\ \hline 2800 \\ 0 \\ \hline 225 \\ \hline 2800 \end{array} \right\}$$

$$= -0.02790 \text{ m} = -27.9 \text{ mm}$$

$$= 27.9 \text{ mm, downward } \textit{Answer}$$

**Example 14.2:** A beam of length 10 m, fixed at one end and supported by a roller at the other end carries a 20 kN concentrated load at the centre of the span. By taking the modulus of elasticity of material as 200 GPa and moment of inertia as  $24 \times 10^{-6} \text{ m}^4$ , determine:

1. Deflection under load
2. Shear force and bending moment at mid span
3. Reactions at supports

**Solution:** The beam is shown in Fig. 14.5 (a). Its finite element idealization is shown in Fig. 14.5 (b). In this problem,

$$E = 200 \text{ GPa} = 200 \times 10^9 \text{ N/m}^2 = 200 \times 10^6 \text{ kN/m}^2$$

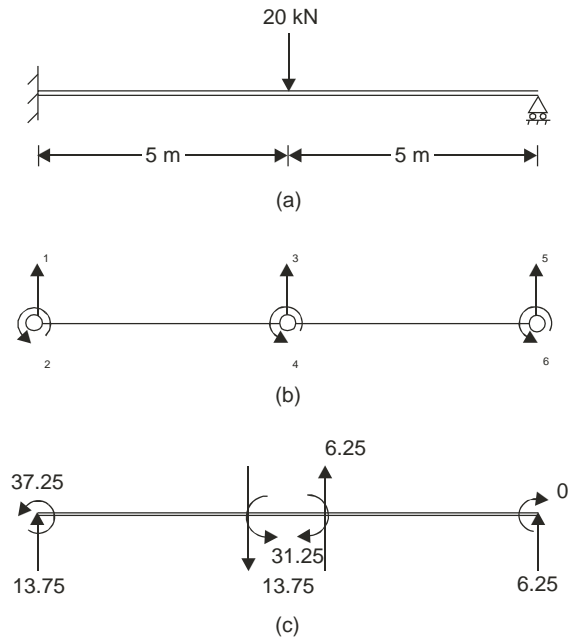


Fig. 14.5

Nodal displacement vector is

$$\{\delta\}^T = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6]$$

Stiffness matrix for an element is given by

$$k_i = \frac{E_i I_i}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$\therefore k_1 = \frac{4800}{125} \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \begin{matrix} 1 & 2 & 3 & 4 \\ \leftarrow \delta, \text{ Global Numbers} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Similarly,

$$k_2 = \frac{4800}{125} \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \begin{matrix} 3 & 4 & 5 & 6 \\ \leftarrow \delta, \text{ Global Numbers} \\ \downarrow \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$$\therefore [k] = \frac{4800}{125} \begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & 50 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \leftarrow \delta, \text{ Global Number} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The consistent load vector is directly available as

$$\{F\}^T = [0 \quad 0 \quad -20 \quad 0 \quad 0 \quad 0]$$

The stiffness equation is

$$[k]\{\delta\} = \{F\}$$

The boundary conditions are,

$$\delta_1 = \delta_2 = \delta_5 = 0$$

Imposing these boundary conditions by elimination method, the stiffness equation reduces to

$$\frac{EI}{125} \begin{bmatrix} 24 & 0 & 30 \\ 0 & 200 & 50 \\ 30 & 50 & 100 \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_4 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix}$$

By using direct inversion, we get

$$\begin{aligned} \begin{Bmatrix} \delta_3 \\ \delta_4 \\ \delta_6 \end{Bmatrix} &= \frac{125}{EI} \times \frac{1}{24(20000 - 2500) + 30(-6000)} \begin{bmatrix} 20000 - 2500 & 1500 & -6000 \\ 1500 & 2400 - 900 & -1200 \\ -6000 & -1200 & 4800 \end{bmatrix} \begin{Bmatrix} 20 \\ 0 \\ 0 \end{Bmatrix} \\ &= \frac{125}{EI} \frac{1}{240000} \begin{bmatrix} 17500 & 1500 & -6000 \\ 1500 & 1500 & -1200 \\ -6000 & -1200 & 4800 \end{bmatrix} \begin{Bmatrix} -20 \\ 0 \\ 0 \end{Bmatrix} \\ &= \frac{1}{EI} \begin{Bmatrix} -182.292 \\ -15.625 \\ 62.5 \end{Bmatrix} = \frac{1}{4800} \begin{Bmatrix} -182.292 \\ -15.625 \\ 62.5 \end{Bmatrix} = \begin{Bmatrix} -0.03798 \\ -3.2552 \times 10^{-3} \\ 0.01302 \end{Bmatrix} \end{aligned}$$

Deflection under the load =  $\delta_3 = -0.03798$  and rotation under the load =  $\delta_4 = -3.2552 \times 10^{-6}$  radians

Shear Force and bending Moment at midspan: Considering element (1), for mid span of beam  $\xi = 1$

$$\begin{aligned} M &= \frac{EI}{l_e^2} \begin{bmatrix} 6 & 2l_e & -6 & 4l_e \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 0 \\ 0 \\ -182.292 \\ -15.625 \end{bmatrix} \\ &= \frac{1}{25} \begin{bmatrix} 6 & 10 & -6 & 20 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -182.292 \\ -15.625 \end{bmatrix} = 31.250 \text{ KN m } \textit{Answer} \end{aligned}$$

$$V = \frac{EI}{125} \begin{bmatrix} 12 & 6 \times 5 & -12 & 6 \times 5 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 0 \\ 0 \\ -182.292 \\ -15.625 \end{bmatrix} = 31.75 \text{ kN } \textit{Answer}$$

Considering element 2, shear force at mid span may be found as 6.25 kN

**End Reactions**

At support on left hand side, from element 1,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{EI}{125} \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 0 \\ 0 \\ -182.292 \\ -15.625 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} 13.75 \\ 37.50 \end{Bmatrix}$$

Considering element 2, the reactions at right hand supports can be obtained

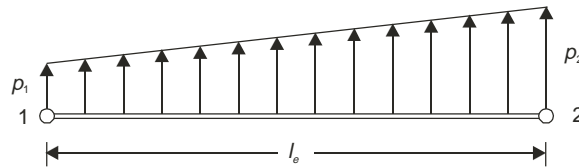
$$\begin{Bmatrix} R_5 \\ R_6 \end{Bmatrix} = \frac{EI}{125} \begin{bmatrix} -12 & -30 & 12 & -30 \\ 30 & 100 & -30 & 100 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} -182.292 \\ -15.625 \\ 0 \\ 62.5 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 6.25 \\ 0 \end{Bmatrix}$$

The reactions and shear force and bending moment at midspan are shown in Fig. 14.5 (c).

**Example 14.3:** Derive the expression for constant load, which varies linearly from  $p_1$  at node 1 to  $p_2$  at node 2 on a beam element of length  $l_e$ .

**Solution:** The element with given load is shown in Fig. 14.6.



**Fig. 14.6**

The load intensity at a point  $x$  distance from node 1 is given by

$$\begin{aligned} p &= p_1 + (p_2 - p_1) \frac{x}{l_e} \\ &= p_1 + (p_2 - p_1) \frac{(1 + \xi)}{l_e} \text{ since } \frac{x}{l_e} = \frac{1 + \xi}{2} \\ &= p_1 \frac{1 - \xi}{2} + p_2 \frac{1 + \xi}{2} \end{aligned} \quad \dots(14.18)$$

$$\begin{aligned}
 \{F\}_e &= \int N^T p dx \\
 &= \int N^T p \frac{l_e}{2} d\xi = \int N^T p_1 \frac{1-\xi}{2} \frac{l_e}{2} d\xi + \int N^T p_2 \frac{1+\xi}{2} \frac{l_e}{2} d\xi \\
 &= \int_{-1}^1 \begin{Bmatrix} \frac{2-3\xi+\xi^3}{4} \\ \frac{1-\xi-\xi^2+\xi^3}{4} \frac{l_e}{2} \\ \frac{2+3\xi-\xi^2}{4} \\ \frac{-1-\xi+\xi^2+\xi^3}{4} \frac{l_e}{2} \end{Bmatrix} \frac{1-\xi}{2} p_1 \frac{l_e}{2} d\xi + \int_{-1}^1 \begin{Bmatrix} \frac{2-3\xi+\xi^3}{4} \\ \frac{1-\xi-\xi^2+\xi^3}{4} \frac{l_e}{2} \\ \frac{2+3\xi-\xi^2}{4} \\ \frac{-1-\xi+\xi^2+\xi^3}{4} \frac{l_e}{2} \end{Bmatrix} \frac{1+\xi}{2} p_2 \frac{l_e}{2} d\xi
 \end{aligned}$$

$$\text{Now} \quad \int_{-1}^1 \left( \frac{2-3\xi+\xi^3}{4} \right) (1-\xi) d\xi = \int_{-1}^1 (2-5\xi+3\xi^2+\xi^3+\xi^4) d\xi$$

$$\text{Noting that} \quad \int_{-1}^1 c d\xi = 2c, \quad \int_{-1}^1 \xi d\xi = \int_{-1}^1 \xi^3 d\xi = 0$$

$$\int_{-1}^1 \xi^2 d\xi = \left[ \frac{\xi^3}{3} \right]_{-1}^1 = \frac{2}{3}$$

$$\text{and} \quad \int_{-1}^1 \xi^4 d\xi = \left[ \frac{\xi^5}{5} \right]_{-1}^1 = \frac{2}{5}, \text{ we get}$$

$$\int_{-1}^1 \frac{2-3\xi+\xi^3}{4} (1-\xi) d\xi = 2 \times 2 - 0 + 3 \times \frac{2}{3} + 0 - \frac{2}{5} = \frac{28}{5}$$

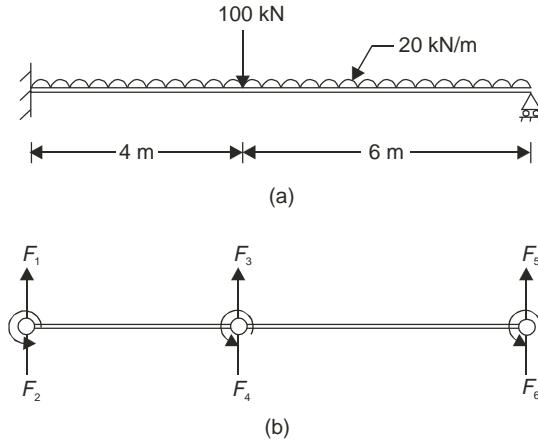
∴ The first term in  $p_1$  is,

$$\frac{p_1 l_e}{16} \times \frac{28}{5} = \frac{p_1 l_e}{60} \quad (21)$$

Similarly the other terms may be evaluated. Finally we get,

$$\{F\}_e = \frac{p_1 l}{60} \begin{Bmatrix} 21 \\ 3l \\ -9 \\ -2l \end{Bmatrix} + \frac{p_2 l}{60} \begin{Bmatrix} 9 \\ 2l \\ 21 \\ -3l \end{Bmatrix} \quad \dots(14.19)$$

**Example 14.4:** Determine the consistent nodal vector due to loads acting on the beam shown in Fig. 14.7.



**Fig. 14.7**

**Solution:** The beam is idealized with two elements as shown in Fig. 14.7 (b). Due to concentrated load, the nodal vector is directly obtained since there is a node directly under the load. It is given by

$$F = \begin{Bmatrix} 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Due to *udl*, it is given by the expression

$$\{F\}_e = \begin{Bmatrix} \frac{pl_e}{2} \\ \frac{pl_e^2}{12} \\ \frac{pl_e}{2} \\ -\frac{pl_e^2}{12} \end{Bmatrix}$$

$$\therefore \{F\}_1 = \begin{Bmatrix} -20 \times \frac{4}{2} \\ -20 \times \frac{16}{12} \\ -20 \times \frac{4}{2} \\ -(-20) \times \frac{16}{12} \end{Bmatrix} = \begin{Bmatrix} -40 \\ -26.667 \\ -40 \\ 26.667 \end{Bmatrix} \begin{matrix} \delta \text{ Global Numbers} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$\therefore \{F\}_e = \begin{Bmatrix} -20 \times \frac{6}{2} \\ -20 \times \frac{36}{12} \\ -20 \times \frac{6}{2} \\ -(-20) \times \frac{36}{12} \end{Bmatrix} = \begin{Bmatrix} -60 \\ -60 \\ -60 \\ 60 \end{Bmatrix} \begin{matrix} \delta \text{ Global Numbers} \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

$\therefore$  Due to *udl*,

$$\{F\} = \begin{Bmatrix} -40 \\ -26.667 \\ -(40 + 60) \\ 26.667 - 60 \\ -60 \\ 60 \end{Bmatrix} = \begin{Bmatrix} -40 \\ -26.667 \\ -(100) \\ -33.333 \\ -60 \\ 60 \end{Bmatrix}$$

Hence due to *udl* and the concentrated load

$$\{F\} = \begin{Bmatrix} -40 \\ -26.667 \\ -200 \\ -33.333 \\ -60 \\ 60 \end{Bmatrix} \text{ Answer}$$

### 14.3 ANALYSIS OF RIGID PLANE FRAMES USING 2 NODED BEAM ELEMENTS

The members of a rigid frame differs from the beam in the following two respect:

- (i) They carry axial loads also and hence their deformation in axial direction also is to be considered.
- (ii) They are oriented in any direction in the plane.

The typical frame element is shown in Fig. 14.8. The element selected is in  $x - y$  plane. The right hand thumb rule is used for the selection of Cartesian coordinates. Let  $x' - y'$  be the local coordinate system.

The nodal displacement vector in local coordinate system is

$$\{\delta'\}_e^T = [\delta'_1 \quad \delta'_2 \quad \delta'_3 \quad \delta'_4 \quad \delta'_5 \quad \delta'_6]_e \quad \dots(14.20)$$

and the nodal displacement vector in global system is,

$$\{\delta\}_e^T = [\delta_1 \quad \delta_2 \quad \delta_3 \quad \delta_4 \quad \delta_5 \quad \delta_6]_e \quad \dots(14.21)$$



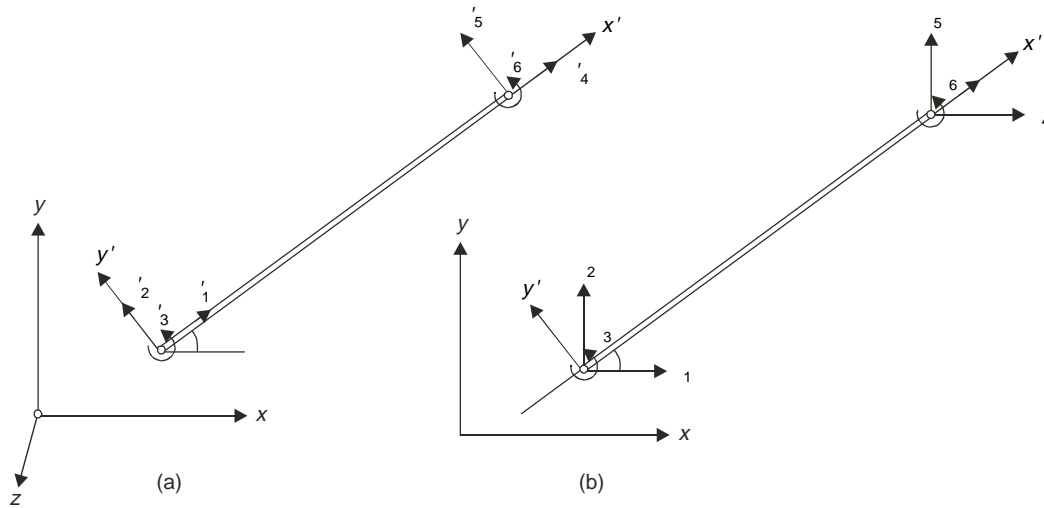


Fig. 14.8 A typical frame element

It may be observed that,

$$\delta'_{1e} = \delta_{1e} \cos \theta + \delta_{2e} \sin \theta = \delta_{1e} l + \delta_{2e} m$$

$$\delta'_{2e} = -\delta_{1e} \sin \theta + \delta_{2e} \cos \theta = -\delta_{1e} m + \delta_{2e} l$$

$$\delta'_{3e} = \delta_{3e}$$

Similarly

$$\delta'_{4e} = \delta_{4e} l + \delta_{5e} m$$

$$\delta'_{5e} = -\delta_{4e} m + \delta_{5e} l$$

$$\delta'_{6e} = \delta_{6e}$$

where  $l, m$  are direction cosines

$$\therefore \{\delta'\}_e = [L]\{\delta\}_e \quad \dots(14.22a)$$

where

$$[L] = \begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \dots(14.22b)$$

$\delta'_1$  and  $\delta'_4$  are like the degrees of freedom of bar element while  $\delta'_2, \delta'_3, \delta'_5$  and  $\delta'_6$  are like the degrees of freedom of beam element. Hence the stiffness matrix of frame element in the local coordinate system can be obtained by appropriately placing the two stiffnesses as

$$[k]_e = \begin{bmatrix} \frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\ 0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & -\frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & -\frac{6EI}{l_e^2} & \frac{2EI}{l_e} \\ -\frac{EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\ 0 & -\frac{12EI}{l_e^3} & -\frac{6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & -\frac{6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & -\frac{6EI}{l_e^2} & \frac{4EI}{l_e} \end{bmatrix} \quad \dots(14.23)$$

From equation 14.10, we know strain energy of the element is given by

$$U_e = \frac{1}{2} (\delta')_e^T [k']_e \{\delta'\}_e$$

But from equation 14.22  $\{\delta'\}_e = [L] \{\delta\}_e$

$$\begin{aligned} \therefore U_e &= ([L] \{\delta\}_e)^T [k']_e [L] \{\delta\}_e \\ &= \{\delta\}_e^T [L]^T [k']_e [L] \{\delta\}_e = \{\delta\}_e^T [k]_e \{\delta\}_e \end{aligned}$$

where

$$[k]_e = [L]^T [k']_e [L] \quad \dots(14.24)$$

$[k]_e$  is element stiffness matrix in global coordinate system.

Using stiffnesses matrices of all elements, system stiffness matrix  $[k]$  can be assembled.

### Load Vector

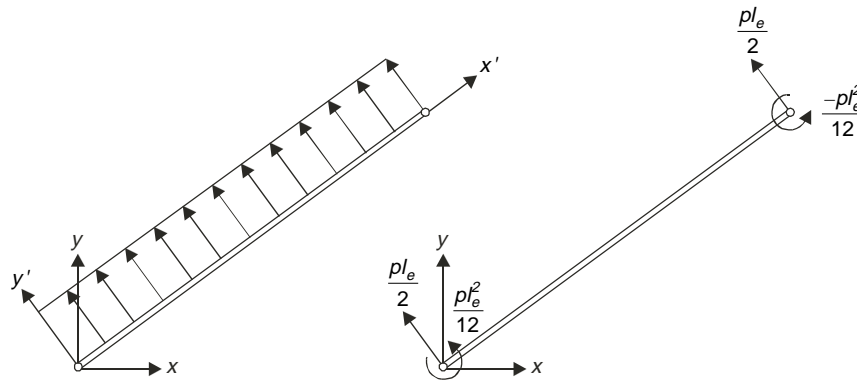


Fig. 14.9 Element load vector due to distributed load

Figure 14.9 shows the uniformly distributed load acting on the typical element. Noting that the load acting is in  $y'$ -direction, the nodal force system in local coordinate system is

$$\{F'\}_e = \begin{bmatrix} 0 & \frac{pl_e}{2} & \frac{pl_e^2}{12} & 0 & \frac{pl_e}{2} & -\frac{pl_e^2}{12} \end{bmatrix} \quad \dots(14.25)$$

By transforming it to global system, we get,  $\{F\}_e = [L]^T \{F'\}_e$ . Using such expressions for all elements the load vector of the system due to the distributed loads can be assembled. The load vector due to concentrated loads and external moments can be directly added to global load vector, since at all such loads nodes are selected.

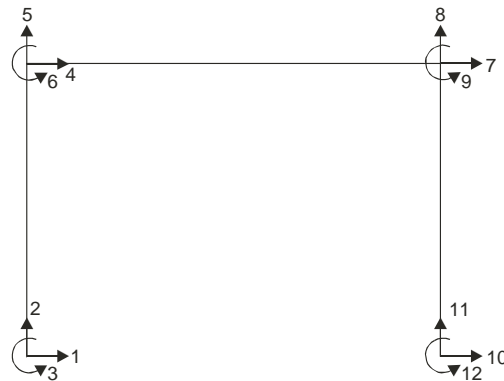
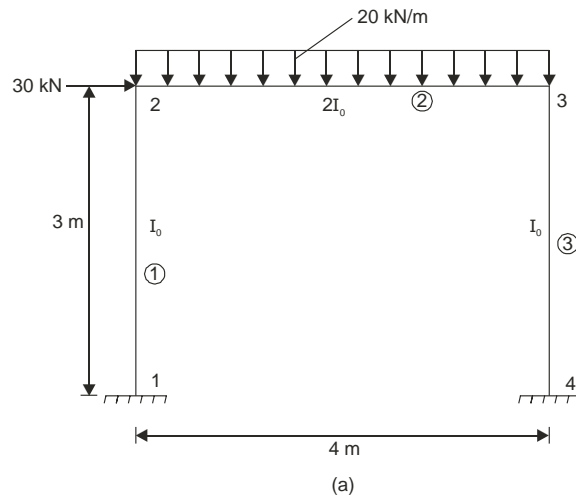
The final system equation is  $[k] \{\delta\} = \{F\}$  as usual. After introducing boundary conditional, the system equations are solved to get nodal displacement vector  $\{\delta\}$ . Then the required stress resultants can be assembled.

**Example 14.5:** Assemble element stiffness matrices for the rigid frame shown in the Fig. 14.10 (a). Explain how do you proceed further to solve the problem. Take,

$$E = 200 \text{ GN/m}^2$$

$$I_o = 40 \times 10^{-6} \text{ m}^4$$

$$A = 4 \times 10^{-3} \text{ m}^2$$



(b)  
**Fig. 14.10**

**Solution:** Figure 14.10 (b) shows the positive directions of the 12 nodal displacements. Thus in this problem

$$\{\delta\}^T = \{\delta_1 \delta_2 \dots \delta_{12}\}$$

For any element, stiffness matrix is given by

$$[k]_e = \begin{bmatrix} \frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\ 0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & -\frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & -\frac{6EI}{l_e^2} & \frac{2EI}{l_e} \\ -\frac{EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\ 0 & -\frac{12EI}{l_e^3} & -\frac{6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & -\frac{6EI}{l_e^2} \\ 0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & -\frac{6EI}{l_e^2} & \frac{4EI}{l_e} \end{bmatrix}$$

Let us take element and nodal connectivity as given below:

Element	Node 1	Node 2
1	1	2
2	2	3
3	4	3

For element No. 1

$$l_e = 3, EA = 200 \times 10^9 \times 4 \times 10^{-3} = 800 \times 10^6 \text{ N} \\ = 800 \times 10^3 \text{ kN}$$

$$EI = EI_0 = 200 \times 10^9 \times 40 \times 10^{-6} = 800 \times 10^3 \text{ N} - \text{m}^2 = 800 \text{ kN} - \text{m}^2$$

$$\therefore [k']_1 = 800 \begin{bmatrix} 333.333 & 0 & 0 & -333.333 & 0 & 0 \\ 0 & 0.444 & 0.667 & 0 & -0.444 & 0.667 \\ 0 & 0.667 & 1.333 & 0 & -0.667 & 0.667 \\ -333.333 & 0 & 0 & 333.333 & 0 & 0 \\ 0 & -0.444 & -0.667 & 0 & 0.444 & -0.667 \\ 0 & 0.667 & 0.667 & 0 & 0.667 & 1.333 \end{bmatrix}$$

For this element  $\theta = 90$ ,  $l = \cos\theta = 0$   $m = \sin\theta = 1$

$\therefore$  The transformation matrix is

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore [k'] [L] = 800 \begin{bmatrix} 0 & 333.333 & 0 & 0 & -333.333 & 0 \\ -0.444 & 0 & 0.667 & 0.444 & 0 & 0.667 \\ -0.667 & 0 & 1.333 & 0.667 & 0 & 0.667 \\ 0 & -333.333 & 0 & 0 & 333.333 & 0 \\ 0.444 & 0 & -0.667 & 0.444 & 0 & -0.667 \\ -0.667 & 0 & 0.667 & 0.667 & 0 & 1.333 \end{bmatrix}$$

$$[L]^T = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[L]^T [k'] [L] = [L]^T ([k'] [L])$$

$$= 800 \begin{array}{cccccc} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} & \begin{matrix} \left[ \begin{array}{cccccc} 0.444 & 0 & -0.667 & -0.444 & 0 & -0.667 \\ 0 & 333.333 & 0 & 0 & -333.333 & 0 \\ -0.667 & 0 & 1.333 & 0.667 & 0 & 0.667 \\ -0.444 & 0 & 0.667 & 0.444 & 0 & 0.667 \\ 0 & -333.333 & 0 & 0 & 333.333 & 0 \\ -0.667 & 0 & 0.667 & 0.667 & 0 & 1.333 \end{array} \right] \end{matrix} & \begin{matrix} \leftarrow \delta_i \\ \downarrow \\ \text{Global Numbers} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

For element No. 2, local and global coordinates are the same, and  $l_e = 4\text{m}$ ,  $I_e = 2I_o$

$$[k]_2 = [k']_2 = 800 \begin{bmatrix} 4 & 5 & 6 & 7 & 8 & 9 & \leftarrow \delta, \text{ Global Numbers} \\ 250 & 0 & 0 & -250 & 0 & 0 & 4 \\ 0 & 0.375 & 0.750 & 0 & -0.375 & 0.750 & 5 \\ 0 & 0.750 & 2.0 & 0 & -0.75 & 1.0 & 6 \\ -250 & 0 & 0 & 250 & 0 & 0 & 7 \\ 0 & -0.375 & -0.75 & 0 & 0.375 & -0.75 & 8 \\ 0 & 0.75 & 1.0 & 0 & -0.75 & 2.0 & 9 \end{bmatrix}$$

For element No. 3, if we take it as member 4–3, we get the stiffness matrix identical to element 1. However nodal displacement vector for it is  $[\delta_{10} \ \delta_{11} \ \delta_{12} \ \delta_7 \ \delta_8 \ \delta_9]$ . Hence

$$[k]_3 = 800 \begin{bmatrix} 10 & 11 & 12 & 7 & 8 & 9 \leftarrow \delta, \text{ Global Numbers} \\ 0.444 & 0 & -0.667 & -0.444 & 0 & -0.667 & 10 \\ 0 & 333.333 & 0 & 0 & -333.333 & 0 & 11 \\ -0.667 & 0 & 1.333 & 0.667 & 0 & 0.667 & 12 \\ -0.444 & 0 & 0.667 & 0.444 & 0 & 0.667 & 7 \\ 0 & -333.333 & 0 & 0 & 333.333 & 0 & 8 \\ -0.667 & 0 & 0.667 & 0.667 & 0 & 1.333 & 9 \end{bmatrix}$$

### Brief Procedure for Further Analysis

- (i) Stiffness matrix of the system  $[k]$  is of size  $12 \times 12$ . It can be assembled by placing the elements of the element stiffness matrices  $[k]_1$ ,  $[k]_2$ , and  $[k]_3$  in the appropriate positions of  $12 \times 12$  matrix.
- (ii) There are six boundary conditions, namely  $\delta_1 = \delta_2 = \delta_3 = \delta_{10} = \delta_{11} = \delta_{12} = 0$ .

They are imposed either by elimination method or by penalty method. When hand calculations are made the elimination method is ideal. In this method, the elements corresponding to rows and columns corresponding to 1, 2, 3, 10, 11 and 12 get eliminated and the stiffness matrix reduces to  $6 \times 6$  size corresponding to displacement vector  $\{\delta\}^T = [\delta_4 \ \delta_5 \ \delta_6 \ \delta_7 \ \delta_8 \ \delta_9]$ . If penalty method is used, which is suitable for computer applications,  $12 \times 12$  size of stiffness matrix is maintained but diagonal elements corresponding to the rows 1, 2, 3, 10, 11 and 12 are increased by very large numbers.

- (iii) The equivalent nodal forces due to the applied loads are as shown in Fig. 14.11.

Apart from these loads, there is a 30 kN load in the direction 4. Hence the load vector is

$\{F\}^T = [0 \ 0 \ 0 \ 30.0 \ -40 \ -26.667 \ 0 \ -40 \ 26.667 \ 0 \ 0 \ 0]$ . In case of elimination method of imposing boundary conditions, we get reduced load vector corresponding to the directions 4, 5, 6, 7, 8 and 9 as

$$\{F\}^T = [30.0 \ -40 \ -26.667 \ 0 \ -40 \ 26.667]$$



Fig. 14.11

- (iv) The system equations may be solved to get the nodal displacement vector.
- (v) The member forces may be calculated as usual. The tensile force as positive is given by,

$$P = \frac{EA}{l_e} (\delta_4 - \delta_1) \quad \dots(14.26)$$

The bending moment and shear forces are obtained by:

$$M = \frac{EI}{l_e^2} \begin{bmatrix} 6\xi & -(1 - 3\xi)l_e & -6\xi & (1 + 3\xi)l_e \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \quad \dots(14.27)$$

$$V = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \end{bmatrix} \begin{Bmatrix} \delta_2 \\ \delta_3 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \quad \dots(14.28)$$

- (vi) The end reaction vector, +ve in the +ve direction of degrees of freedom is obtained by the expression

$$\{R\} = [k]\{\delta\} - \{F\} \quad \dots(14.29)$$

### 14.4 A THREE DIMENSIONAL RIGID FRAME ELEMENT

The typical element is shown in Fig. 14.12. In this, there are six degrees of freedom at each node. Hence total degrees of freedom are 12. It may be noted that the rotations are about the axes but not in the directions of axes. Thus,

$$\begin{aligned} \{\delta\}^T &= [\delta_1 \quad \delta_2 \quad \delta_3 \quad \dots \quad \delta_{12}] \\ &= [u_1 \quad v_1 \quad w_1 \quad \theta_{x1} \quad \theta_{y1} \quad \theta_{z1} \quad u_2 \quad v_2 \quad w_2 \quad \theta_{x2} \quad \theta_{y2} \quad \theta_{z2}] \quad \dots(14.30) \end{aligned}$$

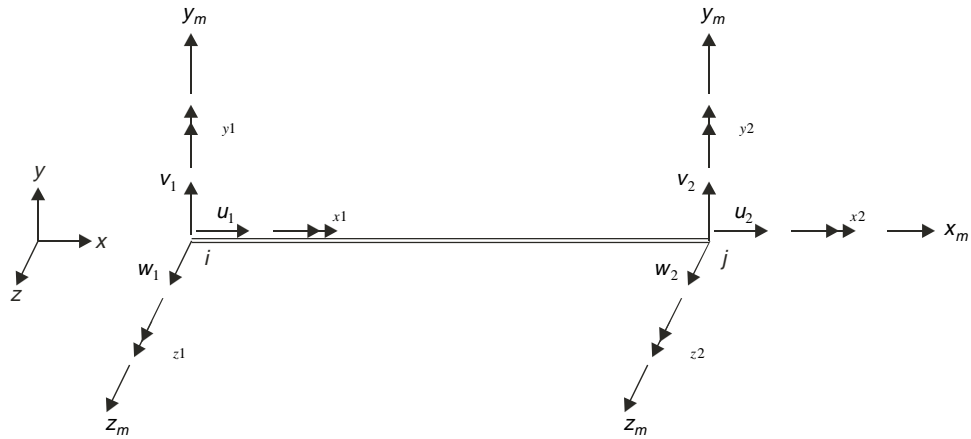


Fig. 14.12

The  $12 \times 12$  stiffness matrix in local coordinate system, may be written down as shown in equation 14.31 by carefully noting the contribution of each nodal displacement.

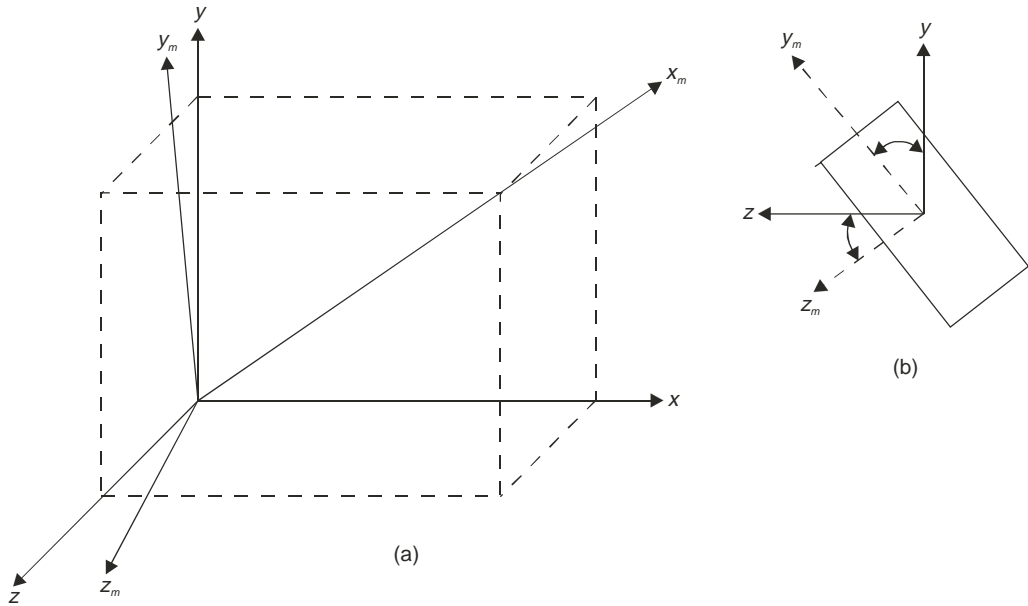


Fig. 14.13 (a) Orientation of beam element (b) Orientation of principle axes about  $x_m$ -axis



$$[k]_e = \begin{bmatrix} \frac{EA}{l_e} & 0 & 0 & 0 & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{l_e^3} & 0 & 0 & 0 & \frac{6EI_z}{l_e^2} & 0 & \frac{12EI_z}{l_e^3} & 0 & 0 & 0 & \frac{6EI_z}{l_e^2} \\ 0 & 0 & \frac{12EI_y}{l_e^3} & 0 & -\frac{6EI_y}{l_e^2} & 0 & 0 & 0 & -\frac{12EI_y}{l_e^3} & 0 & -\frac{6EI_y}{l_e^2} & 0 \\ 0 & 0 & 0 & \frac{GI_x}{l} & 0 & 0 & 0 & 0 & 0 & -\frac{GI_x}{l_e} & 0 & 0 \\ 0 & 0 & -\frac{6EI_y}{l_e^2} & 0 & \frac{4EI_y}{l_e} & 0 & 0 & 0 & \frac{6EI_y}{l_e^2} & 0 & \frac{2EI_y}{l_e} & 0 \\ 0 & \frac{6EI_z}{l_e^2} & 0 & 0 & 0 & \frac{4EI_z}{l_e} & 0 & -\frac{6EI_z}{l_e^2} & 0 & 0 & 0 & \frac{2EI_z}{l_e} \\ -\frac{EA}{l_e} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l_e} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{l_e^3} & 0 & 0 & 0 & -\frac{6EI_z}{l_e^2} & 0 & \frac{12EI_y}{l_e^3} & 0 & 0 & 0 & -\frac{6EI_z}{l_e^2} \\ 0 & 0 & -\frac{12EI_y}{l_e^3} & 0 & \frac{6EI_y}{l_e^2} & 0 & 0 & 0 & \frac{12EI_y}{l_e^3} & 0 & \frac{6EI_y}{l_e^2} & 0 \\ 0 & 0 & 0 & -\frac{GI_x}{l_e} & 0 & 0 & 0 & 0 & 0 & \frac{GI_x}{l_e} & 0 & 0 \\ 0 & 0 & -\frac{6EI_y}{l_e^2} & 0 & \frac{2EI_y}{l_e} & 0 & 0 & 0 & \frac{6EI_y}{l_e^2} & 0 & \frac{4EI_y}{l_e} & 0 \\ 0 & \frac{6EI_z}{l_e^2} & 0 & 0 & 0 & \frac{2EI_z}{l_e} & 0 & -\frac{6EI_z}{l_e^2} & 0 & 0 & 0 & \frac{4EI_z}{l_e} \end{bmatrix} \dots(14.31)$$

If the member axes do not coincide with global axes, we need transformation matrix. Figure 14.13 (a) shows arbitrary orientation of the member axis in global system and Fig. 14.13 (b) shows the orientation of principal axes of the member. Then the transformation matrix  $T$  works out to be

$$[L] = \begin{bmatrix} [L'] & [0] & [0] & [0] \\ [0] & [L'] & [0] & [0] \\ [0] & [0] & [L'] & [0] \\ 0 & 0 & 0 & [L'] \end{bmatrix} \dots(14.32)$$

where  $L$  is a  $12 \times 12$  matrix and each sub matrix on right hand side is a  $3 \times 3$  matrix.  $[L']$  matrix is given by

$$[L'] = \begin{bmatrix} C_x & C_y & C_z \\ \frac{-C_x C_y \cos \alpha - C_z \cos \alpha}{\sqrt{C_x^2 + C_y^2}} & \sqrt{C_x^2 + C_z^2} \cos \alpha & \frac{-C_y C_z \cos \alpha + C_x \sin \alpha}{\sqrt{C_x^2 + C_z^2}} \\ \frac{C_x C_y \sin \alpha - C_z \cos \alpha}{\sqrt{C_x^2 + C_z^2}} & -\sqrt{C_x^2 + C_z^2} \sin \alpha & \frac{C_y C_z \sin \alpha - C_x \cos \alpha}{\sqrt{C_x^2 + C_z^2}} \end{bmatrix} \dots(14.33)$$

In the above expression,

$$C_x = \frac{x_j - x_i}{l_e}, \quad C_y = \frac{y_j - y_i}{l_e} \quad \text{and} \quad C_z = \frac{z_j - z_i}{l_e}$$

$$l_e = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

For the derivation of above transformation matrix reorders may refer to Krishnamurthy[1]

## 14.5 TIMOSHENKO BEAM ELEMENT

In the theory used so far, the assumption was made that the plane section before bending remains plane even after bending. The plane section remain plane even after bending, means shear deformations are neglected. In most of the beams bending is associated with the shear. Elementary bending theory shows that shear stress is zero at extreme fibers and is maximum at the centroid of the cross section. These longitudinal stresses cause varying strain in the plane section. As a result of these shear stresses, plane section will not remain plane after bending. If the beam thickness is small, these stresses are small and hence the assumption that plane section is plane even after bending, gives good results; but in thick beams this assumption will not give good results.

Timoshenko beam theory recognises that the action of the shear force causes a shear strain. This causes warping of the beam element as shown in Fig. 14.14. The shear stress in general can be express as

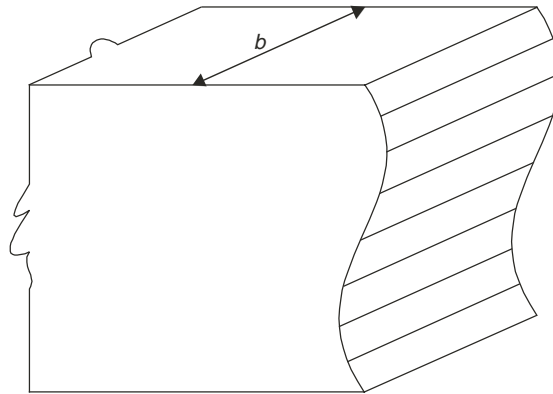


Fig. 14.14 Warped beam element

$$\tau_{xz} = G \phi_{xz} \quad \dots(14.34)$$

and the shear force as

$$Q = \oint \tau_{xz} dz b \quad \dots(14.35)$$

where  $\phi_{xz}$  is the shear strain at distance  $z$  from neutral axis at section  $x$ .

However the use of general equations 14.34 and 14.35 complicates the problem and a simplified approach is possible. In order to account for non-uniform stress distribution at a cross section while still retaining one dimensional approach, the equations 14.34 and 14.35 are modified using a shear correction factor as follows:

$$\tau_{xz} = \alpha G \phi_x \quad \dots(14.36)$$

and 
$$Q = \tau_{xz} A = \alpha AG \phi_x \quad \dots(14.37)$$

The shear correction factor ' $\alpha$ ' is a function of the cross sectional shape and Poisson ratio  $\mu$ . The term ' $\alpha A$ ' is the 'shear area' of the section associated with shear and may be denoted as  $A_s$ . Thus

$$A_s = \alpha A \text{ where } \alpha < 1 \quad \dots(14.38)$$

Values of  $\alpha$  for various cross sectional shapes are given in the solid mechanics books by DYM C.L. & Shames I.H. [2] and Ugural A.C. and Fenster S.K. [3].

The value of  $\alpha$  for a rectangular section is 5/6. Hence,

$$\phi = \frac{Q}{G A_s} \quad \dots(14.39)$$

Timoshenko beam theory averages the effect of shear strain over the cross section i.e. it takes,

$$\phi = \theta - \frac{\partial w}{\partial x} \quad \dots(14.40)$$

where  $\theta$  is the angle through which the face of the cross section rotates after deformation as shown in Fig. 14.15.

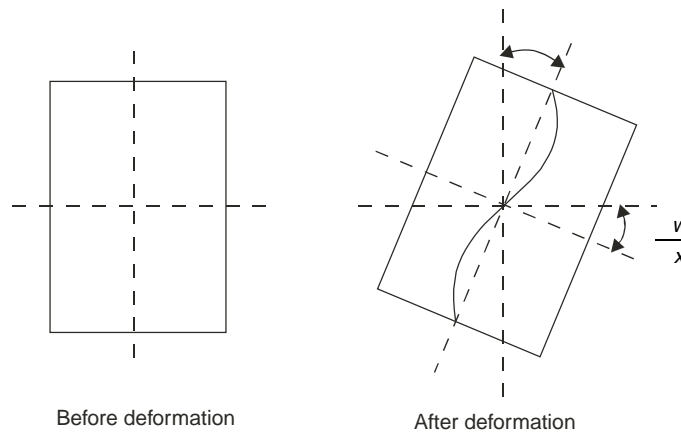


Fig. 14.15

Now, in a beam element

$$M = EI \frac{d\theta}{dx}$$

$$\therefore \text{Strain energy due to flexure} = \int \frac{M^2}{2EI} dx = \frac{EI}{2} \int_0^{l_e} \left( \frac{d\theta}{dx} \right)^2 dx \text{ and strain energy due to shear}$$

$$= \oint \frac{1}{2} \times \text{shear stress} \times \text{shear strain} \times dV$$

$$\begin{aligned}
 &= \oint \frac{1}{2} \tau_{xy} \phi \, dV \\
 &= \oint \frac{1}{2} G \phi^2 \, dV = \frac{1}{2} G A_s \int_0^{l_e} \left( \theta - \frac{\partial w}{\partial x} \right)^2 dx
 \end{aligned}$$

∴ Total strain energy of the element is

$$U = \frac{1}{2} EI \int_0^{l_e} \left( \frac{d\theta}{dx} \right)^2 dx + \frac{1}{2} GA_s \int_0^{l_e} \left( \theta - \frac{\partial w}{\partial x} \right)^2 dx \quad \dots(14.41)$$

### Finite Element Formation

Since according to Timoshenko theory  $\theta = \frac{dw}{dx}$ ,  $\theta$  and  $w$  are decoupled i.e. they are independent of each other, at every node there are two independent displacement components  $\theta$  and  $w$ . In a two noded beam element they vary linearly. Thus,

$$w = N_1 w_1 + N_2 w_2$$

and

$$\theta = N_1 \theta_1 + N_2 \theta_2$$

In matrix form,

$$\begin{Bmatrix} w \\ \theta \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \quad \dots(14.42)$$

where  $N_1$  and  $N_2$  are interpolation functions i.e.  $N_1 = 1 - \frac{x}{l_e}$  and  $N_2 = \frac{x}{l_e}$

Using isoparametric concept, we have

$$x = \sum N_i x_i \quad \dots(14.43)$$

Now,

$$\frac{d\theta}{dx} = \sum_{i=1}^2 \frac{\partial N_i}{\partial x} \theta_i \quad \dots(14.44)$$

$$\therefore \phi = \theta - \frac{\partial w}{\partial x} = \sum_{i=1}^2 N_i \theta_i - \sum_{i=1}^2 \frac{\partial N_i}{\partial x} w_i \quad \dots(14.45)$$

Strain vector

$$\{\varepsilon\} = \begin{Bmatrix} \kappa \\ \phi \end{Bmatrix}$$

where  $\kappa$  is curvature =  $\frac{d\theta}{dx}$

$$\therefore \{\varepsilon\} = \begin{Bmatrix} \kappa \\ \phi \end{Bmatrix} = \begin{bmatrix} 0 & \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} \\ -\frac{dN_1}{dx} & N_1 & -\frac{dN_2}{dx} & x \end{bmatrix} = \frac{1}{l_e} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & l_e - x & -1 & x \end{bmatrix}$$

The stress resultants  $M$  and  $Q$  are related to strain as

$$[\sigma] = \begin{Bmatrix} M \\ Q \end{Bmatrix} = [D]\{\varepsilon\}$$

Now  $M = EI \kappa$

For rectangular section  $I = \frac{1}{12}bh^3$

$$\therefore D_{11} = \frac{E}{12}bh^3$$

and

$$Q = A_s \tau_{xy} = A_s h\phi = \alpha A \frac{E}{2(1+\mu)} \phi = \frac{\alpha bh}{2} E\phi$$

$$\therefore \{\sigma\} = \begin{Bmatrix} M \\ Q \end{Bmatrix} = \begin{bmatrix} \frac{E}{12}bh^3 & 0 \\ 0 & \frac{\alpha bh}{2}E \end{bmatrix} \begin{Bmatrix} \kappa \\ \phi \end{Bmatrix} = \frac{Ebh}{12} \begin{bmatrix} h^2 & 0 \\ 0 & 6\alpha \end{bmatrix} \begin{Bmatrix} \kappa \\ \phi \end{Bmatrix}$$

$\therefore$  Assuming  $\mu = 0$ ,  $D$  matrix for rectangular section is

$$[D] = \frac{Ebh}{12} \begin{bmatrix} h^2 & 0 \\ 0 & 6\alpha \end{bmatrix} \quad \dots(14.48)$$

$$\therefore [k]_e = \int_0^{l_e} [B]^T [D] [B] dx$$

$$= \frac{1}{l_e^2} \int_0^{l_e} \begin{bmatrix} 0 & 1 \\ -1 & l_e - x \\ 0 & -1 \\ 1 & x \end{bmatrix} \frac{Ebh}{12} \begin{bmatrix} h^2 & 0 \\ 0 & 6\alpha \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & l_e - x & -1 & x \end{bmatrix} dx$$

$$= \frac{Ebh}{12l_e^2} \int_0^{l_e} \begin{bmatrix} 0 & 1 \\ -1 & l_e - x \\ 0 & -1 \\ 1 & x \end{bmatrix} \begin{bmatrix} 0 & -h^2 & 0 & h^2 \\ 6\alpha & 6\alpha(l_e - x) & -6\alpha & 6\alpha x \end{bmatrix} dx$$

$$= \frac{Ebh}{12l_e^2} \int_0^{l_e} \begin{bmatrix} 6\alpha & 6\alpha(l_e - x) & -6\alpha & 6\alpha x \\ 6\alpha(l_e - x) & h^2 + 6\alpha(l_e - x)^2 & -6\alpha(l_e - x) & -h^2 + 6\alpha x(l_e - x) \\ -6\alpha & -6\alpha(l_e - x) & 6\alpha & -6\alpha x \\ 6\alpha x & -h^2 + 6\alpha x(l_e - x) & -6\alpha x & h^2 + 6\alpha x^2 \end{bmatrix} dx$$

Separating bending and shear terms, we can write

$$[\kappa]_e = \frac{Ebh}{12l_e^2} \int_0^{l_e} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h^2 & 0 & -h^2 \\ 0 & 0 & 0 & 0 \\ 0 & -h^2 & 0 & h^2 \end{bmatrix} + 6\alpha \int_0^{l_e} \begin{bmatrix} 1 & l_e - x & -1 & x \\ l_e - x & (l_e - x)^2 & -(l_e - x) & x(l_e - x) \\ -1 & -(l_e - x) & 1 & -x \\ x & x(l_e - x) & -x & x^2 \end{bmatrix} dx$$

$$\text{i.e., } [\kappa]_e = [\kappa]_{be} + [\kappa]_{se} \quad \dots(14.49)$$

where  $[\kappa]_{be}$  and  $[\kappa]_{se}$  are the contributions of bending and shear to the total stiffness. The integrations can be performed to get  $[\kappa]_{be}$  and  $[\kappa]_{se}$  as shown below:

$$[\kappa]_{se} = \frac{Ebh^3}{12l_e} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \dots(14.50)$$

$$[\kappa]_{be} = \frac{Gh\alpha}{l_e} \begin{bmatrix} 1 & \frac{l_e}{2} & -1 & \frac{l_e}{2} \\ \frac{l_e}{2} & \frac{l_e^2}{3} & -\frac{l_e}{2} & \frac{l_e^2}{6} \\ -1 & -\frac{l_e}{2} & 1 & -\frac{l_e}{2} \\ \frac{l_e}{2} & \frac{l_e^2}{6} & -\frac{l_e}{2} & \frac{l_e^2}{3} \end{bmatrix} \quad \dots(14.51)$$

The above formulation gives good results for moderately thick beams. For thin beams ( $\frac{l_e}{h}$  very large), the results obtained by this formulation are not correct. The shear term, which should tend to zero in such cases, do not tend to zero. This is called shear locking. The shear stiffness is increasingly constrained. This is called spurious constraint. There are two popular remedies for the elimination of this type of errors:

- (i) Reduced Integration Technique
- (ii) Using field consistency element

(i) *Reduced Integration Technique*: If shear stiffness in equation 14.49 is integrated with one point Gaussian technique, we get

$$[k]_{se} = \frac{\alpha Gh}{l_e} \begin{bmatrix} 1 & \frac{l_e}{2} & -1 & \frac{l_e}{2} \\ \frac{l_e}{2} & \frac{l_e^2}{4} & -\frac{l_e}{2} & \frac{l_e^2}{4} \\ -1 & -\frac{l_e}{2} & 1 & -\frac{l_e}{2} \\ \frac{l_e}{2} & \frac{l_e^2}{4} & -\frac{l_e}{2} & \frac{l_e^2}{4} \end{bmatrix} \quad \dots(14.52)$$

since  $x = \frac{1 + \xi}{2}, dx = \frac{1}{2} d\xi$

and  $f(\xi)d\xi = 2f(\xi = 0)$

In case of two point integration sampling points are at  $\pm \frac{1}{\sqrt{3}}$  and weight function  $W_i = 1$ .

Hence

$$[k]_{se}^2 = \frac{\alpha Gh}{l_e} \begin{bmatrix} 1 & \frac{l_e}{2} & -1 & \frac{l_e}{2} \\ \frac{l_e}{2} & \frac{l_e^2}{3} & -\frac{l_e}{2} & \frac{l_e^2}{6} \\ -1 & -\frac{l_e}{2} & 1 & -\frac{l_e}{2} \\ \frac{l_e}{2} & \frac{l_e^2}{6} & -\frac{l_e}{2} & \frac{l_e^2}{3} \end{bmatrix} \quad \dots(14.53)$$

Now consider the analysis of a cantilever beam with single element (refer Fig. 14.16). The stiffness matrix of the beam is

$$[k] = [k]_e = [k]_{be} + [k]_{se}$$

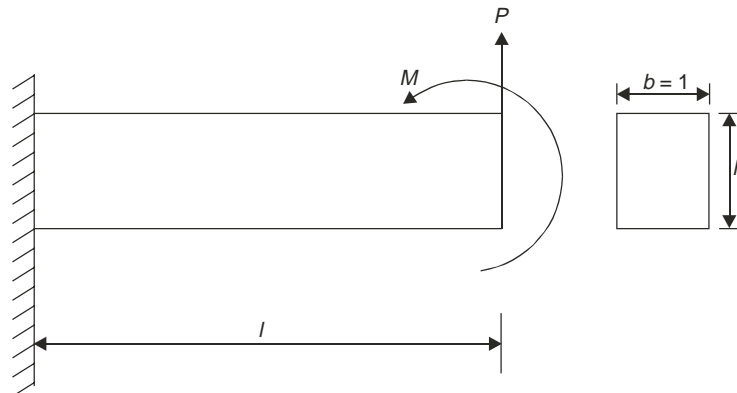


Fig. 14.16

Applying the boundary conditions that  $w_1 = 0$ ,  $\theta_1 = 0$  and defining  $\psi = \frac{Eh^3}{12l_e} = \frac{Eh^3}{12l}$  and

$\beta = \frac{\alpha Gh}{l_e} = \frac{\alpha Gh}{l}$ , we get the equilibrium equation from one point quadrature as,

$$\begin{bmatrix} \beta & \frac{l}{2}\beta & -\beta & \frac{l}{2}\beta \\ \frac{l}{2}\beta & \psi + \frac{l^2}{4}\beta & -\frac{l}{2}\beta & -\psi + \frac{\beta l^2}{4} \\ -\beta & -\frac{l}{2}\beta & \beta & -\frac{l}{2}\beta \\ \frac{l}{2}\beta & -\psi + \frac{l^2}{4}\beta & -\frac{l}{2}\beta & \psi + \frac{l^2}{4}\beta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ P \\ M \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} \beta & -\frac{l}{2}\beta \\ -\frac{l}{2}\beta & \psi + \frac{l^2}{4}\beta \end{bmatrix} \begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ M \end{Bmatrix}$$

Solving the above equation, we get

$$w_2 = \left( \frac{l^2}{4\psi} + \frac{1}{\beta} \right) P + \frac{1}{2\psi} M \quad \dots(14.54)$$

and

$$\theta_2 = \frac{\frac{l}{2}P + M}{\psi} \quad \dots(14.55)$$

In case of thin beam,  $\beta \geq \psi$ . Hence equation 14.54 reduces to

$$w_2 = \frac{1}{2\psi} \left[ \frac{lP}{2} + M \right] \quad \dots(14.56)$$

and  $\theta_2$  remains same as equation 14.55. Thus the beam deformation is solely due to bending as shown by equation 14.55 and 14.56. Hence for thin beams one point integration gives correct results.

If two point quadrature is used for integration, after applying the boundary conditions we get the equations as

$$\begin{bmatrix} \beta & -\frac{l}{2}\beta \\ -\frac{l}{2}\beta & \psi + \frac{l^3}{3}\beta \end{bmatrix} \begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} P \\ M \end{Bmatrix}$$



Solving for  $w_2$  and  $\theta_2$ , we get

$$w_2 = \frac{\psi + \frac{\beta l^2}{3}}{\beta \left( \psi + \frac{\beta l^2}{12} \right)} P + \frac{lM}{2 \left( \psi + \frac{\beta l^2}{12} \right)}$$

and

$$\theta_2 = \frac{M + \frac{l}{2} P}{\psi + \frac{\beta l^2}{12}}$$

In case of thin beams  $\beta \geq \psi$ . Hence we get

$$w_2 = \frac{4P + 6\frac{M}{l}}{\beta} \quad \dots(14.57)$$

and

$$\theta_2 = \frac{6(lP + 2M)}{l^2 \beta} \quad \dots(14.58)$$

The above two equations show that the free end deformation depends on the coefficient  $\beta$  corresponding to shear deformation, which is not true in case of thin beams. Hence two point integration (equations 14.57 and 14.58) lead to erroneous results.

Thus the reduced integration (1 point Gaussian integration instead of 2 point Gaussian integration) gives better results than the exact integration. The reduced integration technique is used to get good results for thin beams.

(ii) *Field Consistent Element Formulation:* Let us first see how the element formulation is field inconsistent in the case of very thin beams. In the formulation we have taken,

$$w = [N_1 \quad N_2] \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = N_1 w_1 + N_2 w_2$$

and

$$\theta = [N_1 \quad N_2] \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = N_1 \theta_1 + N_2 \theta_2$$

where

$$N_1 = \frac{1 - \xi}{2} \quad \text{and} \quad N_2 = \frac{1 + \xi}{2}$$

and

$$\xi = \frac{2x}{l_e} - 1$$

rearranging the terms, we get

$$w = \frac{1 - \xi}{2} w_1 + \frac{1 + \xi}{2} w_2 = \frac{w_1 + w_2}{2} + \frac{w_2 - w_1}{2} \xi = a_1 + a_2 \xi$$

where

$$a_1 = \frac{w_1 + w_2}{2} \quad \text{and} \quad a_2 = \frac{w_2 - w_1}{2}$$

similarly

$$\theta = b_1 + b_2 \xi$$

where

$$b_1 = \frac{\theta_1 + \theta_2}{2} \quad \text{and} \quad b_2 = \frac{\theta_2 - \theta_1}{2}$$

Now bending strain

$$\kappa = \frac{d\theta}{dx} = \frac{d\theta}{d\xi} \frac{d\xi}{dx} = \frac{2}{l} b_2$$

and shear strain

$$\phi = \theta - \frac{dw}{dx} = b_1 + b_2 \xi - \frac{d\xi}{dx} \frac{dw}{d\xi} = b_1 + b_2 \xi - \frac{2}{l} a_2$$

Strain energy due to bending is given by,

$$U_b = \int \frac{EI}{2} \left( \frac{d\theta}{dx} \right)^2 dx = \int \frac{EI}{2} \left( \frac{2}{l} b_2 \right)^2 dx \quad \dots(14.59)$$

and strain energy due to shear is

$$U_s = \int \frac{GA_s}{2} (\phi)^2 dx = \int \frac{GA_s}{2} \left( b_1 - \frac{2}{l} a_2 + b_2 \xi \right)^2 dx \quad \dots(14.60)$$

As the thickness approaches zero, the shear strain energy should vanish and bending strain energy should exist.

$$\text{i.e.,} \quad b_1 - \frac{2}{l} a_2 + b_2 \xi \rightarrow 0$$

$$\text{i.e.,} \quad b_1 - \frac{2}{l} a_2 \rightarrow 0 \quad \dots(14.61)$$

$$\text{and} \quad b_2 \rightarrow 0 \quad \dots(14.62)$$

The terms corresponding to condition 14.61 correspond to both the strain fields flexure and shear. Hence it is called field constant term. The constraint corresponding to equation 14.62 contains the term corresponding only to flexure field. If  $b_2 \rightarrow 0$ , the strain energy due to bending tends to zero, which should not happen. Hence this constraint requirement in the limiting case is spurious and it is this requirement which causes shear locking.

To get rid of this situation in the limiting case, the function smoothening is required i.e. alter the terms to overcome this situation. This is achieved by making  $b_2 = 0$  in the shear strain field i.e. by taking

$$\bar{\theta} = b_1 = \frac{\theta_1 + \theta_2}{1} \quad \text{in the shear field}$$

Thus,

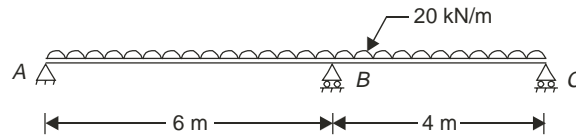
$$\bar{\theta} = b_1 = \frac{\theta_1 + \theta_2}{1} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} \bar{N}_1 & \bar{N}_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

i.e. in shear field interpolation function for  $\theta$  will be taken as  $[\bar{N}_1 \ \bar{N}_2] = \left[ \frac{1}{2} \ \frac{1}{2} \right]$ . This smoothing is required not only in the shear strain field but even in the shear stress field also.

The greatest advantage of Timoshenko beam element is  $C^1$  continuity problem is reduced to  $C^0$  continuity problem. The extension of this concept by Mindlin to plate bending problems, (to be discussed in next chapter) is a great achievement in the plate analysis.

### QUESTIONS

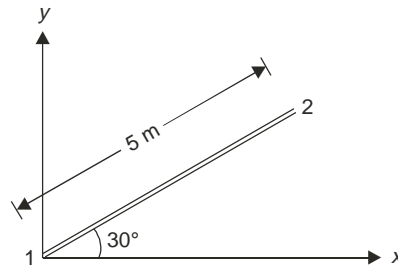
1. Derive the stiffness matrix for a beam element.
2. Analyse the beam shown in Fig. 14.17 using FEM technique. Determine the rotations at the supports. Given  $E = 200\text{GPa}$  and  $I = 4 \times 10^6 \text{ mm}^4$ .



**Fig. 14.17**

(Ans.  $\theta_A = 0.3, \theta_B = -0.375; \theta_C = 0.1625$ )

3. Assemble the stiffness matrix for a plane beam element oriented at angle  $\theta$  to the x-axis. Explain its use in FEA.
4. Assemble element stiffness matrix for the member of plane frame shown in Fig. 14.18, if it is oriented at angle  $30^\circ$  to the x-axis. Take  $E = 200\text{GPa}$ ,  $I = 4 \times 10^6 \text{ m}^4$  and  $A = 4 \times 10^{-3} \text{ m}^2$ .



**Fig. 14.18**

(Ans.)

$$\begin{bmatrix} 150.025 & 86.558 & -0.12 & -150.015 & -086.558 & -0.12 \\ & 50.070 & 0.208 & -86.558 & -50.070 & 0.208 \\ & & 0.8 & 0.12 & -0.208 & 0.4 \\ & SYM & & 150.015 & 86.558 & 0.12 \\ & & & & 50.070 & -0.208 \\ & & & & & 0.8 \end{bmatrix}$$

5. Explain Timoshenko Beam Theory. Discuss its advantages and disadvantages.
6. Explain the term Timoshenko Beam Theory and briefly explain the stiffness formulation for such element.
7. Explain the term 'shear locking' as used in Timoshenko Beam Theory. How this problem is overcome?
8. Taking the example of a single element cantilever beam, show that single point Gaussian integration overcomes the problem of shear locking.
9. Explain what is meant by field consistent formulation to overcome shear locking in Timoshenko Beam Theory. Use the example of a single element cantilever beam.

### **References**

1. Krishnamoorthy C.S., *Finite Element Analysis—Theory and Programming*, Tata McGraw-Hill Publishing Co. Ltd., New Delhi.
2. DYM C.L. and Shames I.H., *Solid Mechanics*, International Students Edition, McGraw-Hill Kogakusha Ltd., 1973.
3. Ugural A.C. and Fenster S.K., *Advanced Strength and Applied Elasticity*, American Elsevier Publishing Co., New York, 1975.

# 15

## Bending of Thin Plates

### 15.1 INTRODUCTION

Plate is a flat surface having considerably large dimensions as compared to its thickness. Slabs in civil engineering structures, bearing plates under columns, many parts of mechanical components are the common examples of plates. In this chapter, we are considering bending of such plates under lateral loads. The bending properties of a plate depend greatly on its thickness. Hence in classical theory we have the following groups

- (i) thin plates with small deflections
- (ii) thin plates with large deflections, and
- (iii) thick plates

In thin plates with small deflections theory, the following assumption are made

- (a) There is no deformation in the middle plane of the plate. This plane remains neutral during bending.
- (b) Points of the plate lying initially on a normal to the middle surface of the plate remain on the normal to the same surface even after bending.
- (c) The normal stresses in the direction transverse to the plate are negligible.

This theory is satisfactory for plates with ratio of thickness to span exceeding  $\frac{1}{10}$  and the ratio of maximum

deflection to thickness less than  $\frac{1}{5}$ . Many engineering problems lie in the above category and satisfactory results are obtained by classical theories of thin plates.

Stresses in the middle plane are negligible, if the deflections are small in comparison with thickness. If the deflections are large, the in plane stresses developed in the so called neutral surfaces are to be considered. This gives rise to theory of thin plates with large deflections, in which geometric non-linearity is incorporated.

If the plate has thickness to span ratio less than  $\frac{1}{10}$  the assumption (a) and (b) listed under theory of thin plates will not hold good. Such plates need three dimensional analysis. Theory developed for the analysis of such plates may be called as thick plate theory.

In this chapter, analysis of thin plates with small deflections with finite element approach is explained. Since many stress-strain-displacement relation in the theory of thin plates with small deflection are used, the limitations of classical theory remains in this finite element analysis also. The advantage of finite element analysis is that it can handle the structures with different end conditions and shapes easily.

## 15.2 BASIC RELATIONS IN THIN PLATE THEORY

For the derivation of basic relations, readers may refer to standard text books on analysis of plates [1, 2]. In this articles the necessary relations are listed taking the notations as indicated in the typical, plate element shown in Fig. 15.1.

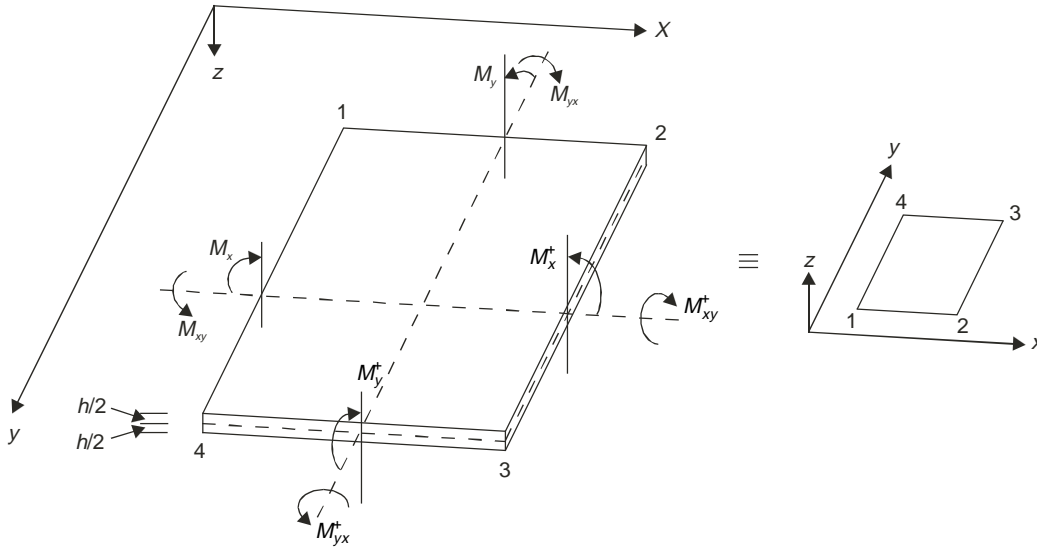


Fig. 15.1

It may be noted that, in Fig. 15.1, the right hand thumb rule of coordinate directions is used. The deflections are taken positive when they are in the positive directions of the coordinates. Stresses are taken as positive when they are in positive directions on positive faces or when they are in negative directions on negative faces. Moments are positive due to positive stress in positive direction of  $z$ .

Let  $u$ ,  $v$  and  $w$  be the displacement at any point  $(x, y, z)$  in the plate. The variation of displacement  $u$  and  $v$  across the thickness can be expressed in terms of displacement  $w$  as,

$$u = -z \frac{\partial w}{\partial x} \quad \text{and} \quad v = -z \frac{\partial w}{\partial y} \quad \dots(15.1)$$

$$\therefore \epsilon_x = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2} = \kappa_x z$$

$$\epsilon_y = \frac{\partial u}{\partial y} = -z \frac{\partial^2 w}{\partial y^2} = \kappa_y z \quad \dots(15.2)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2\kappa_{xy} z$$

where

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \text{and} \quad \kappa_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y} \quad \dots(15.3)$$

For thin plates,  $\gamma_{xz} = \gamma_{yz} = 0$ .

Stress—strain relation for isotropic material is

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{Ez}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \quad \dots(15.4)$$

The moments are given by

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = D \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \quad \dots(15.5)$$

where  $D = EI = \frac{Eh^3}{12(1-\mu^2)}$  and it is called as flexural rigidity

### 15.3 DISPLACEMENT MODELS FOR PLATE ANALYSIS

The development of displacement based finite element method within the frame work of classical plate theory possesses an extra level of difficulty. The fact that classical plate behaviour is characterized by single variable 'w' has considerable advantages in the derivation of governing differential equations for the problem. But the problem of satisfying continuity requirements along the element edges based on the single variable w and its derivatives creates considerable difficulties in the finite element modeling. Considerable research work has taken place to develop suitable plate bending elements. All these works may be grouped into the following three categories:

Category I:  $C^2$ -Continuity element i.e. second order continuity elements in which seconderivates of 'w' are also nodal unknowns.

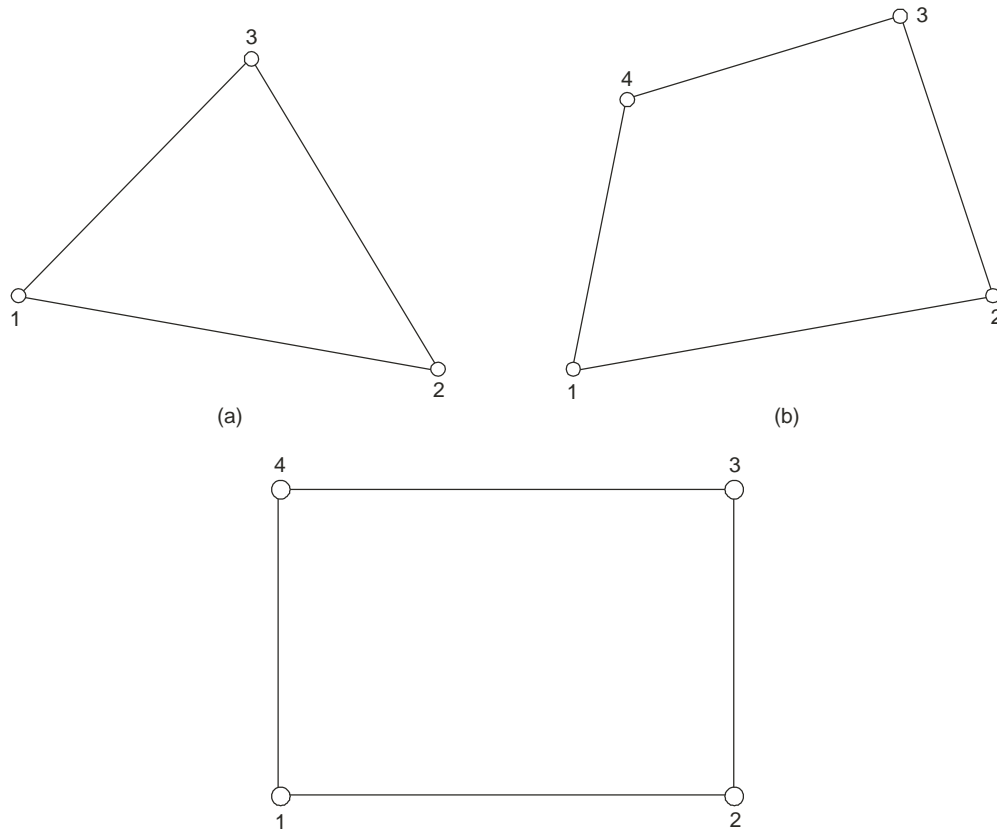
Category II:  $C^1$ -Continuity elements i.e. first order continuity elements in which highest order of derivatives of 'w' is one only.

Category III:  $C^0$ -Continuity element i.e. the elements in which only continuity of nodal variables are to be ensured.

#### 15.3.1 $C^2$ -Continuity Elements

Figure 15.2 shows some of the  $C^2$ -continuity elements. In the three noded triangular plate element nodal

variables considered are  $w$ ,  $\frac{\partial w}{\partial x}$ ,  $\frac{\partial w}{\partial y}$ ,  $\frac{\partial^2 w}{\partial x^2}$ ,  $\frac{\partial^2 w}{\partial y^2}$ , and  $\frac{\partial^2 w}{\partial x \partial y}$ .



**Fig. 15.2**  $C^2$ -continuity plate elements (a) Three noded triangular plate element (b) Four noded quadrilateral plate element (c) Four noded rectangular plate elements

Hence number of nodal unknowns are 18. If fifth order polynomial shape function is used, there are 21 terms. By adding three normal slopes at mid-side as additional unknowns, twenty one degree freedom triangular element has been developed. An exhaustive references to such works may be seen in the books by Zienkiewicz [3], Dawe [4] and Gallagher [5], Krinshnamoorthy [6]. Thus

$$w = \alpha_1 + \alpha_2 x + \dots + \alpha_{21} y^5$$

Using 21 conditions, 21 equations are obtained and the unknowns are expressed as

$$\{\delta\}_e = [C]\{\alpha\}.$$

As it is not easy to obtain inverse of  $[C]$ , stiffness expressions are developed using numerical method. If quadrilateral element is used there are 24 degrees of freedom.

In case of rectangular element at each node only four degrees of freedom may be considered i.e.  $w$ ,

$\frac{\partial w}{\partial x}$ ,  $\frac{\partial w}{\partial y}$ ,  $\frac{\partial^2 w}{\partial x \partial y}$ . Its formulation is explained in detail latter in this chapter. However use of this element is limited.



### 15.3.2 $C^1$ -Continuity Elements

To simplify analysis, many researchers, considered only three degrees of freedom at a node i.e.  $w$ ,  $\frac{\partial w}{\partial x}$  and  $\frac{\partial w}{\partial y}$ .

There is discontinuity of curvature at the corners. These are called non-conforming elements. The performance of such elements have been studied and some researchers have expressed, satisfaction to great extent. Some of them have considered the normal slopes along the edges to improve the performance of such elements. One of such element is 12 degree freedom rectangular elements and its use is explained in detail in the article. 15.4

### 15.3.3 $C^0$ -Continuity Element

Due to Kirchoff's assumption that plane section remains plane even after bending, we have the relations

between the slopes and displacement as  $\theta_x = \frac{\partial w}{\partial x}$  and  $\theta_y = \frac{\partial w}{\partial y}$ . If Kirchoff's assumption is not made, slopes

are independent of deflections and hence  $w$ ,  $\theta_x$ ,  $\theta_y$  as nodal unknowns reduces to  $C^0$ -continuity requirement.

It simplifies a lot in the finite element analysis. Mindlin [7] developed an element of this type. However it is

to be noted that giving up the relationship  $\theta_x = \frac{\partial w}{\partial x}$  and  $\theta_y = \frac{\partial w}{\partial y}$  means permitting shear deformations. Hence

in assembling stiffness expression shear strain energy is also to be considered. This element formulation is discussed in detail latter in this chapter.

## 15.4 Rectangular Plate Element with 12 Degrees of Freedom

In these elements  $C^1$ -continuity is considered, i.e. at each node three degrees of freedom, namely  $w$ ,  $\frac{\partial w}{\partial x}$  and  $\frac{\partial w}{\partial y}$

are treated as basic unknowns. Hence it leads to 12 degrees of freedom per element. This type of element is shown in Fig. 15.3. The typical element has size  $2a \times 2b$ . It may be noted that when seen in  $z$ -direction, the nodes are numbered in clockwise directions.

A complete cubic polynomial has only 10 generalized terms. If we go for complete quartic polynomial, there are 15 generalized terms. Hence to get geometric isotropy with only 12 generalized terms, drop the terms corresponding to  $x^4$ ,  $x^2 y^2$  and  $y^4$ . Hence the generalized form of the displacement selected is,

$$\begin{aligned} w &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \\ &= [P]\{\alpha\} \end{aligned} \quad \dots(14.6)$$

$$\text{Where } [P] = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2 y & xy^2 & y^3 & x^3 y & xy^3 \end{bmatrix}$$

$$\frac{\partial w}{\partial y} = \theta_y = \alpha_3 + \alpha_5 x + 2\alpha_6 y + \alpha_8 x^2 + 2\alpha_9 xy + 3\alpha_{10} y^2 + \alpha_{11} x^3 + 3\alpha_{12} xy^2$$

$$\text{and } \frac{\partial w}{\partial x} = \theta_x = \alpha_2 + 2\alpha_4 x + \alpha_5 y + 3\alpha_7 x^2 + 2\alpha_8 xy + \alpha_9 y^2 + 3\alpha_{11} x^2 y + \alpha_{12} y^3$$

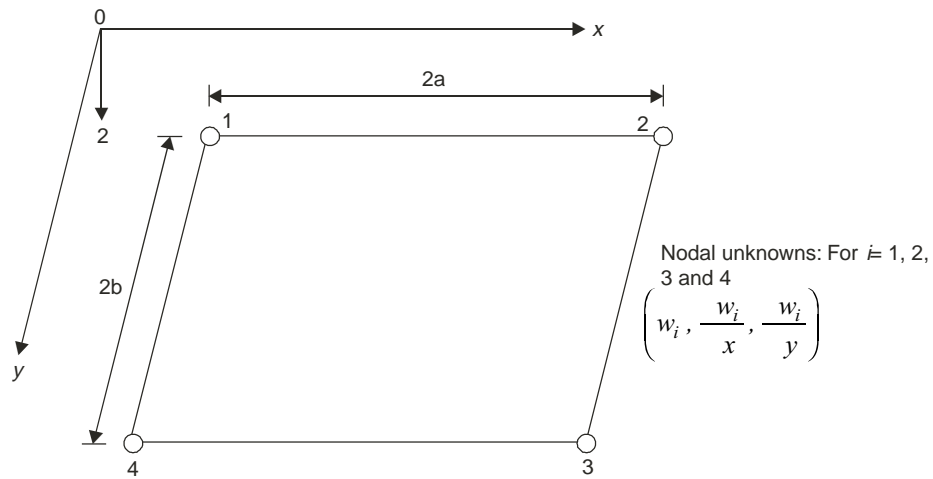


Fig. 15.3 12 degrees of freedom rectangular plate bending element

Substituting the values of  $x_i, y_i$  for nodes 1,2,3 and 4, we get

$$\begin{Bmatrix} w_1 \\ \theta x_1 \\ \theta y_1 \\ w_2 \\ \theta x_2 \\ \theta y_2 \\ w_3 \\ \theta x_3 \\ \theta y_3 \\ w_4 \\ \theta x_4 \\ \theta y_4 \end{Bmatrix} = \begin{matrix} \left[ \right. \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \left. \right] \end{matrix} \begin{matrix} 12 \times 12 \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{12} \end{Bmatrix}$$

i.e.,  $\{\delta\}_e = [G]\{\alpha\}$  ... (15.7)

or  $\{\alpha\} = [G]^{-1}\{\delta\}_e$  ... (15.8)

The displacement within the element can be expressed in the form,

$$w = [P]\{\alpha\} = [P][G]^{-1}\{\delta\}_e \quad \dots (15.9)$$

The deflection field for any point can be expressed in terms of normalized coordinates as

$$w = \sum_{i=1}^4 N_i^I w_i + \sum_{i=1}^4 N_i^{II} \theta_{x_i} + \sum_{i=1}^4 N_i^{III} \theta_{y_i} \quad \dots(15.10a)$$

Where 
$$N_i^I = \frac{1}{8}(1 + \xi\xi_i)(1 + \eta\eta_i)(2 + \xi\xi_i + \eta\eta_i - \xi^2 - \eta^2) \quad \dots(15.10b)$$

$$N_i^{II} = -\frac{b}{8}\eta_i(1 + \xi\xi_i)(1 + \eta\eta_i)^2(1 - \eta\eta_i) \quad \dots(15.10c)$$

$$N_i^{III} = -\frac{a}{8}\xi_i(1 + \xi\xi_i)^2(1 - \xi\xi_i)(1 + \eta\eta_i) \quad \dots(15.10d)$$

Where  $\xi$  and  $\eta$  are dimensionless coordinates, defined as

$$\xi = \frac{x - x_c}{a}, \eta = \frac{y - y_c}{b}, x_c, y_c \text{ being the coordinates of centre of element}$$

$$\therefore w = [N]\{\delta\}^e \quad \dots(15.11)$$

It should be noted that the displacement field obtained by equations 15.6 and 15.10 are precisely the same. The use of dimensionless natural coordinates  $\xi$  and  $\eta$  is for simplifying.

The strain matrix  $[B]$  is obtained by referring to curvature terms as strains and the moments as stress resultants.

$$\{\varepsilon\} = \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \begin{Bmatrix} -2\alpha_4 - 6\alpha_7 x - 2\alpha_8 y - 6\alpha_{11} xy \\ -2\alpha_6 - 2\alpha_9 x - 6\alpha_{10} y - 6\alpha_{12} xy \\ -2\alpha_5 - 4\alpha_8 x - 4\alpha_9 y - 6\alpha_{11} x^2 - 6\alpha_{12} y^2 \end{Bmatrix}$$

$$= [Q]\{\alpha\} = [Q][G]^{-1}\{\delta\}_e = [B]\{\delta\}_e$$

where

$$[B] = [Q][G]^{-1} \quad \dots(15.15)$$

in which,

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -4x & -4y & 0 & -6x^2 & -6y^2 \end{bmatrix} \quad \dots(15.16)$$

In case equation 15.10 are used,

$$[B] = \begin{bmatrix} -\frac{1}{a^2} \frac{\partial^2 N_i}{\partial \xi^2} \\ -\frac{1}{b^2} \frac{\partial^2 N_i}{\partial \eta^2} \\ -\frac{2}{ab} \frac{\partial^2 N_i}{\partial \xi \partial \eta} \end{bmatrix} \quad \dots(15.17)$$

The stiffness matrix relating nodal forces  $M_x, M_y, M_{xy}$  with nodal displacements  $w, \theta_x, \theta_y$  is given by

$$[k]_e = \iint [B]^T [D][B] dx dy = \{[G]^T\} \left( \iint [Q]^T [C][Q] dx dy \right) [G]^{-1} \quad \dots(15.18)$$

or, if natural coordinate system is used

$$[k]_e = \int_{-1}^1 \int_{-1}^1 [B]^T [D][B] |J| d\xi d\eta \quad \dots(15.19)$$

where  $|J|$  is Jacobian determinant and is equal to  $ab$

If generalized coordinate form (equation 15.18) is used for stiffness calculations, the terms within the integration signs can be multiplied and integrated explicitly. An explicit expression for the stiffness matrix  $[k]_e$  has been evaluated for the case of an orthotropic plate and is given in Zienkiewicz book[3].

If equation 15.19 is used, the integration is carried out numerically. It may be noted that the shape function expression contain 4th order terms. Hence the second order differentiation terms in  $[B]$  matrix contain second order terms. The stiffness matrix which has the form  $[B]^T [D] [B]$  will contain 4th order terms in  $\xi$  and  $\eta$ . Hence it needs  $3 \times 3$  Gaussian integration to get exact solution  $(2n - 1)$ th order.

The lateral loads of both surface and body type are assumed to act at the middle surface. The consistent load vector for distributed load on the surface is then given by

$$\begin{aligned} \{F\}_e &= \iint [N]^T \{X_s\} dx dy \quad \text{or} \quad \{F\}_e = \int_{-1}^1 \int_{-1}^1 [N]^T [X_s] |J| d\xi d\eta \\ &= \{[G]^{-1}\}^T \iint [P]^T \{X_x\} dx dy \end{aligned} \quad \dots(15.20)$$

If a concentrated load  $W_c$  is acting at a point whose coordinates are  $x_c$  and  $y_c$ , then its nodal equivalent is obtained as

$$\{F\}_e = \int [N]^T W_c \quad \dots(15.21)$$

With the element stiffness matrix and the consistent load columns vector established, the assembly of the system equations and the solution procedure follows as usual.

### Non-conformity of the Element

Non-conformity of the element can be easily seen by considering two adjacent elements as shown in Fig. 15.4.

In this element, let common edge be along  $y = 0$  and let nodes be 1 and 2. Since  $y = 0$  the displacement along this edge is given by

$$w = \alpha_1 + \alpha_2 x + \alpha_4 x^2 + \alpha_7 x^3$$

$$\theta_y = \frac{\partial w}{\partial x} = \alpha_2 + 2\alpha_4 x + 3\alpha_7 x^2$$

$$\theta_x = \frac{\partial w}{\partial y} = \alpha_3 + \alpha_5 x + \alpha_8 x^2 + \alpha_{11} x^3$$

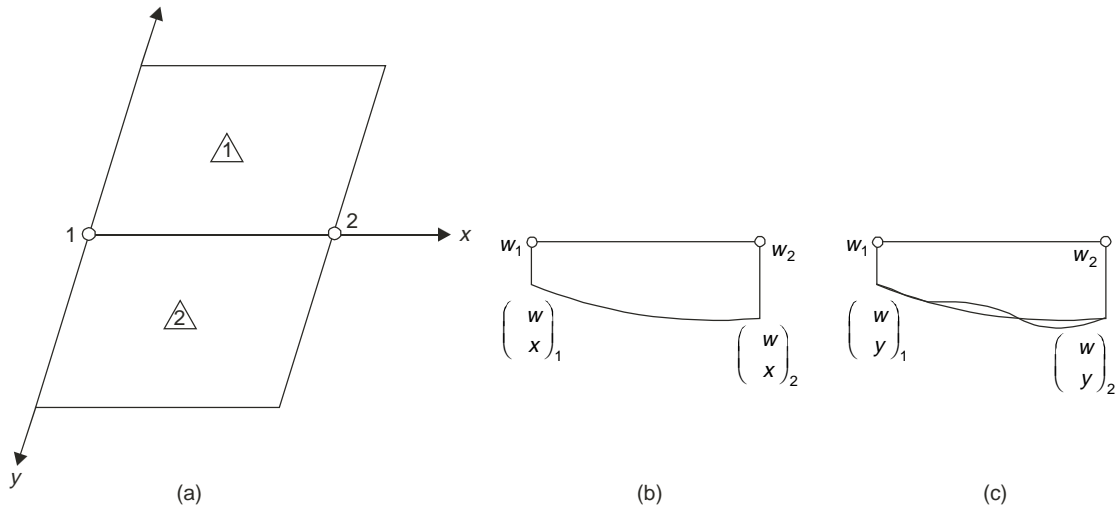


Fig. 15.4 Non-conformity of 12 degrees of freedom rectangular plate bending element

The conditions available are

$$(w_1)_1 = (w_1)_2 \quad \dots(1)$$

$$(w_2)_1 = (w_2)_2 \quad \dots(2)$$

$$\left(\frac{\partial w_1}{\partial x}\right)_1 = \left(\frac{\partial w_1}{\partial x}\right)_2 \quad \dots(3)$$

$$\left(\frac{\partial w_2}{\partial x}\right)_1 = \left(\frac{\partial w_2}{\partial x}\right)_2 \quad \dots(4)$$

and

$$\left(\frac{\partial w_1}{\partial y}\right)_1 = \left(\frac{\partial w_1}{\partial y}\right)_2 \quad \dots(5)$$

$$\left(\frac{\partial w_2}{\partial y}\right)_1 = \left(\frac{\partial w_2}{\partial y}\right)_2 \quad \dots(6)$$

From the conditions 1, 2, 3 and 4, we get four equations in 4 unknowns  $\alpha_1, \alpha_2, \alpha_4$  and  $\alpha_7$ . Hence they are uniquely determined. In other words  $w$  and  $\frac{\partial w}{\partial x}$  satisfy the requirement of uniqueness along the common edge. From the condition 5 and 6, we get two equation in 4 unknowns, namely  $\alpha_3, \alpha_5, \alpha_8$  and  $\alpha_{11}$ . Hence

it is not possible to get uniqueness of  $\frac{\partial w}{\partial y}$  along the common edge. These situations are shown in Figs 15.4

(b) and (c). Thus there is non-conformity of slope  $\frac{\partial w}{\partial y}$  along the edge considered.

However the performance of the element is reasonably good when sufficiently finer mesh are used. The results obtained by Gowdaiah [8] for a simply supported plate and a fixed plate, considering uniformly distributed load are shown in Tables 15.1 and 15.2.

**Table 15.1** Central Deflections and Moment of a Simply Supported Square Plate Using 12 Degrees Freedom Rectangular Plate Element.

Mesh.	Total No. of Nodes	Central Deflection.	% Error	Central Moment $M_x$	% Error
2 × 2	9	0.5063238	24.65	6.601858	37.83
4 × 4	25	0.4328214	6.55	5.216951	8.91
8 × 8	81	0.4129367	1.65	4.891917	2.12
12 × 12	169	0.4091184	0.72	4.833032	0.9
Exact (Timoshenko)		0.4062		4.79	

**Table 15.2** Central Deflections and Moment of a Fixed square Plate Using 12 Degrees Freedom Rectangular Plate Element.

Mesh	Total No. of Nodes	w	% Error	$M_x$ , Centre	% Error	$M_x$ , Support of Middle edge	% Error
2 × 2	9	0.1479614	17.43	4.616477	99.84	-3.551136	30.78
4 × 4	25	0.1493343	11.38	2.778318	20.27	-4.760767	7.19
8 × 8	81	0.1303969	3.48	2.404827	4.10	-5.028381	1.98
12 × 12	169	0.128266	1.8	2.340618	1.33	-5.082736	0.92
Exact(Timoshenko)		0.126		2.31		-5.13	

### 15.5 RECTANGULAR PLATE ELEMENT WITH 16 DEGREES OF FREEDOM

Bogner, Fox and Schmit [9] considered  $\frac{\partial^2 w}{\partial x \partial y}$  also as nodal unknown to overcome non-conformity faced in

12 degrees freedom element. It leads to 16 degrees of freedom, namely  $w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$  and  $\frac{\partial^2 w}{\partial x \partial y}$  at each node.

The generalized form displacement considered by Bogner is,

$$w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 x^{xy} + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 + \alpha_{12} x^2 y^2 + \alpha_{13} xy^3 + \alpha_{14} x^3 y^2 + \alpha_{15} x^2 y^3 + \alpha_{16} x^3 y^3 \quad \dots(15.22)$$

It may be noted that to get geometric isotropy, Bogner et al. [9] have considered the terms appearing in the product  $(1 + x + x^2 + x^3)(1 + y + y^2 + y^3)$ . One can proceed on the lines explained for 12 degrees of freedom element to assemble  $[P]$ ,  $[G]$ ,  $[F]$  and  $[B]$  matrix and then evaluate stiffness matrix. Size of the element stiffness matrix in this case is  $16 \times 16$ .

The above problem can be solved using natural coordinates and numerical integration technique. The deflection field given by equation 15.22 may be expressed in the natural coordinates form using non-dimensionalised coordinates  $\xi$  and  $\eta$

$$w = \sum_{i=1}^4 f_i^I w_i + f_i^{II} \theta x_i + f_i^{III} \theta y_i + f_i^{IV} \theta x_i y_i \quad \dots(15.23)$$

$$\text{Where } \theta_{xy} = \frac{\partial^2 w}{\partial x \partial y}$$

$f_i^I$ ,  $f_i^{II}$ ,  $f_i^{III}$  and  $f_i^{IV}$  are Hermetian shape functions. Dawe D.J.[4] has presented these functions as,

$$N_1 = f_1^I = \frac{1}{16} (4 - 6\xi - 6\eta + 9\xi\eta + 2\xi^3 + 2\eta^3 - 3\xi\eta^3 - 3\xi^3\eta + \xi^3\eta^3)$$

$$N_2 = f_1^{II} = \frac{b}{16} (2 - 3\xi - 2\eta + 3\xi\eta - 2\eta^2 + 3\xi\eta^2 - \xi^3\eta + 2\eta^3 - \xi^3\eta^2 + 3\xi\eta^3 + \xi^3\eta^3 + \xi^3)$$

$$N_3 = f_1^{III} = \frac{a}{16} (2 - 2\xi - 3\eta + 3\xi\eta - 2\xi^2 + 3\xi^2\eta - 3\xi\eta^3 + 2\xi^3 - \xi^2\eta^3 - 3\xi^3\eta + \xi^3\eta^3 + \eta^3)$$

$$N_4 = f_1^{IV} = \frac{ab}{16} (1 - \xi - \eta + \xi\eta - \xi^2 - \eta^2 + \xi\eta^2 + \xi^2\eta + \xi^3 + \eta^3 - \xi\eta^3 - \xi^3\eta - \xi^2\eta^3 + \xi^2\eta^2 - \xi^3\eta^2 + \xi^3\eta^3)$$

$$N_5 = \frac{1}{16} (4 + 6\xi - 6\eta - 9\xi\eta - 2\xi^3 + 2\eta^3 + 3\xi\eta^3 + 3\xi^3\eta - \xi^3\eta^3)$$

$$N_6 = \frac{b}{16} (2 + 3\xi - 2\eta - 3\xi\eta - 2\eta^2 - 3\xi\eta^2 + 3\xi\eta^3 - \xi^3 + 2\eta^3 + \xi^3\eta^2 - \xi^3\eta^3)$$

$$N_7 = \frac{a}{16} (-2 - 2\xi + 3\eta + 3\xi\eta + 2\xi^2 - \xi\eta^3 - 3\xi^2\eta + \xi^2\eta^3 + 2\xi^3 - \eta^3 - 3\xi^3\eta + \xi^3\eta^3)$$

$$N_8 = \frac{ab}{16} (-1 - \xi + \eta + \xi\eta + \xi^2 + \eta^2 + \xi\eta^2 - \xi\eta^3 - \xi^2\eta - \xi^2\eta^2 + \xi^2\eta^3 + \xi^3 - \eta^3 - \xi^3\eta - \xi^3\eta^2 + \xi^3\eta^3)$$

$$N_9 = \frac{1}{16} (4 + 6\xi + 6\eta + 9\xi\eta - 2\xi^3 - 2\eta^3 - 3\xi\eta^3 - 3\xi^3\eta + \xi^3\eta^3)$$

$$N_{10} = \frac{b}{16} (-2 - 3\xi - 2\eta - 3\xi\eta + 2\eta^2 + 3\xi\eta^2 - \xi^3\eta^2 + \xi^3\eta + \xi^3 + 2\eta^3 + 3\xi\eta^3 - \xi^3\eta^3)$$

$$\begin{aligned}
 N_{11} &= \frac{a}{16} (-2 - 2\xi - 3\eta - 3\xi\eta + 2\xi^2 + 3\xi^2\eta - \xi^2\eta^3 + \xi\eta^3 + \eta^3 + 3\xi^3\eta - \xi^3\eta^3) \\
 N_{12} &= \frac{ab}{16} (1 + \xi + \eta + \xi\eta - \xi^2 - \eta^2 - \xi\eta^2 - \xi^2\eta - \xi^3 - \eta^3 + \xi^2\eta^2 - \xi\eta^3 - \xi^3\eta + \xi^2\eta^3 + \xi^3\eta^2 + \xi^3\eta^3) \\
 N_{13} &= \frac{1}{16} (4 - 6\xi + 6\eta - 9\xi\eta + 2\xi^3 - 2\eta^3 + 3\xi\eta^3 + 3\xi^3\eta - \xi^3\eta^3) \\
 N_{14} &= \frac{b}{16} (-2 + 3\xi - 2\eta + 3\xi\eta + 2\eta^2 - \xi^3\eta - 3\xi\eta^2 + \xi^3\eta^2 + 2\xi^3 - \xi\eta^3 - 3\xi\eta^3 + \xi^3\eta^3) \\
 N_{15} &= \frac{a}{16} (2 - 2\xi + 3\eta - 3\xi\eta - 2\xi^2 - 3\xi^2\eta + 3\xi^3\eta - \eta^3 + 2\xi^3 + \xi\eta^3 + \xi^2\eta^3 - \xi^3\eta^3) \\
 N_{16} &= \frac{ab}{16} (-1 + \xi - \eta + \xi\eta + \xi^2 + \eta^2 + \xi^2\eta - \xi\eta^2 - \xi^3\eta - \xi^2\eta^2 + \xi^3\eta^2 - \xi^3 + \eta^3 - \xi\eta^3 - \xi^2\eta^3 + \xi^3\eta^3)
 \end{aligned}$$

...(15.24)

where  $\xi = \frac{x}{a}$  and  $\eta = \frac{y}{b}$

$$\therefore w = [N]\{\delta\}_e \tag{15.25}$$

Where  $[N]$  is the shape function matrix and  $\{\delta\}_e =$  Nodal displacement vector.

Stiffness and load matrices may be assembled on the lines similar to 12 degrees freedom element and system equations assembled. This approach needs fifth order Gaussian integrations, since the stiffness matrix involves 8<sup>th</sup> order terms. Gowdaiah [9] developed a program to analyze plates using 16 degrees freedom rectangular elements and studied the behaviour of square plates subject to uniformly distributed load. Tables 15.3 and 15.4 shows the results for simply supported and fixed plates:

**Table 15.3** Deflections and Moments in a Simply Supported Square Plate Obtained by Using 16 Degrees Freedom Elements.

Mesh	No. of nodes	W centre	% Error	Mx, centre	% Error
2 × 2	9	0.4808683	18.38	5.085667	6.17
4 × 4	25	0.422946	4.123	4.669042	-2.53
8 × 8	81	0.4087133	0.619	4.762794	-0.57
12 × 12	169	0.4072136	0.25	4.781559	-0.17
Exact (Timoshenko)		0.4062		4.79	



**Table 15.4** *Deflections and Moments in a Fixed Square Plate Obtained by Using 16 Degrees of Freedom Elements.*

Mesh	No. of nodes	$W$ centre	% Error	$M_x$ , centre	% Error	$M_x$ , Support at mid span	% Error
$2 \times 2$	9	0.1324794	5.14	4.13357	-78.9	-3.179505	-38.02
$4 \times 4$	25	0.132185	4.94	2.31496	0.215	-3.942102	-23.2
$8 \times 8$	81	0.1279442	1.54	2.283149	-1.162	-4.533	-11.6
$12 \times 12$	169	0.1270355	0.82	2.289747	-0.877	-4.749	-7.43
Exact (Timoshenko)		0.126		2.31		-5.13	

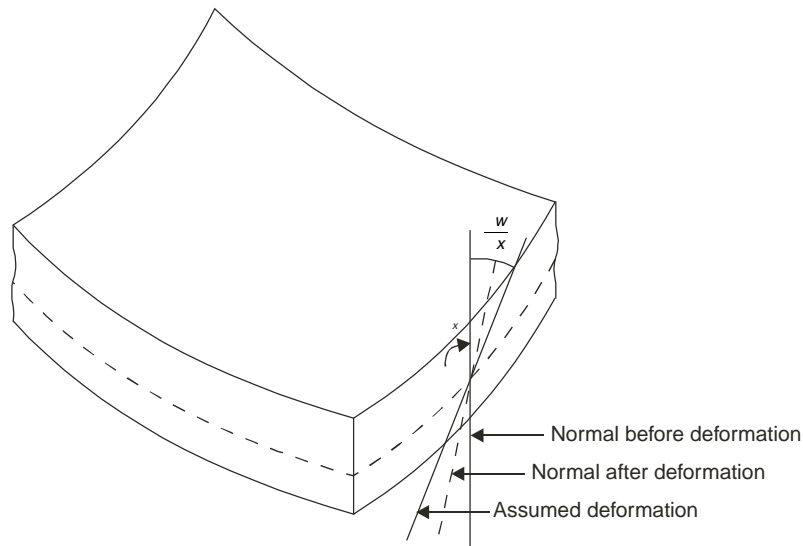
## 15.6 MINDLIN'S PLATE ELEMENT

Mindlin's [7] theory is the extension of Timoshenko theory to the analysis of plates. In this theory the rotation and lateral deflections are decoupled and shear deformations are considered. This resulted into development of  $C^0$ -continuity plate element. This helped in extending isoparametric concept in plate analysis resulting to development of 4-noded quadrilateral and 8-noded quadratic plate bending elements.

Mindlin[7] retained the following assumptions of thin plates small deflection theory:

- (i) The lateral deflections ' $w$ ' are small
- (ii) Stresses normal to the midsurface are negligible

However he gave up Kirchoff's assumption that plane normal to the midsurface remain plane even after bending. Instead of this he assumed normal to the plate midsurface before deformation remains straight but not necessary normal to it after deformation. This is shown in Fig. 15.5. Hence, if,

**Fig. 15.5** *Deformation of the plate in xz-plane*

$\theta_x$  is the final rotation in  $x$ -direction, we get

$$\theta_x = \frac{\partial w}{\partial x} \text{ -average shear strain in } x\text{-direction}$$

i.e.,

$$\theta_x = \frac{\partial w}{\partial x} - \phi_x$$

Similarly

$$\theta_y = \frac{\partial w}{\partial x} - \phi_y \quad \dots(15.26)$$

At any node there are three independent field variables  $w$ ,  $\theta_x$  and  $\theta_y$ . The displacement at any point inside the element is given by

$$\begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^n \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \quad \dots(15.27)$$

where  $n$  is the number of nodes in the element. For quadrilateral element  $n = 4$  and

$$N_1 = \frac{(1-\xi)(1-\eta)}{4}, N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \text{ and } N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

Similarly for quadratic element  $n = 8$ , and the shape functions are as given in equation 5.44. For cubic element the shape functions are as presented in equation 5.46. The position of the point itself is given by,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} \sum N_i x_i \\ \sum N_i y_i \end{Bmatrix} = \sum_{i=1}^n \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \quad \dots(15.28)$$

The measure of strain in this element includes both flexural strain  $k$  and shear strain  $\epsilon$ . The flexural strain is given by.

$$\epsilon_f = \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_y}{\partial x} + \frac{\partial \theta_x}{\partial y} \end{Bmatrix} \quad \dots(15.29)$$

$$= \sum_{i=1}^n \begin{bmatrix} 0 & \frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y} \\ 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \{\delta\}_e \quad \dots(15.30)$$

Shear strain is given by

$$\varepsilon_\phi = \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w}{\partial x} - \theta_x \\ \frac{\partial w}{\partial y} - \theta_y \end{Bmatrix} = \sum_{i=1}^n \begin{bmatrix} \frac{\partial N_i}{\partial x} & -N_i & 0 \\ \frac{\partial N_i}{\partial y} & 0 & -N_i \end{bmatrix} \{\delta\}_e \quad \dots(15.31)$$

The stress resultant curvature (strain) relation is given by

$$\begin{aligned} \{\sigma_f\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= \begin{Bmatrix} -D \left( \frac{\partial \theta_x}{\partial x} + \mu \frac{\partial \theta_y}{\partial y} \right) \\ -D \left( \mu \frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y} \right) \\ -D \frac{(1-\mu)}{2} \left( \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right) \end{Bmatrix} \\ &= -D \sum_{i=1}^n \begin{bmatrix} 0 & \frac{\partial N_i}{\partial x} & \mu \frac{\partial N_i}{\partial y} \\ 0 & \mu \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ 0 & \frac{1-\mu}{2} \frac{\partial N_i}{\partial y} & \frac{1-\mu}{2} \frac{\partial N_i}{\partial x} \end{bmatrix} \{\delta\}_e = [D_f] [B_f] \{\delta\}_e \quad \dots(15.32) \end{aligned}$$

where  $D_f = -D$

and

$$[B_f] = \sum_{i=1}^n \begin{bmatrix} 0 & \frac{\partial N_i}{\partial x} & \mu \frac{\partial N_i}{\partial y} \\ 0 & \mu \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ 0 & \frac{1-\mu}{2} \frac{\partial N_i}{\partial y} & \frac{1-\mu}{2} \frac{\partial N_i}{\partial x} \end{bmatrix} \quad \dots(15.33)$$

The shear forces, referred as shear stress resultants are given by

$$\{\sigma_s\} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = GA \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

where  $A$  is area per unit length, since  $\theta_x$  and  $\theta_y$  are defined as the forces per unit length and  $\gamma_{xz}$ ,  $\gamma_{yz}$  are shear strains and  $G$  is modulus of rigidity.

For isotropic plates the relation is

$$\{\sigma_s\} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gh \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = Gh \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix} = \frac{Eh}{2(1+\mu)} \begin{Bmatrix} \phi_x \\ \phi_y \end{Bmatrix}$$

since

$$G = \frac{E}{2(1+\mu)}$$

$$\begin{aligned} \therefore \{\sigma_s\} &= \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = Gh \begin{bmatrix} \theta_x - \frac{\partial w}{\partial x} \\ \theta_y - \frac{\partial w}{\partial y} \end{bmatrix} \\ &= Gh \sum_{i=1}^n \begin{bmatrix} -\frac{\partial N_i}{\partial x} & N_i & 0 \\ -\frac{\partial N_i}{\partial x} & 0 & N_i \end{bmatrix} \{\delta\}_e = [D_s] \sum_{i=1}^n \begin{bmatrix} -\frac{\partial N_i}{\partial x} & N_i & 0 \\ -\frac{\partial N_i}{\partial y} & 0 & N_i \end{bmatrix} \{\delta\}_e \dots (15.34) \end{aligned}$$

However for  $[D_s]$  a correction factor  $\alpha$  is suggested by Proger et. al.[10]. They suggest  $D_s$  may be taken as  $Gh\alpha$  to represent the restraint of the cross section against warping. The value of  $\alpha$  commonly used is 5/6. Equation 14.31 may be written as

$$\{\sigma_s\} = [D_s][B_s]\{\delta\}_e$$

Where  $B_s$  is given by,

$$B_s = \sum_{i=1}^n \begin{bmatrix} -\frac{\partial N_i}{\partial x} & N_i & 0 \\ -\frac{\partial N_i}{\partial y} & 0 & N_i \end{bmatrix} \dots (15.35)$$

The local coordinates and the global coordinate systems can be related by using isoparametric concept

$$\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} \dots (15.36)$$

Where  $[J]$  is Jacobian matrix

$$\text{i.e., } [J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i \\ \sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i \end{bmatrix} \quad \dots(15.37)$$

Now the stiffness due to flexure and shear together is given by

$$\begin{aligned} k &= k_f + k_s \\ &= \oint [B_f]^T [D_f] [B_f] dx dy h + \oint [B_s]^T [D_s] [B_s] dx dy h \\ &= \oint [B_f]^T [D_f] [B_f] |J| h d\xi d\eta + \oint [B_s]^T [D_s] [B_s] |J| h d\xi d\eta \quad \dots(15.38) \end{aligned}$$

Gaussian integration is used to evaluate  $[k]$ . Consistent load vector is obtained as usual, for uniformly distributed load of intensity 'q'

$$\{F\} = \int_{-1}^1 \int_{-1}^1 [N]^T q |J| d\xi d\eta \quad \dots(15.39)$$

The remaining part of finite element analysis follows the steps as usual.

Mindlin plate element gives good results for moderately thick plates. They can be used even for odd shaped plates.

Unfortunately these elements behave very erratically in extreme thin plates. This has been the subject of interest in 1970s, 1980s and lot of research [11–14] has gone in to get rid of the problem faced in extremely thin plates. Zienkiewicz et al. [11]. showed that the lack of field consistency in field definition introduces spurious constraints. This can be removed by an appropriate integration strategy.

Ideally, for a consistent definition of shear strain, one must have associated with each term of the polynomial expansion of the shear strain field, contribution from both  $w$  and  $\theta_x$  and  $w$  and  $\theta_y$  interpolations. This can be achieved only if unequal order interpolations are used for  $w$  and the rotation terms. If equal order is used it turns out that some of the term in shear strain field have contributions only from the interpolations for face rotations  $\theta_x$  and  $\theta_y$ . In case of extreme thinness, these terms severely constrain the behaviour of the rotations  $\theta_x$  and  $\theta_y$ . This situation is termed as shear locking.

Hughes et al. [12] has successfully used uniform one point integration for shear strain energy terms and  $2 \times 2$  integration for bending energy terms. The element behaves very well in this thin plate situations but restricts the use of the element beyond a moderately thick situation. Pratap and Viswanath [10] suggested an optimal integration strategy for 4 noded Mindlin plate bending element. The strategy suggested incorporates a  $2 \times 2$  Gaussian integration of the bending energy and separate  $1 \times 2$  and  $2 \times 1$  integration rules for the shear

energy contributions from the  $\left(\frac{\partial w}{\partial x} - \theta_x\right)$  and  $\left(\frac{\partial w}{\partial x} - \theta_y\right)$  terms respectively. The results are good for

moderately thick to thick plate situations, especially if local axes  $\xi$  and  $\eta$  are aligned with global  $x$  and  $y$  axes.

It may be noted that the Kirchoff's constraint in thin plate theory is dependent on  $x$  and  $y$  orthogonal Cartesian system. With the distortion to non-rectangular forms or with the arbitrary orientation system, it is impossible to devise a simple integrations strategy that will correctly retain all valid constraints. This deficiency felt more severely in thin plate situations rather than thick plate situations.

The derivation of element properties based on Mindlin plate theory is incorporated in many standard Finite Element Analysis packages. It allows the use of isoparametric concept, hence more general element shapes, such as quadrilateral and quadratic can be devised. The shape functions and element routines are also simple compassed to  $C^1$ -continuity elements. With the judicial selection of the order of integration the same element may be used both in the analysis of thin plates and moderately thick plates.

### Need for Stress Smoothing

In the finite element analysis it is proved that

1. Stresses are discontinuous at nodes.
2. Stresses at interior of the elements are more accurate. And
3. Stresses obtained at Gaussian points are accurate.

The stress discontinuities obtained at node points are shown in Fig. 15.6. Hence there is need for stress smoothing.

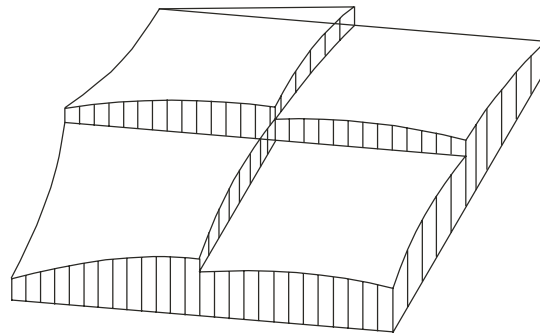


Fig. 15.6 Stress discontinuities at nodes

**Stress Smoothing Technique:** Initially it was thought that at node point the average of the values obtained from different elements may be taken. But this technique has draw back that it is not considering the size effect of the various elements meeting at that node. Finally it is accepted that since Gauss point values are accurate, they may be used for bilinear extrapolation and nodal values calculated. This technique was suggested by Hinton and Campbell [15].

For a  $2 \times 2$  Gauss point integration bilinear extrapolation may be done as explained below:

Let  $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3,$  and  $\bar{\sigma}_4$  be nodal values to be obtained by extrapolations and  $\sigma_I, \sigma_{II}, \sigma_{III}$  and  $\sigma_{IV}$  be the values at Gauss points as shown in Fig. 15.7. In two point Gauss integrations the Gauss point are at

$(0.57735, 0.57735)$  i.e. at  $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$  From the centre. To find  $\bar{\sigma}_1$ , first we can get  $\sigma'_1$  and  $\sigma'_2$  values by

extrapolating  $\sigma_I, \sigma_{II}$  and  $\sigma_{IV}$  and  $\sigma_{III}$  values as follows:

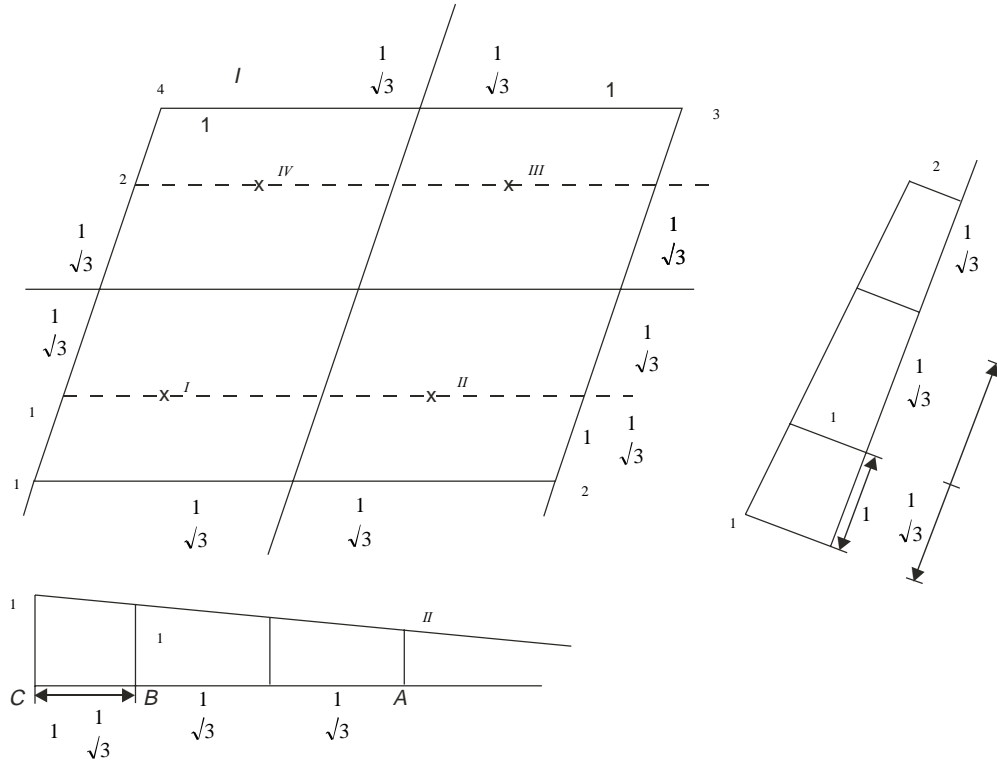


Fig. 15.7 Bilinear extrapolation of stresses

$$\sigma'_1 = \sigma_I + (\sigma_I - \sigma_{II}) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right)$$

$$\sigma'_2 = \sigma_{IV} + (\sigma_{IV} - \sigma_{III}) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right)$$

Now linearly extrapolating between  $\sigma'_1$  and  $\sigma'_2$  to get  $\bar{\sigma}_1$ , we get

$$\begin{aligned} \bar{\sigma}_1 &= \sigma'_1 + (\sigma'_1 - \sigma'_2) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \\ &= \sigma_I + (\sigma_I - \sigma_{II}) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) + \left[ \sigma_I + (\sigma_I - \sigma_{II}) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) - \sigma_{IV} - (\sigma_{IV} - \sigma_{III}) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \right] \\ &\quad \times \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sigma_I \left[ 1 + \frac{\sqrt{3}}{2} - \frac{1}{2} + \frac{\sqrt{3}}{2} - \frac{1}{2} + \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \right] + \sigma_{II} \left[ - \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) - \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \right] + \\
 &\sigma_{III} \left[ \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \right] - \sigma_{IV} \left[ 1 + \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \right] \left( \frac{\sqrt{3}}{2} - \frac{1}{2} \right) \\
 &= \sigma_I \left( 1 + \frac{\sqrt{3}}{2} \right) - \frac{1}{2} \sigma_{II} + \sigma_{III} \left( 1 - \frac{\sqrt{3}}{2} \right) - \frac{1}{2} \sigma_{IV}
 \end{aligned}$$

Similarly the values at other node points may be obtained by bilinear extrapolation to get the following relations:

$$\begin{Bmatrix} \overline{\sigma}_1 \\ \overline{\sigma}_2 \\ \overline{\sigma}_3 \\ \overline{\sigma}_4 \end{Bmatrix} = \begin{bmatrix} 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} \\ 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 + \frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{III} \\ \sigma_{IV} \end{Bmatrix}$$

### QUESTIONS

1. Discuss the use of triangular plate bending elements.
2. Discuss the conforming and non-conforming rectangular plate bending analysis.
3. Explain the term Mindlin's  $C^0$ -continuity plate element and briefly explain stiffness matrix formulation for such elements.
4. Explain the term 'Shear locking'. How this problem is overcome?
5. Write short notes on numerical integration and stress smoothening in the case of four noded quadrilateral plate element.
6. Write short notes on
  - (i) Triangular plate bending elements
  - (ii) Conforming and non-conforming plate bending elements
  - (iii) Mindlin's  $C^0$ -continuity element
  - (iv) Shear locking and
  - (v) Stress smoothening as applied to plate bending analysis.

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# 16

## Analysis of Shells

### 16.1 INTRODUCTION

A shell is a curved surface. Due to their shape they transfer most of the load applied on their surface as inplane forces (membrane forces) rather than by flexure. Hence the shells are examples of strength through form rather than mass. Civil engineers use them as roofs to get large column free areas covered. Cylindrical shells, domes hyperbolic parabolic shells etc. are common examples of shell roofs. Cooling towers, conical shells are also commonly used shells. Mechanical and chemical engineers use shells as pressure vessels and as components of many machines.

Shells may be classified as singly curved or doubly curved. Classification of shell surfaces is attempted on the basis of Gauss curvature (product of principle curvature in two perpendicular directions). If the Gauss Curvature is positive, zero, negative the surface will be classified as synclastic, developable, anticlastic respectively. Further classification is possible depending upon whether a shell is translational surface, a ruled surface or a surface of revolution. Indian standard code 2210[1] gives the classification of shells and folded plates by various criteria.

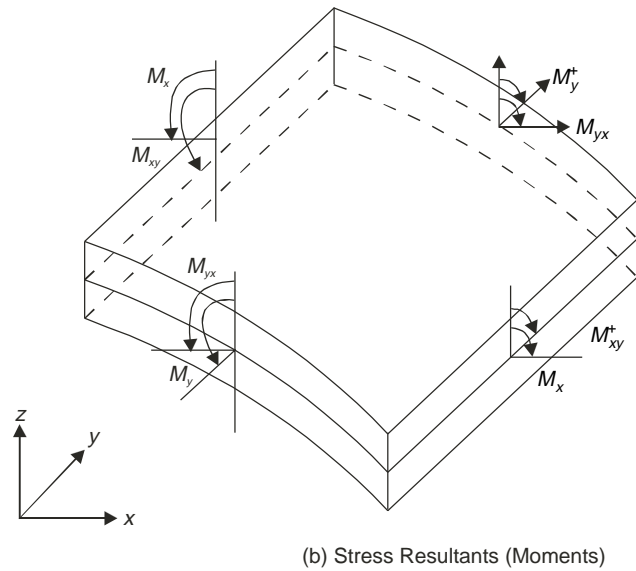
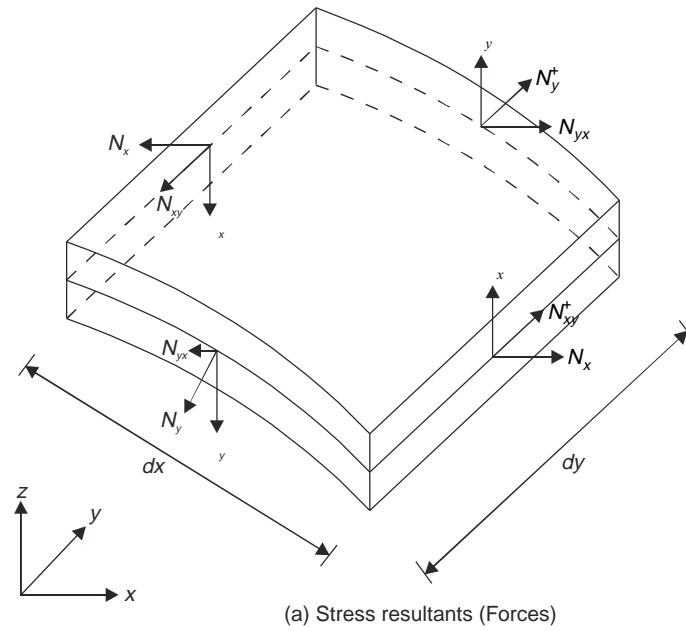
As most of the load is transformed as inplane, the shells can be thin resulting into considerable reduction in material cost. In addition shells have aesthetic advantage. However the cost of form work required for reinforced concrete make shell roof expensive. Unless there is chance of several reuse of form work shell roofs are not used. For details of classical shell theory one can refer the book by Ramaswamy [2]

In this chapter, first various forces developed in a shell elements are explained. Then various finite elements developed are briefly explained. Finite element formulation for 4-noded degenerated shell element is shown and it is hoped that reader, if need be, will be able to extend the finite element formulation to 8-noded degenerated element also.

### 16.2 FORCES ON SHELL ELEMENT

Fig. 16.1 shows a typical shell element and various stress resultants acting on it. It may be noted that the sign convention is:

- (i) Coordinate direction are as per right hand thumb rule
- (ii) A force acting on +ve face in +ve direction or -ve face -ve direction is +ve
- (iii) A +ve force acting on +ve z-direction produces +ve moment, about mid surface



**Fig. 16.1** *Stress resultants*

### 16.3 FINITE ELEMENTS FOR SHELL ANALYSIS

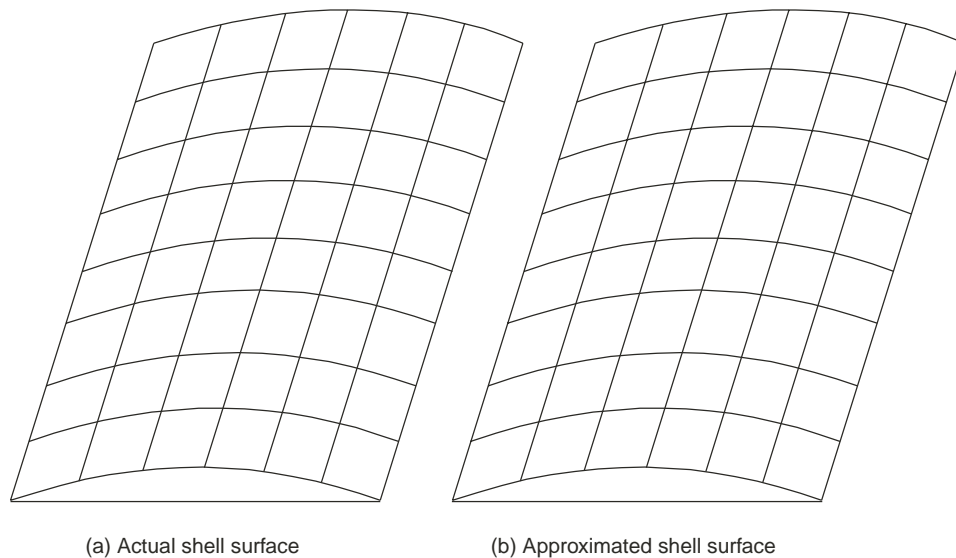
Curved shell structures constitute possibly the most difficult class of structures to analysis by the finite element method and the difficulties involved have lead to the development of considerable variety of approaches to the problems and a large number of element types. The following are the four different approaches used to generate the shell elements:

1. Flat Elements
2. Curved Elements
3. Solid Elements
4. Degenerated Solid Elements.

The above elements are briefly explained below and their performance is commented.

## 1. Flat Elements

The earliest method to analysis shells by finite element method was to approximate the curved surface with a number of flat elements. Fig. 16.2 shows approximation of a cylindrical shell roof by a number of flat elements. Since shell, have bending as well as in plane forces, for flat element stiffness matrix should be assembled using both plate bending consideration and considering in plane forces. Fig. 16.3 shows in plane and bending forces to be considered. One can use triangular, rectangular or quadrilateral plate elements. Smaller the element size, better is the result. The development of such shell elements progressed along with the development of plate elements. Using such elements arch dams, cylindrical shell roofs and cooling towers have been successfully analysed by zienkiewicz et al.[3, 4, 5]

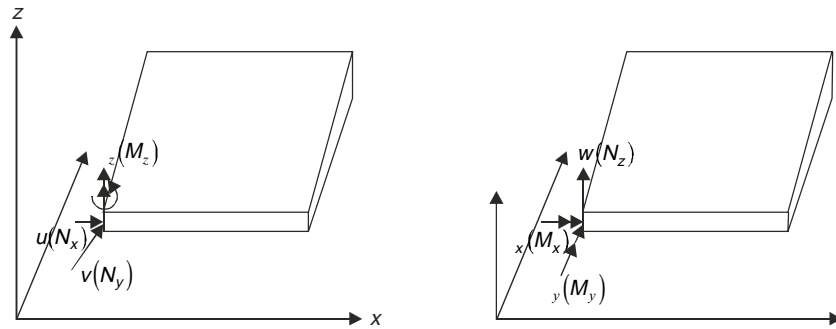


**Fig. 16.2** Flat elements used for shell analysis

The shortcomings of these flat elements (also called as Facet Elements) are as listed below:

- (i) The curvature of the elements is absent within the element.
- (ii) The discontinuities of slope between the plate elements produce spurious moments.
- (iii) The plate elements themselves have limitations in the analysis of plates, which continues to stay in the shell analysis too.

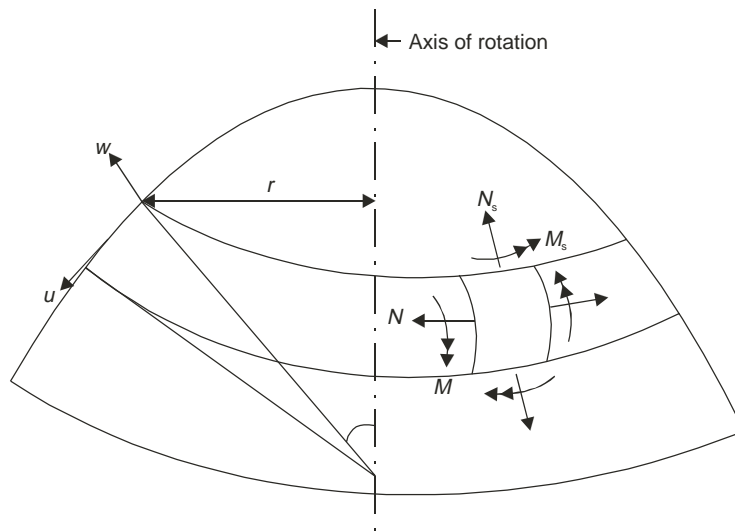
However singly curved shells may be analysed satisfactorily by taking refined meshes.



**Fig. 16.3** (a) *Inplane forces and deformations* (b) *Bending forces and deformations*

## 2. Curved Shell Elements

There are a number of practical problems in which we come across axi-symmetric shell analysis. Fig. 16.4 shows one such case.



**Fig. 16.4** *Axisymmetric shell*

In this problem of thin shell analysis, the displacement and stress resultants may be defined with respect to meridional directions  $(u, N_s, M_\theta)$  and circumferential directions  $(w, N_\theta, M_s)$ . Thus the strain vector is given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_u \\ \varepsilon_\theta \\ k_s \\ k_\theta \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial s} \\ (w \cos \phi + u \sin \phi) \\ \frac{-\partial^2 w}{\partial s^2} \\ \frac{-\sin \phi}{r} \frac{dw}{ds} \end{Bmatrix} \quad \dots(16.1)$$

and stress resultant is given by

$$\{\sigma\} = \begin{Bmatrix} N_s \\ N_\theta \\ M_s \\ M_\theta \end{Bmatrix} = [D]\{\varepsilon\} \quad \dots(16.2)$$

For a straight edged element, suitable for the analysis of conical shells  $[B]$  matrix can be assembled easily and stiffness matrix  $\int [B]^T [D][B] dV$  can be assembled. For curved shells one has to ensure common tangent between adjacent elements which needs  $C^1$ —continuity elements. Using isoparametric concept axi-symmetric shells have been analyzed [Jones et al. (6), Stricklin (7)]. After getting nodal displacement, the stresses at required points are obtained. Gallagher [8] lists the following difficulties in the development of such shell elements:

1. For assembling  $[D]$  matrix suitable shell theory is to be used, but there are a number of shell theories.
2. It is difficult to achieve inter element compatibility as seen in plate elements.
3. Describing the geometry using given element data is difficult.
4. The satisfaction of rigid body modes of behaviour is acute.

### 3. Solid Shell Element

Another approach for shell analysis is to use three dimensional solid elements. One can think of using 4 noded tetrahedron, 8 noded hexahedron or 20 noded curved solid elements for the analysis of shells. To take care of bending behaviour more than one layer of elements are to be used across the thickness. However this approach for shell analysis is found not satisfactory because of the following reasons:

- (i) As the thickness reduces the strain normal to the mid surface is associated with very large stiffness coefficients and hence the equations become ill conditioned
- (ii) These elements carry too many degrees of freedom making the computation uneconomic.

### 4. Degenerated Solid Elements

In 1970 Ahmad et al. [9] introduced the concept of degenerating 3-D-elements to 2-D-elements for finite element analysis while using 3D- elastic theory. For example, a 3-D brick element is reduced to shell element by deleting the intermediate nodes in the thickness direction and then by projecting the nodes on each surface to the mid surface as shown in Fig. 16.6. Similarly 20 noded solid element may be degenerated to 8 noded element on the mid surface which is also shown in Fig. 16.6. However the nodes on the 2 outer surfaces corresponding to each mid-surface nodes are defined so as to keep the analysis in 3-D. The theory is developed with the following assumptions:

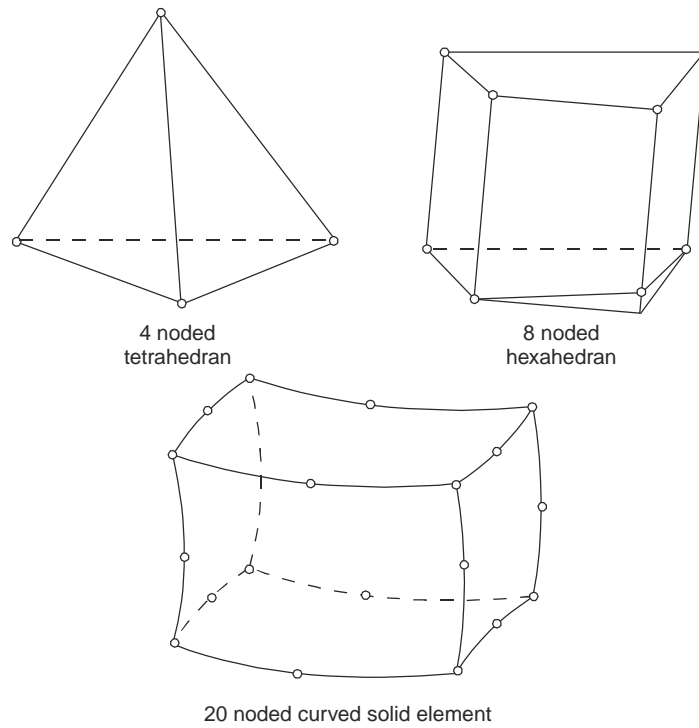


Fig. 16.5

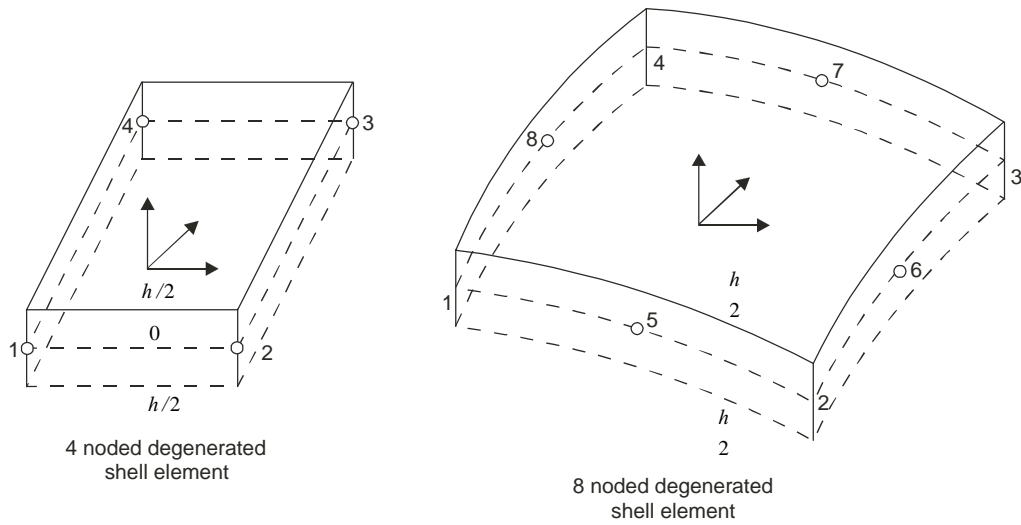


Fig. 16.6 Degenerated shell elements

- (i) The normal stresses and strain in the direction of thickness is zero i.e.  $\sigma_{zz} = 0, \epsilon_{zz} = 0$

(ii) The normal to mid surface may not remain normal after deformation, but remains straight (as used in Mindlin plate theory).

Hence the strain energy associated with stresses perpendicular to the middle surface are neglected but due to assumption (ii), the shear strain energy is to be considered.

The elements need only  $C^0$  continuity. This simplifies the analysis. However the problem of shear locking is associated; particularly when the shell is thin. This is to be overcome by using reduced integration technique as explained in using Timoshenko beam element and Mindlin plate element. The author guided M.Tech thesis [10 – 13] in this field at K.R.E.C. surathkal, reference [11] being jointly with Dr. Pratap G of NAL, Bangalore,

### 16.4 FINITE ELEMENT FORMULATION USING FOUR NODDED DEGENERATED QUADRILATERAL SHELL ELEMENT

Finite element formulation for 4 noded degenerated quadrilateral is presented below. On the same line one can extend it to the FEM formulation for 8 noded degenerated shell elements also.

Fig. 16.7 shows the typical four noded degenerated quadrilateral shell element.  $x, y, z$  are the global coordinates and  $\xi, \eta, \zeta$  are the natural coordinates.  $\xi = 0$  represents the mid surface.  $\xi = 1$  represents outer surface of the shell and  $\xi = -1$  represents the inner surface. For defining geometry as well as displacement, the shape functions for quadrilateral elements are

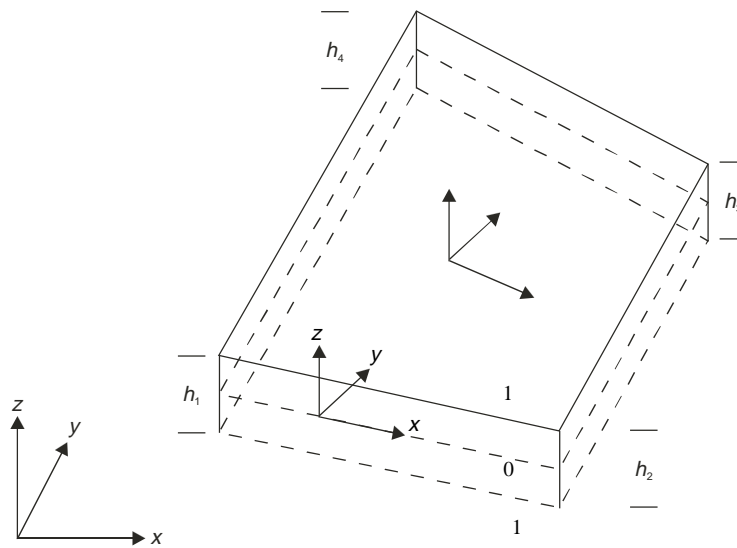


Fig. 16.7 Coordinate system for 4 noded degenerated shell element

$$N_i(\xi, \eta) = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i) \quad \dots(16.3)$$

If  $l_{3i}$ ,  $m_{3i}$  and  $n_{3i}$  are the unit normal vector at  $i$  and  $x_p, y_p, z_p$  are the global coordinates of the middle surface node  $i$ , then the global coordinates of any point in the element at distance  $\zeta$  on normal are given by



$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^4 N_i(\xi, \eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \zeta \frac{h_i}{2} \begin{Bmatrix} l_{3i} \\ m_{3i} \\ n_{3i} \end{Bmatrix} \quad \dots(16.4)$$

### Local Direction Cosines

The determination of the direction cosines is the important process in the finite element analysis of shell structures. At any point having the coordinates  $(\xi, \eta)$  on the middle surface, an orthogonal set of local coordinates  $x', y', z'$  are constructed  $e'_1, e'_2$  be the tangents to the middle surface. From vector algebra, we know that the cross product of two vectors gives a vector oriented normal to the plane given by the two vectors, we also know that unit vector is obtained by dividing the vector by scalar length. Hence  $e'_1, e'_2$  and  $e'_3$  can be found from the following relations:

$$e'_3 = \begin{Bmatrix} l_3 \\ m_3 \\ n_3 \end{Bmatrix} = \frac{\{V_1 \times V_2\}}{\{|V_1 \times V_2|\}}_{\xi, \eta} \quad \dots(16.5)$$

$$e'_3 = \begin{Bmatrix} l_3 \\ m_3 \\ n_3 \end{Bmatrix} = \frac{\{e'_3\} \times \{V_1\}_{0,0}}{\{|e_3\}_{\xi, \eta} \times \{V_1\}_{0,0}|}} \quad \dots(16.6)$$

and

$$e'_3 = \begin{Bmatrix} l_1 \\ l_2 \\ l_3 \end{Bmatrix} = \{e'_2\}_{\xi, \eta} \times \{e'_1\}_{(\xi, \eta)} \quad \dots(16.7)$$

where

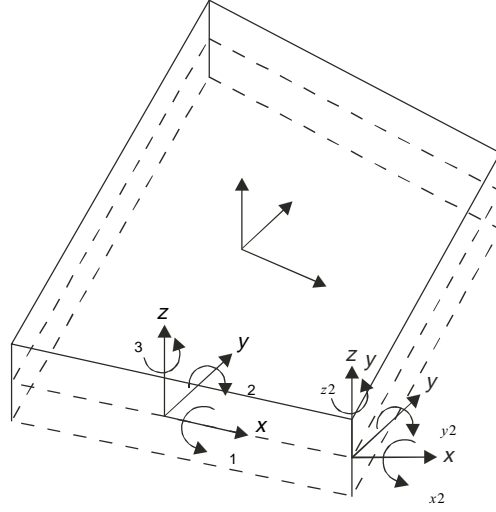
$$V_1 = \begin{Bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{Bmatrix} \quad \text{and} \quad V_2 = \begin{Bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{Bmatrix}$$

$V_1$  and  $V_2$  can be obtained from equation 16.3. Now the direction cosines of the new local coordinates  $x', y', z'$  with respect to global axes  $x, y, z$  are given by

$$[DC] = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \quad \dots(16.8)$$

## Displacement Field

Let  $u, v, w$  be displacement of a point having its local coordinate;  $u_i, v_i, w_i$  be the displacement of corresponding mid surface which is having local coordinates  $\xi, \eta$  (ref. Fig. 16.8)



**Fig. 16.8** Displacement field in 4 noded degenerated shell element

Let  $u_i^*, v_i^*, w_i^*$  be the relative displacement along  $x, y, z$  directions due to rotation of normal at node  $i$ . i.e.  $\theta_{xi}, \theta_{yi}, \theta_{zi}$  about the global axes. Then

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^4 N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \begin{Bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{Bmatrix} \quad \dots(16.9)$$

If  $\alpha'_{1i}, \alpha'_{2i}, \alpha'_{3i}$  are the normal rotations at ' $i$ ' about axes  $x', y', z'$  with the shell assumption of straight normal to middle surface remain straight even after deformation,  $\alpha'_{3i}$  becomes zero.

$$\therefore \begin{Bmatrix} u'_i \\ v'_i \\ w'_i \end{Bmatrix} = \zeta \frac{h_i}{2} \begin{Bmatrix} \alpha_{2i} \\ -\alpha_{1i} \\ 0 \end{Bmatrix} \quad \dots(16.10)$$

in which  $u'_i, v'_i, w'_i$  are displacements along axes  $x', y', z'$  respectively.

If the direction cosines between global and local axes are  $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ , then

$$u_i^* = l_{1i} u'_i + l_{2i} v'_i$$

$$v_i^* = m_{1i} u'_i + m_{2i} v'_i$$

$$w_i^* = n_{1i} u_i' + n_{2i} v_i'$$

$$\text{i.e.,} \quad \begin{Bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{Bmatrix} = \begin{bmatrix} l_{1i} & l_{2i} \\ m_{1i} & m_{2i} \\ n_{1i} & n_{2i} \end{bmatrix} \begin{Bmatrix} u_i^1 \\ v_i^1 \end{Bmatrix} \quad \dots(16.11)$$

The rotation  $\alpha_{1i}$  and  $\alpha_{2i}$  are given by the relation,

$$\begin{Bmatrix} \alpha_{1i} \\ \alpha_{2i} \end{Bmatrix} = \begin{bmatrix} l_{1i} & m_{1i} & n_{1i} \\ l_{2i} & m_{2i} & n_{2i} \end{bmatrix} \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} \quad \dots(16.12)$$

From equation 16.11, 16.10 and 16.12 we get,

$$\begin{aligned} \begin{Bmatrix} u_i^* \\ v_i^* \\ w_i^* \end{Bmatrix} &= \begin{bmatrix} l_{1i} & l_{2i} \\ m_{1i} & m_{2i} \\ n_{1i} & n_{2i} \end{bmatrix} \frac{h_i}{2} \zeta \begin{bmatrix} l_{2i} & m_{2i} & n_{2i} \\ -l_{1i} & -m_{2i} & -n_{2i} \end{bmatrix} \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} \\ &= \zeta \frac{h_i}{2} \begin{bmatrix} 0 & n_{1i} & -m_{3i} \\ -n_{3i} & 0 & l_{1i} \\ n_{1i} & -l_{3i} & 0 \end{bmatrix} \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} = \zeta \frac{h_i}{2} \begin{Bmatrix} n_{3i} \theta_{yi} - m_{3i} \theta_{zi} \\ -n_{3i} \theta_{xi} + l_{3i} \theta_{zi} \\ m_{3i} \theta_{xi} - l_{3i} \theta_{yi} \end{Bmatrix} \quad \dots(16.13) \end{aligned}$$

Substituting equation 16.13 in equation 16.9 we get

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^4 N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \frac{\zeta h_i}{2} \begin{Bmatrix} n_{3i} \theta_{yi} - m_{3i} \theta_{zi} \\ -n_{3i} \theta_{xi} - l_{3i} \theta_{zi} \\ m_{3i} \theta_{xi} - l_{3i} \theta_{yi} \end{Bmatrix} \quad \dots(16.14)$$

## Strains and Stresses

Assuming the component of strain normal to the middle surface of the shell element is zero, the strain components along the local axes of the shell element are given by,

$$\{\epsilon'\} = \begin{Bmatrix} \epsilon_{x'} \\ \epsilon_{y'} \\ \gamma_{x'y'} \\ \gamma_{x'z'} \\ \gamma_{y'z'} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u'}{\partial x'} \\ \frac{\partial v'}{\partial y'} \\ \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'} \\ \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} \\ \frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'} \end{Bmatrix} \quad \dots(16.15)$$

For the convenience, the strain matrix is split into two matrices  $[\varepsilon'_m]$  and  $[\varepsilon'_s]$  as

$$\{\varepsilon'_m\} = \begin{Bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \gamma_{x'y'} \end{Bmatrix} \quad \dots(16.16)$$

and

$$\{\varepsilon'_s\} = \begin{Bmatrix} \gamma_{x'z'} \\ \gamma_{y'z'} \end{Bmatrix} \quad \dots(16.17)$$

The stress components corresponding to these strain components are defined by the matrix,

$$\{\sigma'\} = \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \\ \tau_{x'z'} \\ \tau_{y'z'} \end{Bmatrix} \quad \dots(16.18)$$

$$= [D] \{\varepsilon^1\} = [D] [B] \{\delta\} \quad \dots(16.19)$$

where  $[D]$  is the constitute matrix of size  $5 \times 5$ , given by

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-\mu)\alpha}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\mu)\alpha}{2} \end{bmatrix} \quad \dots(16.20)$$

The factor  $\alpha$  included in the last two shear terms is taken as  $\frac{5}{6}$  and its purpose is to improve shear displacement approximation as explained in Timoshenko beam theory and Mindlin's plate theory.

The constitutive matrix  $[D]$  can be split into

$$[D] = \begin{bmatrix} D_m & 0 \\ 0 & D_s \end{bmatrix} \quad \dots(16.21)$$

where

$$[D_m] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \quad \dots(16.22)$$

and

$$[D_s] = \frac{E\alpha}{1-\mu^2} \begin{bmatrix} \frac{1-\mu}{2} & 0 \\ 0 & \frac{1-\mu}{2} \end{bmatrix} \quad \dots(16.23)$$

### Jacobian Matrix

To establish the transformation of derivatives from local to global system we need the Jacobian matrix, which is given by

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \\ \frac{\partial N_i}{\partial \zeta} \end{Bmatrix} = [J] \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{Bmatrix} \quad \dots(16.24)$$

Jacobian  $[J]$  is given by,

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad \dots(16.25)$$

### Strain Displacement Matrix

The general relation between strain and displacement is given by,

$$\{\varepsilon\} = [B] \{\delta\}$$

$[B]$  matrix is defined in terms of the displacement derivatives with respect to the local Cartesian coordinates  $x'$ ,  $y'$ ,  $z'$  by equation 16.15. Now we require two sets of transformations before the stiffness matrix can be assembled with respect to the coordinates  $\xi$ ,  $\eta$ ,  $\zeta$ .

Firstly, the derivatives with respect to the global  $x$ ,  $y$ ,  $z$  directions are obtained by using the matrix relation

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial v}{\partial \xi} & \frac{\partial w}{\partial \xi} \\ \frac{\partial u}{\partial \eta} & \frac{\partial v}{\partial \eta} & \frac{\partial w}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} & \frac{\partial v}{\partial \zeta} & \frac{\partial w}{\partial \zeta} \end{bmatrix} \quad \dots(16.26)$$

The derivatives  $\frac{\partial u}{\partial \xi}, \frac{\partial v}{\partial \eta}$  ... etc. are obtained using the equation 16.14.

Secondly the direction cosines of the local axes are to be established. Then the global derivatives of displacements  $u, v, w$  are transformed to the local derivatives of the local orthogonal displacements by a standard operation,

$$\begin{bmatrix} \frac{\partial u'}{\partial x'} & \frac{\partial v'}{\partial x'} & \frac{\partial w'}{\partial x'} \\ \frac{\partial u'}{\partial y'} & \frac{\partial v'}{\partial y'} & \frac{\partial w'}{\partial y'} \\ \frac{\partial u'}{\partial z'} & \frac{\partial v'}{\partial z'} & \frac{\partial w'}{\partial z'} \end{bmatrix} = [DC]^{-1} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} [DC] \quad \dots(16.27)$$

Making use of the equations 16.6 and 16.27 all the derivatives necessary to compute  $\{\epsilon'\}$  of equation 16.15 can be obtained. After simplifying, the equation 16.16 and 16.17, we get.

$$\{\epsilon'_m\} = \sum_{i=1}^4 \left[ [B_{1mi}] \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \left( [B_{2mi}] + \zeta [B_{3mi}] \right) \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} \right] \quad \dots(16.28)$$

and

$$\{\epsilon'_s\} = \sum_{i=1}^4 \left[ [B_{1si}] \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \left( [B_{2si}] + \zeta [B_{3si}] \right) \begin{Bmatrix} \theta_{xi} \\ \theta_{yi} \\ \theta_{zi} \end{Bmatrix} \right] \quad \dots(16.29)$$

where  $[B_{1mi}]$  and  $[B_{1si}]$  are strain-displacement matrices formed by considering only the in plane displacements  $u', v', w'$  and  $[B_{2mi}], [B_{2si}], [B_{3mi}], [B_{3si}]$  are strain displacement matrices formed by considering only relations  $\theta'_{xi}, \theta'_{yi}, \theta'_{zi}$ . By orthogonality condition, the strain displacement matrix  $[B_{2mi}]$  will be zero. The other terms are given by

$$[B_1 m_i] = \begin{bmatrix} l_1 B'(1, i) & m_1 B'(1, i) & n_1 B'(1, i) \\ l_2 B'(2, i) & m_2 B'(2, i) & n_2 B'(2, i) \\ l_1 B'(2, i) + l_2 B'(1, i) & m_1 B'(2, i) + m_2 B'(1, i) & n_1 B'(2, i) + n_2 B'(1, i) \end{bmatrix} \quad \dots(16.30)$$

in which  $B'(1, i) = \frac{\partial N_i}{\partial x} l_1 + \frac{\partial N_i}{\partial y} m_1 + \frac{\partial N_i}{\partial z} n_1$

$$B'(2, i) = \frac{\partial N_i}{\partial x} l_2 + \frac{\partial N_i}{\partial y} m_2 + \frac{\partial N_i}{\partial z} n_2 \quad \dots(16.31)$$

$$[B_{3mi}] = \frac{h_i}{2} \begin{bmatrix} B'(1, i)(m_{3i} n_1 - n_{3i} m_1) & B'(1, i)(n_{3i} l_1 - l_{3i} n_1) & B'(1, i)(l_{3i} m_1 - m_{3i} l_1) \\ B'(2, i)(m_{3i} n_2 - n_{3i} m_2) & B'(2, i)(n_{3i} l_2 - l_{3i} n_2) & B'(2, i)(l_{3i} m_2 - m_{3i} l_2) \\ B'(2, i)(m_{3i} n_1 - n_{3i} m_1) & B'(2, i)(n_{3i} l_1 - l_{3i} n_1) & B'(2, i)(l_{3i} m_1 - m_{3i} l_1) \\ + B'(1, i)(m_{3i} n_2 - n_{3i} m_2) & + B'(1, i)(n_{3i} l_2 - l_{3i} n_2) & + B'(1, i)(l_{3i} m_1 - m_{3i} l_2) \end{bmatrix} \dots(16.32)$$

$$[B_{1si}] = \begin{bmatrix} l_1 B'(3, i) + m_1 B'(3, i) + n_1 B'(3, i) + \\ l_3 B'(1, i) \quad m_3 B'(1, i) \quad n_3 B'(1, i) \\ l_2 B'(3, i) + m_2 B'(3, i) + n_2 B'(3, i) + \\ l_3 B'(2, i) \quad m_3 B'(2, i) \quad n_3 B'(2, i) \end{bmatrix}$$

$$\text{Where } B'(3, i) = \frac{\partial N_i}{\partial x} l_3 + \frac{\partial N_i}{\partial y} m_3 + \frac{\partial N_i}{\partial z} n_3 \quad \dots(16.33)$$

$$[B_{2si}] = \frac{h_i}{2} N_i B'' \begin{bmatrix} m_{3i} n_i - n_{3i} m_i & n_{3i} l_i - l_{3i} n_i & l_{3i} m_i - m_{3i} l_i \\ m_{3i} n_2 - n_{3i} m_2 & n_{3i} l_2 - l_{3i} n_2 & l_{3i} m_2 - m_{3i} l_2 \end{bmatrix} \quad \dots(16.34)$$

$$\text{where } B'' = l_3 J_{13}^* + m_3 J_{23}^* + n_3 J_{33}^* \quad \dots(16.35)$$

$$[B_{3si}] = \frac{h_i}{2} \begin{bmatrix} B'(3, i)(m_{3i} n_1 - n_{3i} m_1) & B'(3, i)(n_{3i} l_1 - l_{3i} n_1) & B'(3, i)(l_{3i} m_1 - m_{3i} l_1) \\ + B'(1, i)(m_{3i} n_3 - n_{3i} m_3) & + B'(1, i)(n_{3i} l_3 - l_{3i} n_3) & + B'(1, i)(l_{3i} m_3 - m_{3i} l_3) \\ B'(3, i)(m_{3i} n_2 - n_{3i} m_2) & B'(3, i)(n_{3i} l_2 - l_{3i} n_2) & B'(3, i)(l_{3i} m_1 - m_{3i} l_2) \\ + B'(2, i)(m_{3i} n_3 - n_{3i} m_3) & + B'(2, i)(n_{3i} l_3 - l_{3i} n_3) & + B'(2, i)(l_{3i} m_3 - m_{3i} l_3) \end{bmatrix} \dots(16.36)$$

Thus,

$$\{\varepsilon'\} = \begin{Bmatrix} \varepsilon'_m \\ \varepsilon'_s \end{Bmatrix} = \sum_{i=1}^4 \begin{bmatrix} B_{1mi} & \zeta[B_{3mi}] \\ B_{1si} & B_{2si} + \zeta[B_{3si}] \end{bmatrix} \{\delta\}_e \quad \dots(16.37)$$

$$\text{Where } \{\delta\}_e^T = [u_i \quad v_i \quad w_i \quad \theta_{xi} \quad \theta_{yi} \quad \theta_{zi}] \quad \dots(16.38)$$

## Element Stiffness Matrix

As usual stiffness matrix is given by

$$[k] = \oint [B]^T [D] [B] dV \quad \dots(16.39)$$

It is convenient to split stiffness matrix into two parts

$[k]_m$  – contribution due to bending and membrane effect

$[k]_s$  – contribution due to transverse shear

i.e.,

$$[k] = [k]_m + [k]_s$$

$$= \sum_{i=1}^4 \sum_{j=1}^4 \left[ [k_{ij}]_m + [k_{ij}]_s \right] \quad \dots(16.40)$$

where

$$[k_{ij}]_m = \oint [B_{mi}]^T [D_m] [B_{mi}] dV \quad \dots(16.41)$$

and

$$[k_{ji}]_s = \oint [B_{si}]^T [D_s] [B_{si}] dV \quad \dots(16.42)$$

In natural coordinates form, the above equations will be,

$$[k_{ij}]_m = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [B_{mi}]^T [D_m] [B_{mi}] |J| \partial\xi \partial\eta \partial\zeta \quad \dots(16.43)$$

$$[k_{ij}]_s = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 [B_{si}]^T [D_s] [B_{si}] |J| \partial\xi \partial\eta \partial\zeta \quad \dots(16.44)$$

Where  $|J|$  is the determinant of the Jacobian matrix. To be consistent with the shell assumption instead of  $|J(\xi, \eta, \zeta)|$  we can take it as  $|J(\xi, \eta, 0)|$ .

The size of each sub matrix in equation 16.43 and 16.44 is  $6 \times 6$ . Hence

$$[k]_m \text{ or } [k]_s = \begin{bmatrix} [k_{11}] & [k_{12}] & [k_{13}] & [k_{14}] \\ [k_{21}] & [k_{22}] & [k_{23}] & [k_{24}] \\ [k_{31}] & [k_{32}] & [k_{33}] & [k_{34}] \\ [k_{41}] & [k_{42}] & [k_{43}] & [k_{44}] \end{bmatrix} \quad \dots(16.45)$$

A  $2 \times 2$  Gaussian integration is used to evaluate  $[k]_m$ . To avoid shear locking effect one point Gaussian integration is used to evaluate  $[k]_s$ .

## Equivalent Load

For a shell the major loads are vertical gravity loads, uniform vertical pressure and uniform normal surface



pressures. The equivalent nodal loads for these can be found by variational principle as shown below:

(i) **Gravity Load:**

$$\{F_i\} = \int_{-1}^1 \int_{-1}^1 \rho \begin{Bmatrix} 0 \\ 0 \\ N_i \\ 0 \\ 0 \\ 0 \end{Bmatrix} |J| \partial\xi \partial\eta \quad \dots(16.46)$$

where  $\rho$  is unit weight of the material of the shell.

(ii) **Uniform Vertical Pressure:** If intensity of this load is  $P_v$  on top surface of the shell, then

$$\{F_i\} = \int_{-1}^1 \int_{-1}^1 P_v \begin{Bmatrix} 0 \\ 0 \\ N_i \\ 0.5h_i N_i m_{3i} \\ -0.5h_i N_i l_{3i} \\ 0 \end{Bmatrix} |J| |a| \partial\xi \partial\eta \quad \dots(16.47)$$

$$\text{Where } |a| = \sqrt{\left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \xi}\right)^2} \quad \dots(16.48)$$

(iii) **Uniform Normal Surface Pressure:** If  $p_n$  is the uniformly distributed normal surface pressure, applied at top surface, then due to this load equivalent nodal forces are,

$$\{F_i\} = \int_{-1}^1 \int_{-1}^1 P_n \begin{Bmatrix} N_i l_3 \\ N_i m_3 \\ N_i n_3 \\ 0.5h_i (n_3 m_{3i} - m_3 n_{3i}) \\ 0.5h_i (l_3 n_{3i} - n_3 l_{3i}) \\ 0.5h_i (m_3 l_{3i} - l_3 m_{3i}) \end{Bmatrix} |J| |\alpha| \partial\xi \partial\eta \quad \dots(16.49)$$

In any problem global stiffness matrix  $[k]$  and right hand side (load) vector  $[F]$  are assembled after calculating them element by element and then placing them in global system. The standard procedure is followed in solving the simultaneous equations after imposing the boundary conditions and in calculating the required stress resultants.

The above procedure may be extended to 8 noded degenerated shell elements also.

Using degenerated shell elements lot of studies have been carried out and satisfactory performance is reported. 4 Noded elements approximate the curved surface by straight edges. Hence to get better results we need more elements. 8 noded degenerated shell elements approximate curved surface by quadratic curve. Since most of the shell surface are having quadratic surfaces, 8 noded shell elements are used commonly.

## QUESTIONS

1. List and sketch the various flat elements used in the analysis of shells mentioning the nodal degrees of freedom in each element.
2. Explain briefly the various factors to be considered in the development of curved shell elements.
3. Explain with neat sketch the various three dimensional elements used in the analysis of shells.
4. Write short notes on the following shell elements
  - (i) Facet elements
  - (ii) Curved elements
  - (iii) Solid elements
  - (iv) Degenerated elements.

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# 17

## Non Linear Analysis

### 17.1 INTRODUCTION

The finite element formulation discussed so far was based on the following assumptions:

- (i) Stress-strain relation is linear and
- (ii) Strain displacement relation is also linear.

It resulted into stiffness equation  $[k] \{\delta\} = [F]$  in which  $[k]$  and  $\{F\}$  were independent of displacement  $\{\delta\}$ . Hence after forming the stiffness equation we could get required displacements by solving the set of linear equations only once.

In engineering we come across many problems in which stress strain and strain displacement relations are not linear. As the computer facility is increasing, the researchers are taking up rigorous analysis of structures incorporating actual stress strains curves and changes in geometry due to loading. In 1993 International Association of shells and spatial structures, conducted an international seminar on Non-linear analysis at Tokyo[1] in which the author also published a paper [2]. In all 68 papers were presented in the conference. In this chapter different types of non-linearities encountered are discussed first and the methods of solving them presented later.

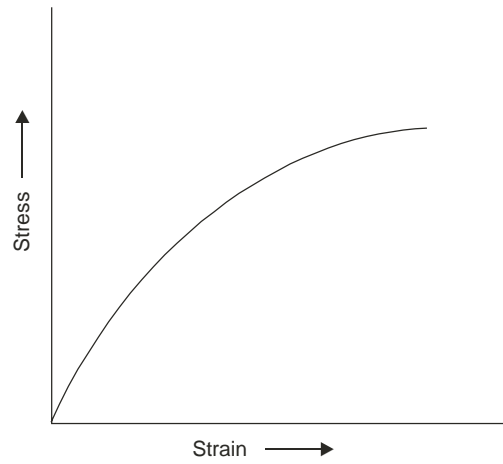
### 17.2 NON-LINEAR PROBLEMS

Various non-linear problems in finite element analysis may be grouped into the following three categories, the basis being the sources of non-linearities:

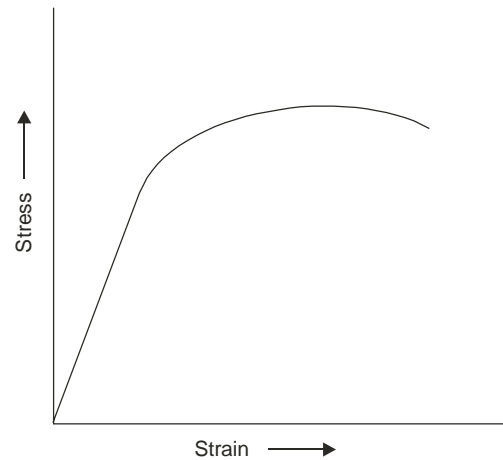
1. Material Non-Linearity Problems
2. Geometric Non-Linearity Problems and
3. Both material and Geometric Non-linearity Problems.

#### 1. Material Non-Linearity Problems

The stress-strain relation for the material i.e. the constitutive law may not be linear and may be some times time-dependent too. For example, for concrete actual stress strain curve is as shown in Fig. 17.1. Even for steel, if one is interested to study the actual behaviour of the structure beyond yielding, the stress strain relation is non-linear. Hence Young's Modulus depends upon the deformation. Apart from these basic non-linear relations, there are time dependent complex constitutive relations like plasticity, creep which make the problem non-linear. In soil mechanics problems, almost all soils need consideration of plasticity.



**Fig. 17.1** *Stress-strain relation for concrete*



**Fig. 17.2** *Stress-strain relation for steel*

## 2. Geometric Non-Linear Problems

In many problems strains – displacement relations are not linear. They need consideration of actual strain displacement relations (equation 2.5) rather than the linear strain displacement (equation 2.6). Large deflection problems like the analysis of tension structures and post buckling studies of beams, plates and shells also fall under this category.

## 3. Both Material and Geometric Non-Linearity Problems

If the large deflection, post buckling studies, etc. involve non-linear constitutive laws, then we need to study both material and geometric non-linearity effect simultaneously.

### 17.3 ANALYSIS OF MATERIAL NON-LINEAR PROBLEMS

As explained earlier material non-linearity is due to non-linear constitutive matrix  $[D]$ . The material properties are to be evaluated experimentally. For many materials non-linear stress strain curves have been obtained experimentally by conducting uniaxial tests. These results are enough to carry out non linear analysis of homogeneous materials. For any material, if non-linear analysis is to be carried out, experimental results are necessary to find non-linear relations of all the terms in general constitutive matrix. This has become a major constraint in achieving non-linear solutions for all materials. In this article only non-linear analysis for isotropic materials is explained which can be easily extended to other materials also if the non-linear material properties are known.

Consider a material with typical non-linear stress strain and corresponding load deformation curve as shown in Fig. 17.3. When a structure is loaded stresses are different at different points. Hence Young's Modulus is different from point to point. For finite element analysis a single value is assumed for an element and element to element the value may be different. Since stiffness matrix is assembled element by element, the procedure of accounting non-linearity is explained below for a element stiffness matrix. For simplicity the letter 'e' is dropped from all notations. The following three methods are available in literature for handling material non-linear problems:

1. Incremental procedure
2. Iterative procedure and
3. Mixed procedure

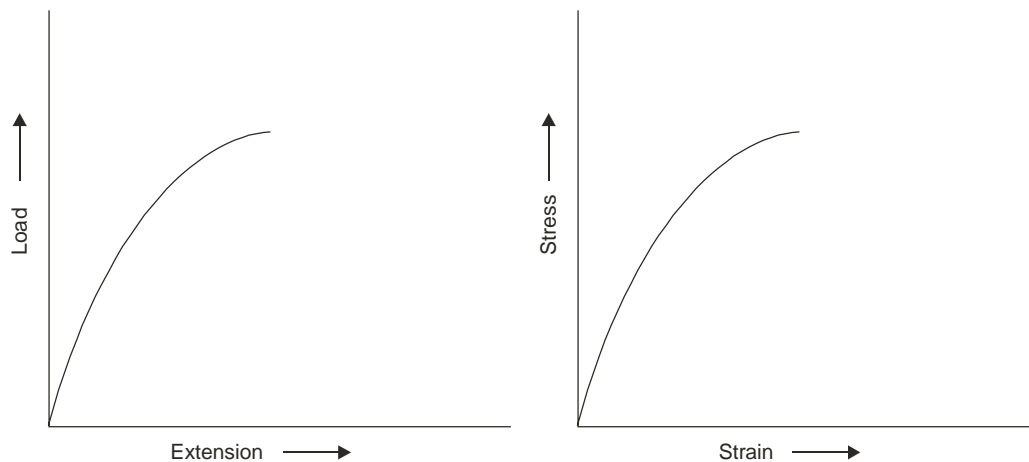


Fig. 17.3

#### 1. Incremental Procedure

In this method load is applied in a number of equal increments. It may be in 8 to 10 increments. More the increments better is the solution. For each incremented load the stiffness equation is assembled and incremented displacements are found. Fig. 17.4 shows the procedure. At the time of considering  $i$ th increment load  $\Delta F_i$ , we are at point A on the load deflection curve. At this stage we know the stress level corresponding to  $i-1$ th iteration. Corresponding to this stress level, we can pickup material properties and assemble the stiffness matrix  $[k_{i-1}]$ . Usually tangent modulus is considered. Incremented load  $\Delta F_i$ , can be written as

$$[k_{i-1}] \{\Delta\delta_i\} = \{\Delta F_i\} \quad \dots(17.1)$$

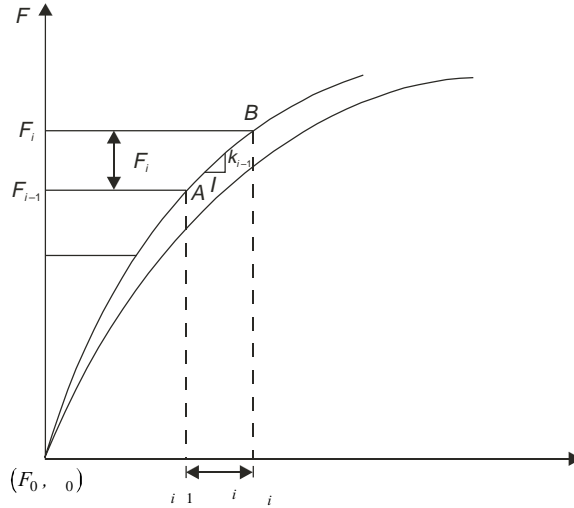


Fig. 17.4 Incremental procedure

Solving the above linear equation we get the incremented deflections  $\Delta\delta_i$ . Then we reach the next point B, at which

$$\delta_i = \delta_0 + \sum_{j=1}^i \Delta\delta_j \quad \text{and} \quad \dots(17.2)$$

$$F_i = F_0 + \sum_{j=1}^i \Delta F_j \quad \dots(17.3)$$

In the above equations  $\delta_0$  and  $F_0$  correspond to initial deflections and loads. Usually these values are zero in many problems. Thus with piecewise linearization of the material property we move from one point to another point till full load is considered.

**Midpoint Runge-Kutta Incremental Procedure**

The above basic incremental scheme may be improved to get better results by using midpoint Runge-Kutta incremental procedure. This scheme is shown in Fig. 17.5. In this, point A corresponds to the end of (i-1)th incremental stage. We are seeking solution for  $i^{\text{th}}$  incremental load  $\Delta F_i$ . In basic scheme we reach point B and select displacements corresponding to  $B'$  as displacement. In mid point Runge-Kutta method we first pick up point C which correspond to increment load  $\frac{\Delta F_i}{2}$  by basic scheme. Corresponding to the stress level at this point material properties are picked up and stiffness matrix  $k_{i-\frac{1}{2}}$  is found. Then we go back to point A and calculate incremental displacement for the load  $\Delta F_i$  using stiffness matrix  $k_{i-\frac{1}{2}}$ . This leads to point E and displacements corresponding to  $E'$ . From the scheme it may be easily seen that result obtained is better.

However it involves additional computational effort to find  $k_{i-\frac{1}{2}}$ . This scheme is better than having the load increment and using basic method which would have given point  $D$  and  $D'$  as the solution.

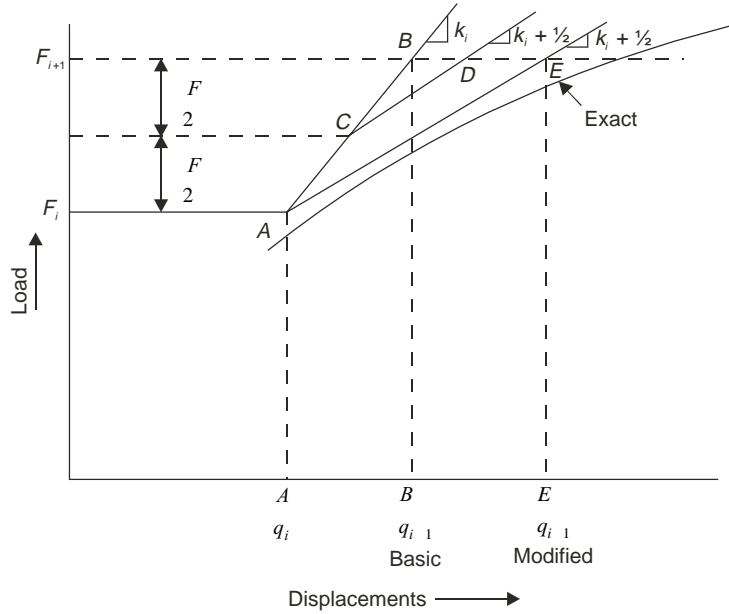


Fig. 17.5 Mid point Runge-Kutta incremental procedure

## 2. Iterative Procedure

This procedure is developed on the concept of finding load corresponding to initial strains. Hence let us see how to calculate initial load corresponding to initial stress in a material.

Let  $\epsilon_0$  be initial strain and  $\epsilon$  the final strain (ref. Fig. 17.6). Hence elastic strain is given by

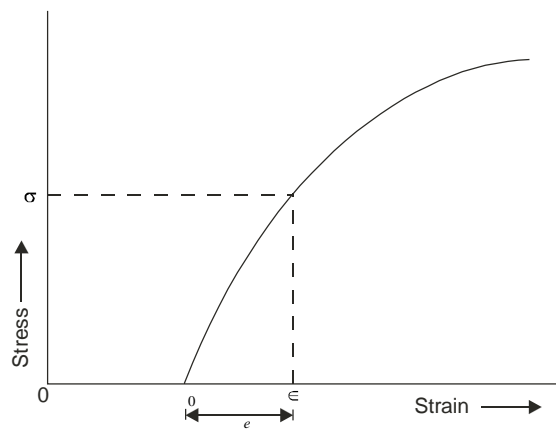


Fig. 17.6

$$\{\varepsilon^e\} = \{\varepsilon\} - \{\varepsilon_0\} \quad \dots(17.3)$$

Let  $\Delta\delta$  be the virtual displacement vector. Then

$$\Delta\varepsilon = [B]\{\Delta\delta\} \quad \dots(17.4)$$

and stresses are,

$$\{\sigma\} = [D]\{\varepsilon^e\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) \quad \dots(17.5)$$

$\therefore$  Work done by internal stresses,

$$\begin{aligned} & \oint \{\Delta\varepsilon\}^T [\sigma] dV \\ &= \oint \{\Delta\delta\}^T [B]^T [D] (\{\varepsilon\} - \{\varepsilon_0\}) dV \\ &= \{\Delta\delta\}^T \left\{ \oint [B]^T [D] \{\varepsilon\} - \oint [B]^T [D] \{\varepsilon_0\} \right\} dV \\ &= \{\Delta\delta\}^T \left\{ \oint [B]^T [D] [B] \{\delta\} - \oint [B]^T [D] \{\varepsilon_0\} \right\} dV \end{aligned} \quad \dots(17.6)$$

work done by external loads

$$= \{\Delta\delta\}^T \{F\} \quad \dots(17.7)$$

Equating work done by internal stresses to work done by external loads we get,

$$\oint [B]^T [C] [B] \{\delta\} dV - F + \oint [B]^T [D] \{\varepsilon_0\} dV$$

i.e.,

$$[k]\{\delta\} = F + F_0$$

where

$$F_0 = \oint [B]^T [D] \{\varepsilon_0\} dV \quad \dots(17.8)$$

This term may be called as load corresponding to initial strain. This concept of finding initial load corresponding to initial strain is used in iterative procedure as explained below:

Let  $F_0$  be initial load and  $\delta_0$  be initial strain. Total load to be applied be  $F$  (ref. Fig. 17.7)

Using tangent modulus initial value of material can be found and stiffness matrix  $k_0$  assembled. Let the displacement obtained for full loading  $F$  be  $\Delta\delta_1$ .  $\Delta\delta_1$  is obtained by the stiffness equation

$$[k_0]\{\Delta\delta_1\} = \{F\} \quad \dots(17.9)$$

In the scheme shown in Fig. 17.7 it corresponding to the finding  $A_1$

$$\therefore \delta_1 = \delta_0 + \Delta\delta_1 \quad \dots(17.10)$$

Using equation 17.8, we can find the equivalent load  $F_{e1}$  corresponding to displacement  $\delta_1$ , as

$$F_{e1} = \oint [B]^T [D] \{\varepsilon_1\} = \oint [B]^T [D] [B] \{\delta_1\} \quad \dots(17.11)$$



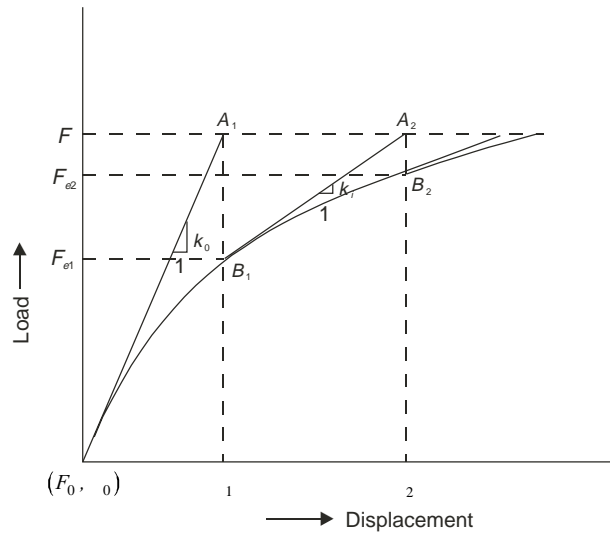


Fig. 17.7 Iterative procedure

Then we find difference between total load and  $F_{e1}$  calculated. Corresponding to  $F_{e1}$  we get the point  $B_1$ . The unbalanced force is  $F - F_{e1}$ . At point  $B$  tangent modulus may be obtained from material property and calculation may be made with  $[k_j]$  stiffness, to reach point  $A_2$  for the applied load  $F - F_{e1}$ . Then find the equivalent

load  $F_{e2}$  corresponding to  $\delta_2 = \delta_0 + \sum_{i=1}^2 \Delta\delta$ . The procedure is repeated till unbalanced force is negligible.

A modified iterative procedure by Oden [3], is shown schematically in Fig. 17.8. In this method throughout stiffness matrix  $k_0$  is used. This may take more iterations to reach final value, but surely there is considerable saving in time since the repeated calculation of stiffness matrix is avoided.

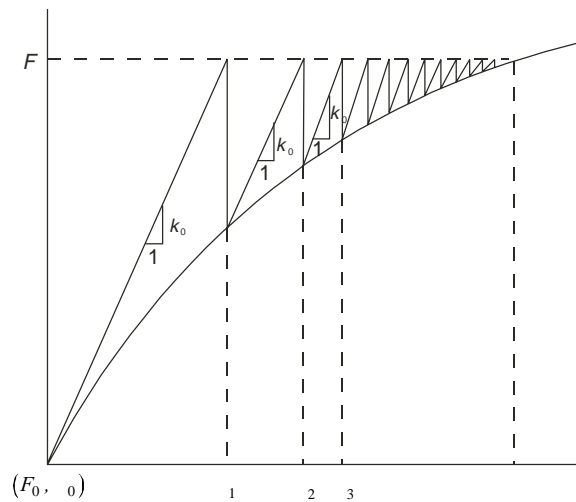


Fig. 17.8 Modified iterative procedure

### 3. Mixed Procedure

The mixture of iterative and incremental produce may be used to get better accuracy. In this method total load is divided into a number of incremental load. For each load increment, iterative procedure is employed to get the displacement. By this method accuracy is improved at the cost of computation time.

### 17.4 ANALYSIS OF GEOMETRIC NON-LINEAR PROBLEMS

So far we analysed the structures assuming that there is no appreciable change in the geometry of the structure after loading. Hence the transformation matrix used to connect local and global values remained constant throughout and the following relations could be used:

$$\{\delta_g\} = [L]^T \{\delta\} \quad \dots(17.12)$$

$$\{k_g\} = [L]^T [k_e] [L] \text{ and} \quad \dots(17.13)$$

$$\{F_g\} = [L]^T \{F_e\} \quad \dots(17.14)$$

where, subscript  $g$  refers to global values and  $l$  refers to local values.  $[L]$  is the rotation (transformation) matrix.

In cable structures, the deflections are large. Hence change of geometry with loads are not negligible. There are attempts to study the effect of changes in geometry on structures like shells also. Consider a bar element shown in Fig. 17.9. The line 1-2 shows initial position. After loading the element takes position 1' - 2'. Hence its inclination to global x-axis changes from  $\theta$  to  $\theta + \Delta\theta$ . Hence the rotation matrix  $L$  changes. Thus  $L$  is not constant throughout but it is a function of displacement. We can represent this by writing  $[L] = [L\{\delta\}]$ . Hence the stiffness matrix varies with displacements. This type of non-linear problems may be handled by incremental iterative or mixed method similar to handling material non-linearity problems.

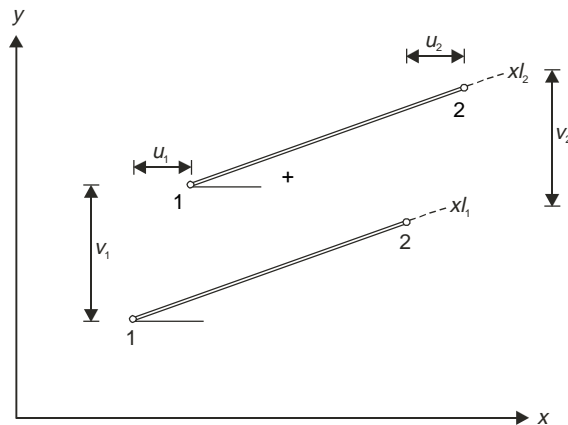


Fig. 17.9 Geometric non-linearity

## 1. Incremental Procedure

If the load is incremented from  $\{F_g\}$  to  $\{F_g + \Delta F_g\}$ , the equation 17.14 can be written as

$$\begin{aligned}\{F_g\} + \{\nabla F_g\} &= ([L] + [\Delta L])^T (\{F_l\} + \{\Delta F_l\}) \\ &= [L]^T \{F_l\} + [L]^T \{\Delta F_l\} + [\Delta L]^T \{F_l\} + [\Delta L]^T \{\Delta F_l\}\end{aligned}$$

Neglecting the small quantity of higher order, the above equation will be

$$[F_g] + \{\Delta F_g\} = [L]^T \{F_l\} + [L]^T \{\Delta F_l\} + [\Delta L]^T \{F_l\}$$

since  $[F_g] = [L]^T \{F_l\}$ , the above equation reduces to

$$\{\Delta F_g\} = [L]^T \{\Delta F_l\} + [\Delta L]^T \{F_l\} \quad \dots(17.15)$$

From the stiffness equation, we know

$$[L]^T \{\Delta F_l\} = [k_g] \{\Delta \delta_g\}$$

∴ Equation 17.15 reduces to

$$\{\Delta F_g\} = [k_g] \{\Delta \delta_g\} + [\Delta L]^T \{F_l\} \quad \dots(17.16)$$

In the above equation, the term  $[\Delta L]^T \{F_l\}$  represents the change in stiffness equation due to change in the geometry. Let us consider this term further

$$[\Delta L]^T \{F_l\} = \sum_{i=1}^n \{F_l\}_i \{\Delta L\}_i \quad \dots(17.17)$$

but

$$\begin{aligned}\Delta L_i &= \frac{\partial \{L\}_i}{\partial (\delta_g)} (\Delta \delta_g) = \frac{\partial \{L\}_i}{\partial (\delta_g)} \delta_g \\ &= \left[ \frac{\partial \{L\}_i}{\partial (\delta_g)_1} \frac{\partial \{L\}_i}{\partial (\delta_g)_2} \dots \frac{\partial \{L\}_i}{\partial (\delta_g)_n} \right] \quad \dots(17.18)\end{aligned}$$

$$\therefore [\Delta L]^T \{F_l\} = \sum_{i=1}^n (F_l)_i [G_i] \{\Delta \delta_g\}$$

Substituting it in equation 17.16, we get

$$\{\Delta F_g\} = [k_g] \{\Delta \delta_g\} + \sum_{i=1}^n (F_l)_i [G_i] \{\Delta \delta_g\}$$

$$\begin{aligned}
&= k_g \Delta\delta_g + \left( \sum_{i=1}^n (F_e)_i [G_i] \{ \Delta\delta_g \} \right) \\
&= [k_g + k_G] \{ \Delta\delta_g \} \quad \dots(17.19)
\end{aligned}$$

where

$$k_G = \sum_{i=1}^n (F_e)_i [G_i]$$

Since we normally talk about global values in the final analysis, for simplicity we can drop subscript 'g' and write equation 17.19 as

$$\{ \Delta F \} = ([k] + [k_G]) \{ \Delta\delta \} \quad \dots(17.20)$$

From  $i$ th stage, if we want to proceed to  $i + 1$ th stage, the equation 17.20 is

$$([k] + [k_G])_i \{ \Delta\delta_{i+1} \} = \Delta F_{i+1} \quad \dots(17.21)$$

Thus to get additional deflections due to geometric non-linearity we need stiffness matrices  $[k]$  and  $[k_G]$  at the beginning of an increment. Hence evaluation effort required is more. However it may be noted that to find  $[k_G]$  there is no need to evaluate the stiffness matrix afresh. We need only modifications to transformation matrix  $[L]$ .

## 2. Iterative Procedure

This method is straight forward. For the initial geometry, the transformation matrix is assembled. Using this we find,

$$[k_0] = [L_0]^T [k_l] [L_0] \quad \dots(17.22)$$

$$[F_0] = [L_0]^T [F_l] \quad \dots(17.23)$$

Then after solving stiffness equation,

$$[k_0] \{ \delta_1 \} = [F_0] \quad \dots(17.24)$$

we get  $\delta$  values of 1st stage. Using these displacements, the new coordinates of the nodes are determined. For the new geometry the above process is repeated to get displacements  $\delta_2$  of second stage. The process is repeated until the displacements no longer change significantly. Though the process is simple, it has limitations of the iterative techniques i.e. convergence is slow. It is time consuming.

## 3. Mixed Procedure

Instead of applying total load in each iteration, if we apply load in the increments and for every incremented load carryout the iterative procedure, better results may be obtained. It involves lot of computational effort.

## 17.5 ANALYSIS OF BOTH MATERIAL AND GEOMETRIC NON-LINEAR PROBLEMS

Researchers have attempted some problems, treating them as both material and geometric non-linear problems. This gives real behaviour of structures under load. Oden [3] has given a generalized mathematical basis for incremental and iterative procedure and has given an exhaustive list of references on non-linear analysis

### QUESTIONS

1. Explain the different types of non-linearities encountered in structural analysis
2. Explain incremental procedure to handle material non-linear problems.
3. Explain mid point Runge-Kutta incremental scheme and discuss its advantages and disadvantages over the incremental procedure.
4. Explain iterative procedure and modified iterative procedure for the analysis of material non-linearity problems.
5. Explain the iterative procedure of handling geometric non-linearity problems in structural mechanics.

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2. Bhavikatti S.S. and Siddamal J.U., Non-linear Analysis of Reinforced Concrete Shells Using 8-Noded Degenerated Shell Element, Proceedings SEIKEN – IASS Symposium on Shells and Spatial Structures, Tokyo, Japan Oct 20–22, 1993 pp. 315–322.
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# 18

## Standard Packages and Their Features

### 18.1 INTRODUCTION

Finite element analysis involves lot of numerical calculations. Hence it is not a suitable method for hand calculations. The method is ideally suited for computer applications and has developed along with developments in computer technology. The development of finite element programs is time consuming. Many universities and software companies have spent several man hours to develop general purpose finite element analysis packages. The packages are continuously being updated by incorporating more and more elements and adding new modules like non-linear analysis, dynamic analysis, optimization techniques. The cost of development of packages is very high. Hence their cost is high. In 1996, the cost of NASTRAN package was as high as Rs.4.5 lakhs. As the number of users are increasing, the cost of these packages is continuously coming down. Now the same package with additional features are available at around Rs.1.50 lakhs. In this chapter the list of standard finite element packages is presented and then the structure of finite element package explained. The need for pre and post processors is pointed out and a features of these processors is presented. Finally desirable features of general purpose packages is listed.

### 18.2 COMMERCIALLY AVAILABLE STANDARD PACKAGES

The following are some of the general purpose finite element analysis packages now available in the market.

1. Structural Design Language (Integrated Civil Engineering System, M.I.T, USA) STRUDEL
2. NASA Structural Analysis (U.S. National Aeronautical and Space Administration) NASTRAN. Now this has split into five separate groups and each group is continuously improving the package. CSA NASTRAN and MSC NASTRAN are popular in India.
3. Non-linear Incremental Structural Analysis (developed by E Ramm, Institute of Biostatic University of Stuttgart, W Germany) NISA.
4. Engineering Analysis System (Swanson Analysis System Inc.) ANSYS.
5. Structural Analysis Program (developed by EL. Wilson, University of California, USA) SAP.

Continuous up-gradations of all the packages is going on in the form of

- (a) Increasing the variety of elements
- (b) Provision for using different types of elements at a time
- (c) Addition of dynamic analysis

- (d) Addition of non-linear analysis
- (e) Addition of optimizations
- (f) Developing more and more users friendly programs (pre and post processors) to handle input data and output informations.

### 18.3 STRUCTURE OF A FINITE ELEMENT ANALYSIS PROGRAM

The program starts with lot of INPUT information to define the problem. Then the data given is processed and required result is printed out. Finite Element Analysis Program using a particular element is explained below and the flow chart is presented.

#### Input Information

The various input required to define a problem may be grouped into the following:

1. **Geometric Data:**
  - (a) General information like total number of elements, total number of nodal points, type of element (number of nodes, degrees of freedom for each node), are to be supplied.
  - (b) Coordinates of each node to be supplied or generated.
  - (c) For each element nodal connectivity is to be supplied.
2. **Load Data** is to be given. It consists of total types of loads and for each load its magnitude, point of application (coordinate or line or surface of application) etc.
3. **Material Properties** to be supplied consists of total number of materials used, for each material required material property like Young's Moduli, Poisson's ratio etc. and material number of each element.
4. Next input required is about total number of **boundary conditions** and for each boundary condition specified displacements.
5. Number of **Gaussian points** to be used and for each Gaussian point weight function and coordinates in local system should be supplied.

#### Processing

Data supplied is to be processed to complete the analysis. It starts with initializing global stiffness matrix and load matrix. Then element loop is entered to assemble element stiffness matrix. Element stiffness matrix is initialized. When numerical integration is used, there should be Gauss point loop inside the element. Contribution of each Gauss point to stiffness matrix is to be found. It needs entering shape function subroutine to get shape function, and shape function derivatives. Assemble Jacobian matrix Jacobian inverse and determinants are found.  $[B]$  matrix is assembled and the then contribution of the Gauss point i.e.  $[B]^T [D][B] J |d\xi d\eta$  type terms are found and added to the existing values of element stiffness matrix. When Gauss loop is completed we get element stiffness matrix. With the help of nodal connectivity details, the position of each value of element stiffness matrix in global matrix is identified and added to existing value. When element loop is completed global stiffness matrix is available.

Using load details nodal loads are to be assembled. Usually first nodal values are initialized. Then one by one load case is taken up to get final load vector  $\{F\}$ .

The next step in processing is to impose the boundary conditions. Penalty method is used to impose the boundary conditions.

Now, the stiffness equations are ready. Standard solution package is used to solve the equation to get nodal variables. Using these nodal variables at Gauss point of each element, strains and stresses are found. If principal stresses and strains are required they may be assembled. Any other required stress resultant may be assembled.

## Output

Various values calculated after the analysis, may be printed out.

The structure of the finite element analysis program is shown by flow chart in Fig. 18.1. The flow chart is for a single element. Suitable changes are to be made to incorporate various elements and for the calculations other than static linear analysis.

### 18.4 PRE AND POST PROCESSORS

A practical problem to be solved by finite element analysis needs hundreds of elements and nodal degree of freedom may exceed over 1000. Hence the preparation of data, such as numbering nodes, generating nodal coordinates, supplying nodal connectivity, load and material information, specifying boundary conditions, is too lengthy process. It needs considerable efforts and is laborious. If handled manually by looking at INPUT statements, possibility of errors creeping in are discouraging factors. Hence many softwares have been developed to handle the data graphically and display it for the verification. Such softwares developed exclusively to assist in generating finite element analysis INPUT are known as **Pre Processors** of FEA packages. They use Graphic User Interfaces (GUI) for the following:

1. Generate finite element mesh.
2. Number the elements automatically.
3. Number the nodes automatically so as to keep band width least.
4. Generate nodal coordinates, using the values supplied at salient points.
5. Develop nodal connectivity details.
6. Display the standard tables to specify various loads and load informations.
7. Display the tables to specify material numbers and material properties, boundary conditions and other details.

The provisions are made to select only part of the structure to check the INPUT data and alter if necessary. HELP commands are available for the users. PRINT commands are also available to print out data for documentation.

Display III / IV, FEMAP and PATRON are some of the commercially available preprocessor. Upgraded versions are coming up regularly to make preprocessor as user friendly as possible.

The preprocessors develop data file required by main FEA program, which is known as Processors. The processors use the data file, analyses and stores the final results.

The output of a FEA consist of nodal displacements, the calculated values of stresses, strains, moments etc. in each element at all Gauss points. The output values calculated are in all global coordinated directions and also in principal directions. It is time consuming to go through entire output file before picking up the required one. Hence the user friendly, Graphic users Interface software have been developed which may be named as **Post Processors**. Normally pre and post processors are clubbed and commercial packages are



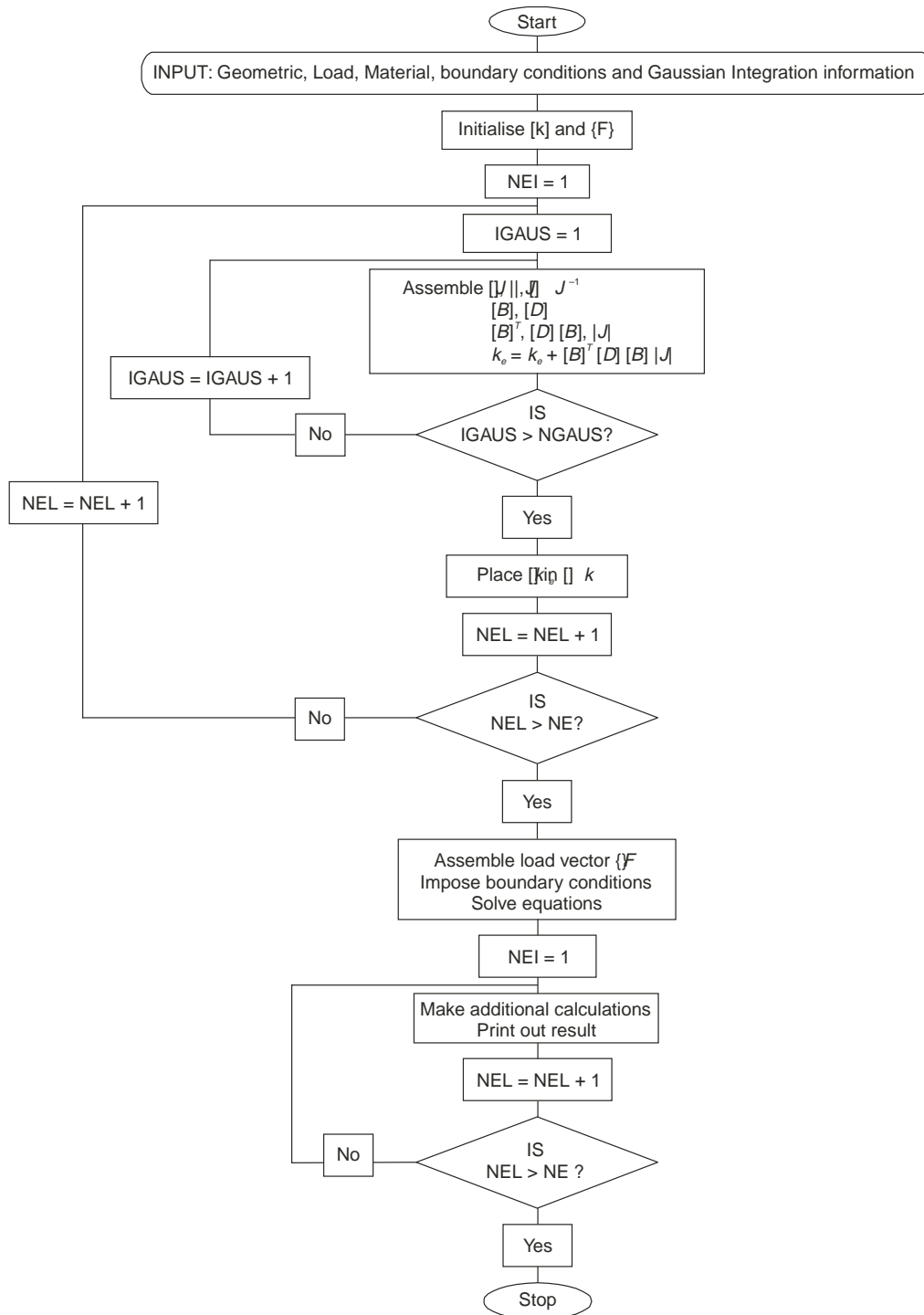


Fig. 18.1 Flow chart

developed. With the help of post processor, making use of various windows users can:

1. Pick up absolute maximum stress resultants
2. Plot the graph of stresses along a selected cross section
3. Plot stress resultant or strain contours.
4. Pick up values for specified element or point.

There is competition among software developers to make pre and post processors as users friendly as possible.

## 18.5 DESIRABLE FEATURES OF FEA PACKAGES

The desirable features of a good FEA package are listed below:

1. The package should be supported by an excellent pre and post processor.
2. The element library should incorporate all types of elements like 1D, 2D, 3D elements, plate elements, shell elements.
3. The package should have a capability to handle different types of loads like concentrated, uniformly distributed, uniformly varying, internal and external pressures, centrifugal forces, moving loads, temperatures stresses.
4. It should be possible to impose boundary conditions of all types the user may encounter in his problems.
5. The limitation on degrees of freedom that can be handled on a specified hardware should be as high as possible. This is possible if the processor makes use of banded nature and symmetry in stiffness matrix. Another point where processor can be made efficient is by avoiding repeated calculations of strains  $\left(\oint BdV\right)$  and stresses  $\left(\oint DBdV\right)$  which are assembled while assembling the stiffness matrix. In efficient program these values are written on hard disc element by element while assembling stiffness matrix and read while assembling strains and stresses.
6. The package should include various features like dynamic analysis, buckling analysis and non-linear analysis.
7. If design is also incorporated in the package, there should be choice for the user to select required codal provisions like Indian standard practice, British codes, American codes etc.
8. User may even expect an excellent optimizer in the package.
9. The package should be economical and there should be choice for user to pick up the package to suit his requirement and budget.
10. For large users multi users network version licenses should be available.

NISA has brought out NISA / CIVIL which is users friendly for civil engineers and gives facility for the analysis and design of various R.C.C and steel designs. ANSYS has workshop supplement which specializes on analysis and design of mechanical components. All established software developers are coming out with latest version of finite element packages with more features and more user friendly versions to suit various users.

### QUESTIONS

1. Briefly explain the structure of a finite element analysis program.
2. Name some of the standard FEA packages.

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3. Write short note on pre and post processors.
4. List the desirable features of FEA packages.
5. Write short note on commercially available FEA packages.