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Fundamentals of Finite Element Methods: Variational methods for the Laplace and Poisson equations

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Outline

- Introduction to FEM
- FEM Process
- Fundamentals - FEM
- Interpolation Functions
- Variational Method
- Rayleigh-Ritz Method - Example
- Method of Weighted Residuals
- Galerkin Method
- Galerkin FEM
- Assembly of Element Equations
- Laplace & Poisson Equations
- Discretization of Poisson Equation – Variational Method
- Numerical Integration
- Summary
- References

Introduction – Finite Element Method (FEM)

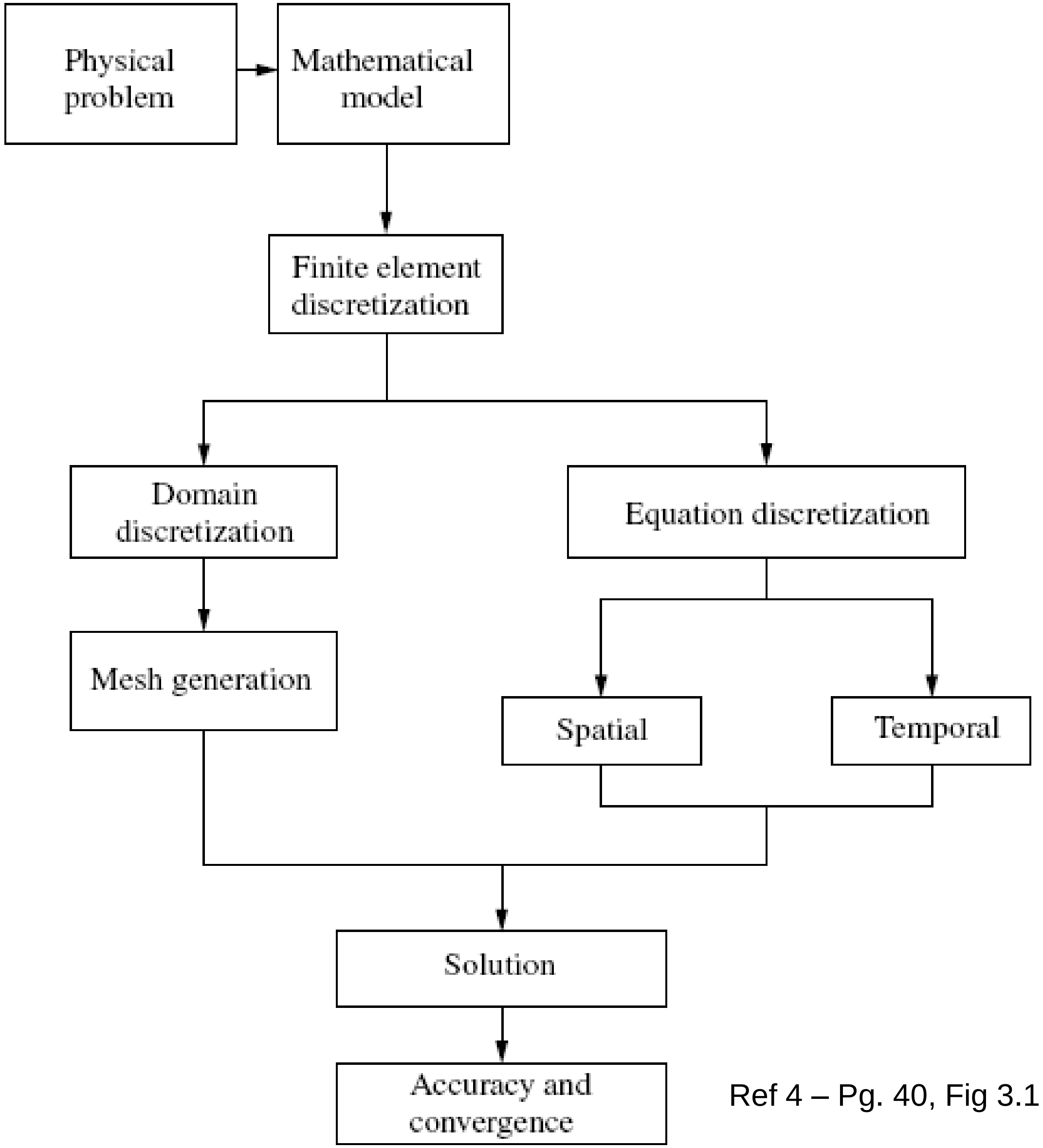
- Numerical solution of complex problems – in Fluid Dynamics, Structural Mechanics
- General discretization procedure of continuum problems posed by mathematically defined statements – differential equations
- Difference in approach between Mathematician & Engineer
- Mathematical approaches –
  1) Finite Differences
  2) Method of Weighted Residuals
  3) Variational Formulations
- Engineering approaches –

  Create analogy between finite portions of a continuum domain and real discrete elements

  – Example: Replace finite elements in an elastic continuum domain by simple elastic bars or equivalent properties

FEM Process

- Finite Element Process based on following two conditions–
  1) Finite number of parameters determine behavior of finite number of elements that completely make up continuum domain
  2) Solution of the complete system is equivalent to the assembly of the individual elements
- The process of solving governing equations using FEM –
  1) Define problem in terms of governing equations (differential equations)
  2) Choose type and order of finite elements and discretize domain
  3) Define Mesh for the problem / Form element equations
  4) Assemble element arrays
  5) Solve resulting set of linear algebraic equations for unknown
  6) Output results for nodal/element variables



Ref 4 – Pg. 40, Fig 3.1

Fundamentals – FEM

$$\mathbf{A}(\mathbf{u}) = \begin{Bmatrix} A_1(\mathbf{u}) \\ A_2(\mathbf{u}) \\ \vdots \end{Bmatrix} = \mathbf{0}$$

Differential Equation

$$\mathbf{B}(\mathbf{u}) = \begin{Bmatrix} B_1(\mathbf{u}) \\ B_2(\mathbf{u}) \\ \vdots \end{Bmatrix} = \mathbf{0}$$

Boundary Conditions

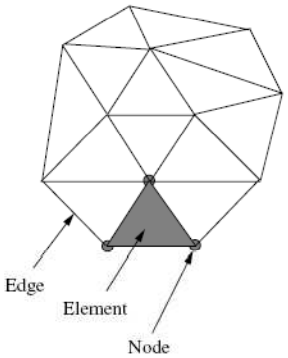
Being an approximate process – will seek solution of form →
where: N_i - shape functions or trial functions
a_i – unknowns or parameters to be obtained to ensure "good fit"

$$\mathbf{u} \approx \hat{\mathbf{u}} = \sum_{i=1}^{n} \mathbf{N}_i \mathbf{a}_i = \mathbf{N}\mathbf{a}$$

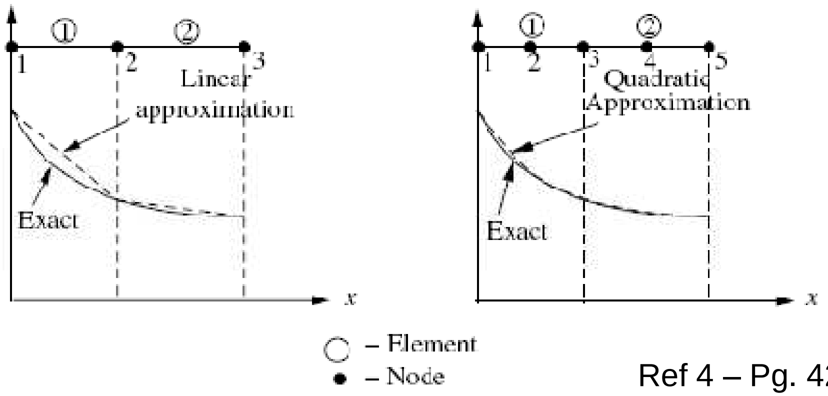
Defined: $N_i = 0$ at boundary of domain

Discretizing Domain

- Domain is broken up into number of non-overlapping elements
- Functions used to represent nature of solution in elements – trial / shape / basis / interpolation functions
- These serve to form a relation between the differential equation and elements of domain



Ref 4 – Pg. 40, Fig 3.2



Ref 4 – Pg. 42, Fig 3.3

Interpolation Functions – Introduction

- N_m are independent trial functions

- Properties:
  1. N_m is chosen such that u" → u as m → ∞ (Completeness requirement)
  2. N_m depends only on geometry and no. of nodes

$$\mathbf{u} \approx \hat{\mathbf{u}} = \sum_{i=1}^{n} \mathbf{N}_i \mathbf{a}_i = \mathbf{N}\mathbf{a}$$

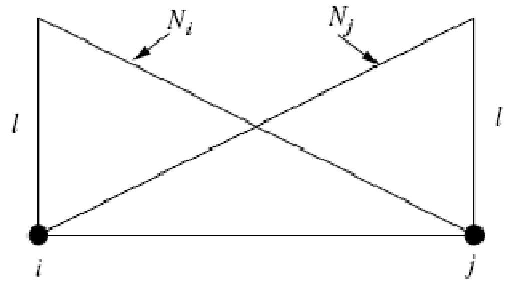
Interpolation Functions – 1-d Linear

- Piecewise defined trial functions: one dimensional linear

  Varies linearly across each element

- Properties:
  1. N_m = 1 at node m
  2. N_m = 0 at all other nodes
  3. $\sum N_m^e$ = 1 for element e
  4. Number of nodes = number of functions
  5. If N_m is a polynomial of order n-1, then

$$N_k^e = \prod_{i=1}^{n} \frac{x - x_i}{x_k - x_i}$$, node k, element e, $k \neq i$



Shape function values within an element

Ref 4 – Pg. 44, Fig 3.4

Interpolation Functions – 2 – D Mesh: Triangular Element

- Simplex element
- Simplest geometric shape used to approximate an irregular surface

$$T(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$T_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i$$

$$T_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j$$

$$T_k = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k$$

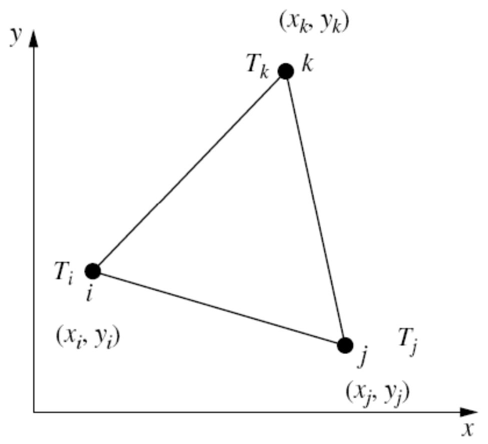


Figure 3.7 A linear triangular element

Ref 4 – Pg. 49, Fig 3.7

Solving for α_1, α_2, α_3 in terms of nodal coordinates, nodal values & rewriting expression for T(x,y)

Interpolation Functions – 2 – D Mesh: Triangular Element

→→ $$T = N_i T_i + N_j T_j + N_k T_k = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \begin{Bmatrix} T_i \\ T_j \\ T_k \end{Bmatrix}$$

where →

$$a_i = x_j y_k - x_k y_j; \quad b_i = y_j - y_k; \quad c_i = x_k - x_j$$
$$a_j = x_k y_i - x_i y_k; \quad b_j = y_k - y_i; \quad c_j = x_i - x_k$$
$$a_k = x_i y_j - x_j y_i; \quad b_k = y_i - y_j; \quad c_k = x_j - x_i$$

and interpolation functions →

$$N_i = \frac{1}{2A}(a_i + b_i x + c_i y)$$
$$N_j = \frac{1}{2A}(a_j + b_j x + c_j y)$$
$$N_k = \frac{1}{2A}(a_k + b_k x + c_k y)$$

A – Area of triangle

Interpolation Functions – 2-D Mesh: Quadrilateral

- Similarly – quadratic triangular elements are also possible for better accuracy

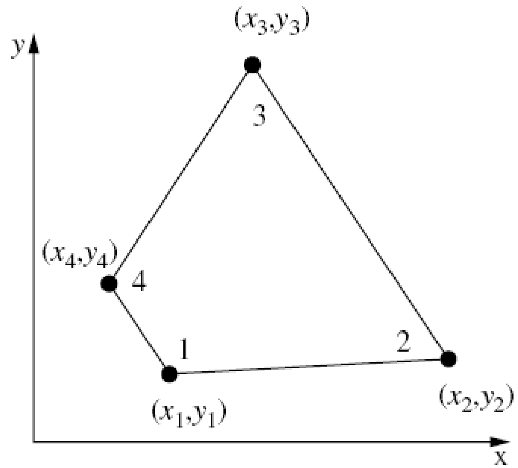
$$T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy$$

- Higher orders are also possible

2 – D Mesh: Quadrilateral Element

$$T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

  - In simplest form → rectangular element



Ref 4 – Pg. 58, Fig 3.13

Element Equations

$$[\mathbf{K}]\{\mathbf{T}\} = \{\mathbf{f}\}$$

where:
[K] – stiffness matrix;
{ T } – Vector of unknowns (like temperature);
{ f } – forcing or loading vector

$$[\mathbf{K}]_e = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{\mathbf{f}\}_e = \begin{Bmatrix} Q_i \\ Q_j \end{Bmatrix}$$

Example (Heat Transfer) –
Consider a single element on a one dimensional domain with nodes i, j.

Q_i – Heat flux through node i;
e – element;
l – length of element;
k – thermal conductivity;
A – Area.
Now, $\{ T \}_e$ is the unknown temperature at either node

Variational Method – Principles

$$\Pi = \int_{\Omega} F\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x}, \ldots\right) d\Omega + \int_{\Gamma} E\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial x}, \ldots\right) d\Gamma$$

u is solution to continuum problem
F,E are differential operators
Π is variational integral

Now, u is exact solution if for any arbitrary δu → $\delta\Pi = 0$

ie. if variational integral is made "stationary"

Now, the approximate solution can be found by substituting trial function expansion → $\mathbf{u} \approx \hat{\mathbf{u}} = \sum_{1}^{n} \mathbf{N}_i \mathbf{a}_i$

Variational Method – Principles

$$\delta\Pi = \frac{\partial\Pi}{\partial\mathbf{a}_1}\delta\mathbf{a}_1 + \frac{\partial\Pi}{\partial\mathbf{a}_2}\delta\mathbf{a}_2 + \cdots + \frac{\partial\Pi}{\partial\mathbf{a}_n}\delta\mathbf{a}_n = 0$$

Since above holds true for any δa →

$$\frac{\partial\Pi}{\partial\mathbf{a}} = \begin{Bmatrix} \dfrac{\partial\Pi}{\partial\mathbf{a}_1} \\ \vdots \\ \dfrac{\partial\Pi}{\partial\mathbf{a}_n} \end{Bmatrix} = \mathbf{0}$$

Parameters a_i are thus found from above equations

Note:
The presence of symmetric coefficient matrices for above equations is one of the primary merits of this approach

Natural & Contrived – Variational Principles

- Natural:
  - Variational forms which arise from physical aspects of problem itself
  - Example – Min. potential energy → equilibrium in mechanical systems

However, not all continuum problems are governed by differential equations where variational forms arise "naturally" from physical aspects of problem

- Contrived:
  1. Lagrange Multipliers: extending number of unknowns by addition of variables
  2. Least square problems: Procedures imposing higher degree of continuity requirements

Rayleigh – Ritz Method (Variational) – Process

- Method depends on theorem from theory of the calculus of variations –

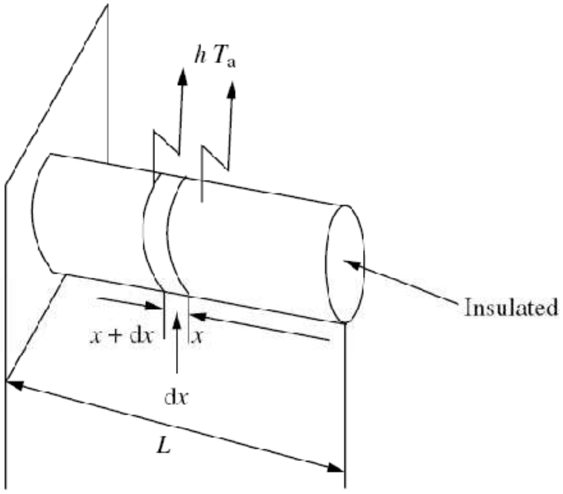
The function *T (x)* that extremises the variational integral corresponding to the governing differential equation (called Euler or Euler–Lagrange equation) is the solution of the original governing differential equation and boundary conditions'

- Process –

1) Derive Variational Integral from governing differential equation
2) Vary the solution function until Variational Integral is made stationary with respect to all unknown parameters (a_i) in the approximation

Rayleigh – Ritz Method (Variational) – Example

Example –



Ref 4 – Pg. 75, Fig 3.24

$$\frac{d^2\theta}{d\zeta^2} - \mu^2\theta = 0 \qquad \frac{d\theta(0)}{d\zeta} = 0 \quad \text{and} \quad \theta(1) = \theta_b$$

Using the governing equation as the Euler-Lagrange equation →
- where I is the Variational Integral

$$\delta I = \int_0^1 \left(\frac{d^2\theta}{d\zeta^2} - \mu^2\theta \right) \delta\theta d\zeta = 0$$

Rayleigh – Ritz Method (Variational) – Example

Integrating by parts →

$$\left[\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\delta\theta\right]_0^1 - \int_0^1 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)\frac{\mathrm{d}}{\mathrm{d}\zeta}(\delta\theta)\mathrm{d}\zeta - \mu^2\int_0^1 \theta\delta\theta\mathrm{d}\zeta = 0$$

And using the relation →
And applying boundary conditions

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}(\delta\theta) = \delta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)$$

Variational Integral is =

$$I = \int_0^1 \frac{1}{2}\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)^2 + \mu^2\theta^2\right]\mathrm{d}\zeta$$

Note: order of derivative in integrand

Rayleigh – Ritz Method (Variational) – Example

Substitute the approximation into integral and forcing I to be stationary with respect to unknown parameters yields set of linear equations to be solved for the unknown parameters

$$\mathbf{u} \approx \hat{\mathbf{u}} = \sum_{i=1}^{n} \mathbf{N}_i \mathbf{a}_i = \mathbf{N}\mathbf{a}$$

Let I = Π

Now, checking for stationarity of the variational integral by differentiating with unknown parameters →

$$\frac{\partial \Pi}{\partial \mathbf{a}} = \left\{ \begin{array}{c} \frac{\partial \Pi}{\partial \mathbf{a}_1} \\ \vdots \\ \frac{\partial \Pi}{\partial \mathbf{a}_n} \end{array} \right\} = \mathbf{0}$$

Set of linear algebraic equations → K a = f

Variational Formulation

- One can observe from the variational integral that despite the governing equation being a second derivative equation, the integrand is only of the first derivative
- In cases where the second derivative tends to infinity or does not exist, this formulation is very useful as it does not require the second derivative

Example – Change in rate of change of temperature where two different materials meet might lead to such a case

- Hence, the **variational formulation** of a problem is often called the **Weak formulation**
- However, this formulation may not be possible for all differential equations
- An alternative approach is - Method of Weighted Residuals

Method of Weighted Residuals (MWR)

- Residual = A(T") – A(T) where: T – exact; T" – approximate; A – Governing Equation
- Since, A(T) = 0 → Residual (R) = A(T")
- Method of weighted residuals requires that a_i be found by satisfying following equation -

$$\int_{\Omega} w_i(x) R \, dx = 0 \quad \text{with} \quad i = 1, 2, \ldots, n \qquad \rightarrow \qquad K\,a = f$$

where: $w_i(x)$ are n arbitrary weighting functions

Essentially, the average Residual is minimized → Hence, minimizing error in approximation. R → 0 when n → ∞

- weighting functions can take any values
- However, depending on the weighting functions certain special cases are defined and commonly used –

Method of Weighted Residuals – Special Cases

1) Point Collocation: Dirac Delta Function

$$w_i = \delta(x - x_i) \qquad \int_\Omega R\delta(x - x_i)\mathrm{d}x = R_{x=x_i} = 0$$

Equivalent to making the residual R equal to zero at a number of chosen points

2) Sub-Domain:

$$w_i = 1 \qquad \int_{\Omega_i} R\,\mathrm{d}x = 0 \quad \text{with} \quad i = 1, 2, \ldots, n$$

Integrated error over N sub-domains should each be zero

Method of Weighted Residuals – Special Cases

3) Galerkin:

$$w_i(x) = N_i(x) \qquad \int_\Omega N_i(x) R \,\mathrm{d}x = 0 \quad \text{with} \quad i = 1, 2, \ldots, n$$

Advantages include –

i. Better accuracy in many cases
ii. Coefficient Matrix is symmetric which makes computations easier

4) Least Squares: Attempt to minimize sum of squares of residual at each point in domain

$$w_i = \partial R / \partial a_i \qquad \int_\Omega \frac{\partial R}{\partial a_i} R \,\mathrm{d}x = 0 \quad \text{with} \quad i = 1, 2, \ldots, n$$

Galerkin Method (MWR)

$$w_i(x) = N_i(x) \qquad \int_\Omega N_i(x) R \, \mathrm{d}x = 0 \quad \text{with} \quad i = 1, 2, \ldots, n$$

- One of the most important methods in using Finite Element Analysis.
- Here, the weighting function is the same as the trial function at each of the elements/nodes
- Interestingly, the solution obtained via this method is exactly the same as that obtained by the variational method
- It can further be shown that, if a physical problem has a natural variational principle attached to it or in other words if a governing equation can be written as a variational integral, then the Galerkin and Variational methods are identical and thus provide same solution

Weak formulation (MWR)

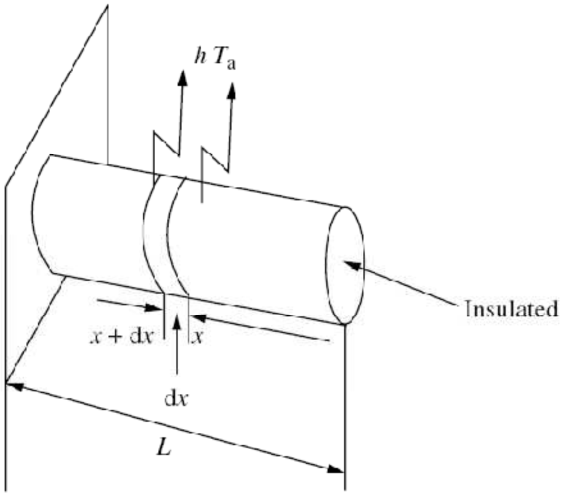
Now consider the integral of this form → $$\int_{\Omega} \mathbf{w}_j^{\mathrm{T}} \mathbf{A}(\mathbf{Na})\, \mathrm{d}\Omega$$

Here, the weighted residual integrated over domain

Now, on integration by parts → $$\int_{\Omega} \mathbf{C}(\mathbf{w}_j)^{\mathrm{T}} \mathbf{D}(\mathbf{Na})\, \mathrm{d}\Omega$$ + Boundary Terms

C & D are operators with lower order of differentiation as compared to A, Hence lower orders of continuity are demanded from the trial functions.

Galerkin Finite Element method – Example



Ref 4 – Pg. 75, Fig 3.24

Governing Equation → $$\frac{d^2\theta}{d\zeta^2} - \mu^2\theta = 0$$

Consider domain to consist of 5 linear elements & 6 nodes

Approximate solution from Elements → $$\overline{\theta} = N_i\theta_i + N_j\theta_j$$

N_i, N_j are interpolation functions across node i

θ_i , θ_j are nodal unknowns

Galerkin Finite Element method – Example

Galerkin method requires →
Since weighting function = shape function

$$\int_\zeta N_k \left(\frac{\mathrm{d}^2\overline{\theta}}{\mathrm{d}\zeta^2} - \mu^2\overline{\theta} \right) \mathrm{d}\zeta = 0$$

Integrating by parts and
applying boundary conditions →

$$\tilde{n}\left[N_i \frac{\mathrm{d}\theta}{\mathrm{d}\zeta} \right]_0^{\zeta_e} - \int_0^{\zeta_e} \frac{\mathrm{d}N_i}{\mathrm{d}\zeta}\frac{\mathrm{d}N_j}{\mathrm{d}\zeta}\mathrm{d}\zeta\{\theta\} - \int_0^{\zeta_e} N_i\mu^2(N_i\theta_i + N_j\theta_j)\mathrm{d}\zeta$$

Algebraic Equation →
Equation of the elements

$$\frac{1}{\zeta_e}[1 \quad -1]\begin{Bmatrix}\theta_i \\ \theta_j\end{Bmatrix} + \frac{\mu^2\zeta_e}{6}[2 \quad 1]\begin{Bmatrix}\theta_i \\ \theta_j\end{Bmatrix} + \begin{Bmatrix}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} \\ 0\end{Bmatrix}$$

Galerkin Finite Element method – Example

Assembling elements together and solving for Unknown parameters = nodal temperatures

$$\begin{bmatrix} 5.2 & -4.9 & 0.0 & 0.0 & 0.0 & 0.0 \\ -4.9 & 10.4 & -4.9 & 0.0 & 0.0 & 0.0 \\ 0.0 & -4.9 & 10.4 & -4.9 & 0.0 & 0.0 \\ 0.0 & 0.0 & -4.9 & 10.4 & -4.9 & 0.0 \\ 0.0 & 0.0 & 0.0 & -4.9 & 10.4 & -4.9 \\ 0.0 & 0.0 & 0.0 & 0.0 & -4.9 & 5.2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{Bmatrix} = \begin{Bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \\ \dfrac{d\theta}{d\xi} \end{Bmatrix}$$

Interestingly, this method provides better results when compared to approximate methods that use a function profile that satisfies BC and is assumed before hand – MWR or Rayleigh-Ritz

Assembly of element arrays

- Based on two criteria for interpolation functions:
  1. Compatibility
  2. Completeness

- Compatibility:
  - Field Variable and any of its partial derivatives up to one order less than highest in variational integral should be continous
  - Example – Continuity of Temperature with reference to heat conduction governing equation

- Completeness:
  - Within each element continuity must exist up to order of highest derivative in variational integral
  - Essentially, as number of nodes → ∞, Residual → 0

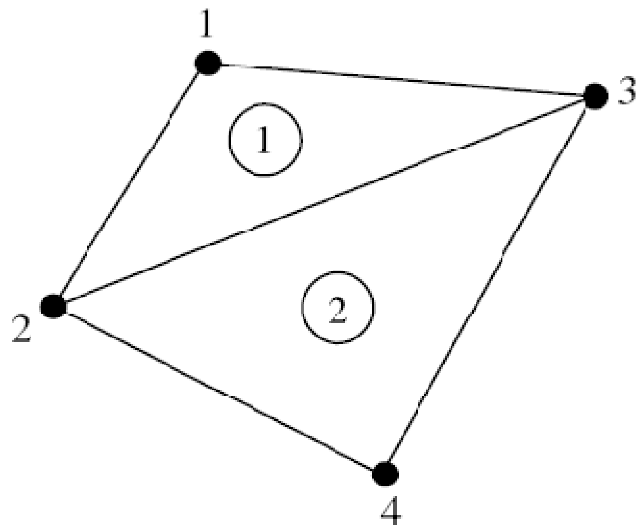
Assembly – Example

Assembly of two triangular elements

$$\mathbf{K}_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathbf{K}_2 = \begin{bmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$\mathbf{f}_1 = \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} \qquad \mathbf{f}_2 = \begin{Bmatrix} d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

$$[\mathbf{K}]\{\boldsymbol{\phi}\} = \{\mathbf{f}\}$$



Ref 4 – Pg. 324, Fig C.1

Assembly – Example

$$[\mathbf{K}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22}+b_{22} & a_{23}+b_{23} & b_{24} \\ a_{31} & a_{32}+b_{32} & a_{33}+b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix} \qquad \{\boldsymbol{\phi}\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} \qquad \{\mathbf{f}\} = \begin{Bmatrix} c_1 \\ c_2+d_2 \\ c_3+d_3 \\ d_4 \end{Bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22}+b_{22} & a_{23}+b_{23} & b_{24} \\ a_{31} & a_{32}+b_{32} & a_{33}+b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2+d_2 \\ c_3+d_3 \\ d_4 \end{Bmatrix}$$

Laplace and Poisson Equations

Poisson Equation → $\nabla^2\varphi = f.$

In 3-d Cartesian coordinates → $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi(x,y,z) = f(x,y,z).$

Laplace Equation → $\nabla^2\varphi = 0$

In 3-d Cartesian coordinates → $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0.$

Essentially, Laplace equation is a special case of the Poisson Equation where the RHS = 0

A physical example is the Steady-State Heat Conduction equation; Others include fluid mechanics, electrostatics etc.

Using the Steady-state heat equation, FEM discretization is carried out in the following slides

FEM Discretization – Variational Method

Poisson Equation →

$$\frac{\partial}{\partial x}\left(k\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial \phi}{\partial y}\right)+Q=0$$

Boundary Conditions →

$$B(\phi)=k\frac{\partial \phi}{\partial n}-\bar{q}=0 \qquad \text{on } \Gamma_q$$

Finiding the Variational Integral

$$\delta\Pi=\int_{\Omega}\left[k\frac{\partial \phi}{\partial x}\delta\left(\frac{\partial \phi}{\partial x}\right)+k\frac{\partial \phi}{\partial y}\delta\left(\frac{\partial \phi}{\partial y}\right)-Q\,\delta\phi\right]\mathrm{d}\Omega-\int_{\Gamma_q}(\bar{q}\,\delta\phi)\,\mathrm{d}\Gamma$$

Integrating by parts &
Using relation →

$$\delta\left(\frac{\partial \phi}{\partial x}\right)=\frac{\partial}{\partial x}(\delta\phi)$$

FEM Discretization – Governing Equation

→→ $$\Pi = \int_\Omega \left[\frac{1}{2}k\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{2}k\left(\frac{\partial\phi}{\partial y}\right)^2 - Q\phi\right] \mathrm{d}\Omega - \int_{\Gamma_q} \bar{q}\phi \,\mathrm{d}\Gamma$$

Approximating integral to achieve algebraic equations using shape functions →
Shape functions are determined based on domain and accuracy required in analysis

$$\phi \approx \hat{\phi} = \sum N_i a_i = \mathbf{Na}$$

→→ $$\Pi = \int_\Omega \frac{1}{2}k\left(\sum \frac{\partial N_i}{\partial x}a_i\right)^2 \mathrm{d}\Omega + \int_\Omega \frac{1}{2}k\left(\sum \frac{\partial N_i}{\partial y}a_i\right)^2 \mathrm{d}\Omega - \int_\Omega Q\sum N_i a_i \,\mathrm{d}\Omega - \int_{\Gamma_q} \bar{q}\sum N_i a_i \,\mathrm{d}\Gamma$$

FEM Discretization – Governing Equation

On differentiating with respect to individual parameters a_i →

$$\frac{\partial \Pi}{\partial a_j} = \int_{\Omega} k\left(\sum \frac{\partial N_i}{\partial x} a_i\right) \frac{\partial N_j}{\partial x}\,\mathrm{d}\Omega + \int k\left(\sum \frac{\partial N_i}{\partial y} a_i\right) \frac{\partial N_j}{\partial y}\,\mathrm{d}\Omega$$

$$- \int_{\Omega} Q N_j\,\mathrm{d}\Omega - \int_{\Gamma_q} \bar{q} N_j\,\mathrm{d}\Gamma$$

We get system of linear algebraic equations to be solved for unknown parameters →

$$\mathbf{K}\mathbf{a} + \mathbf{f} = \mathbf{0}$$

Note:

Galerkin and Variational procedures must give the same answer for cases where natural variational principles exist – example is the case above

Numerical Integration

- As elements increase in order, the integrals required to solve for the element matrices become complex and algebraically tedious
- Hence, methods of numerical integration are used – especially in computational techniques like finite element

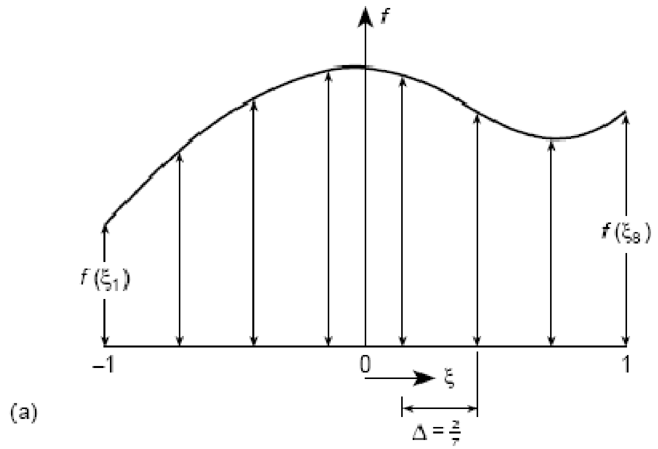
$$I = \int_{-1}^{1} f(\xi)\, \mathrm{d}\xi = \sum_{1}^{n} H_i f(\xi_i)$$

- This simple summation of terms of the integrand evaluated at certain points and weighted with certain functions serves as the approximation

Numerical Integration – Newton-Cotes Quadrature

Here, a polynomial of degree n-1 is used to approximate the integrand by sampling at n points on the integrand curve

Resulting polynomial is integrated instead to approximate value of the integral



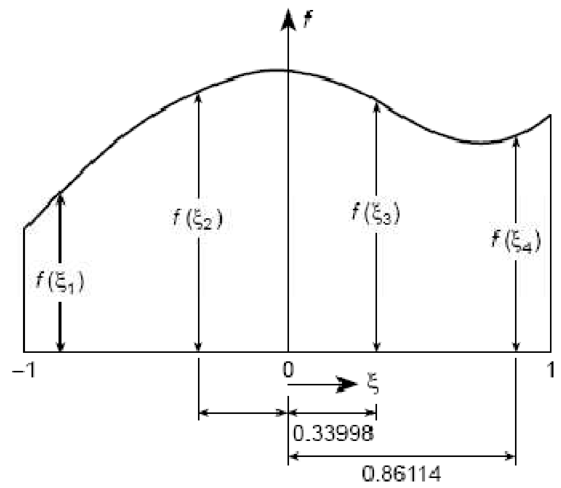
Ref 1 – Pg. 218, Fig 9.12a

At n = 1 → Trapezoidal Rule $I = f(-1) + f(1)$

At n = 2 → Simpson's Rule $I = \frac{1}{3}[f(-1) + 4f(0) + f(1)]$

Numerical Integration – Gauss Quadrature

Here, sampling points are not assigned but are rather evaluated; thus, we get exact value of integral when integrand is of order $\leq$ p ($\geq$ n), also to be determined

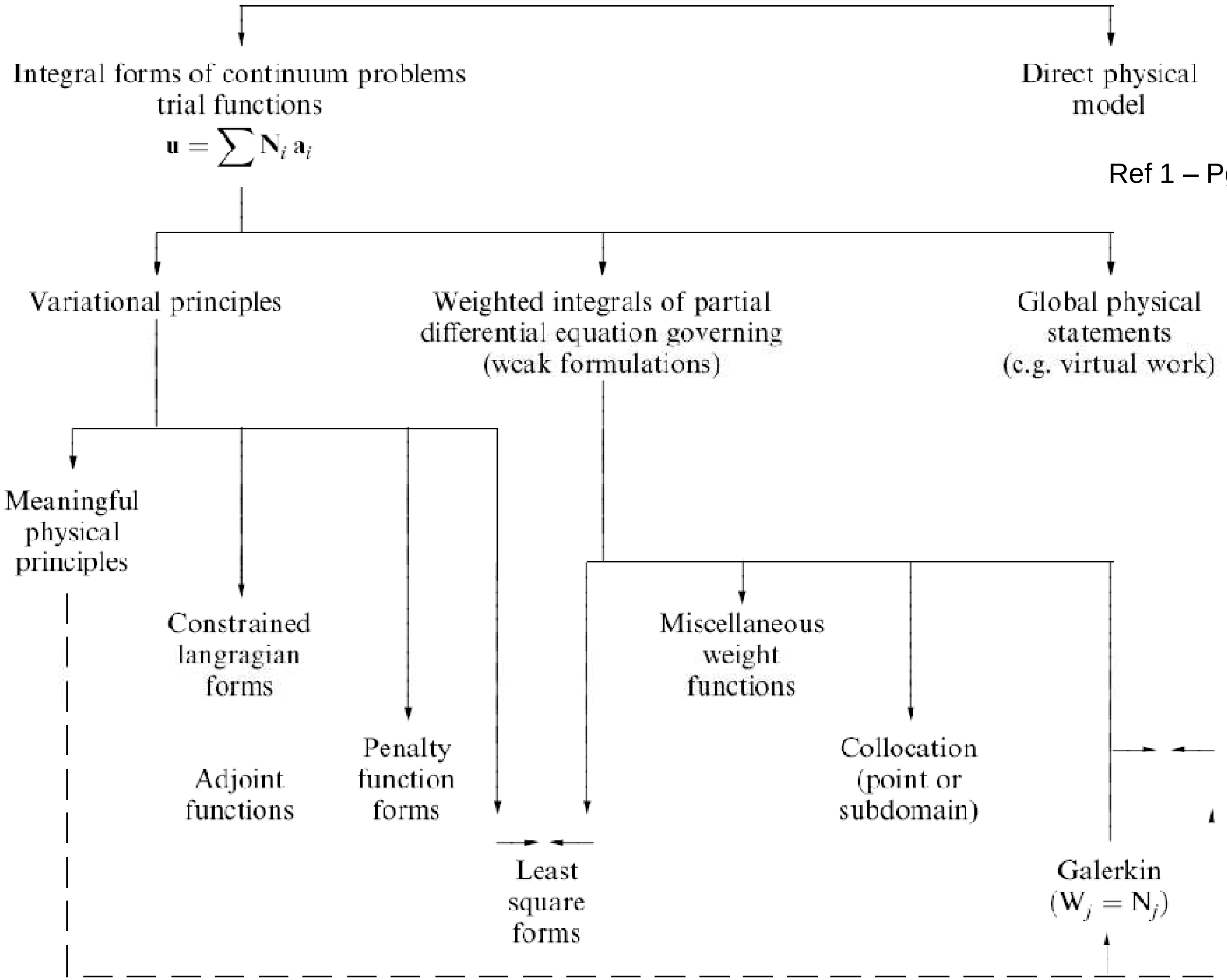


Ref 1 – Pg. 218, Fig 9.12b

Consider polynomial of order p; Use above summation approximation to evaluate integral with this polynomial as integrand and comparing coefficients with exact value we get → $p + 1 = 2\,(n+1)$

Thus, after sampling at n + 1 points → order of accuracy = 2n + 1; In earlier approach, order of accuracy = n only achieved

Table 3.2 Finite element approximation



Ref 1 – Pg. 83, Table 3.2

Summary

- FEM – Numerical solution to complex continuum problems
- Breaks down domain into discrete elements which are then piecewise approximated and assembled back to get complete solution
- Differential Equation → Integral Form → Linear Algebraic Equations
- Variational Method : Rayleigh – Ritz
- Method of Weighted Residuals : Galerkin Method
- Laplace & Poisson Equations
- FEM Discretization – Poisson Equation
- Numerical Integration – simplify complex integrals

References

1) The Finite Element Method – Its Basis and Fundamentals, 2005, Sixth Edition – O.C. Zienkiewicz, R.L. Taylor & J.Z. Zhu
2) Finite Elements and Approximation, 1983 - O.C. Zienkiewicz, K. Morgan
3) Finite Element Procedures in Engineering Analysis, 2001 – K. Bathe
4) Fundamentals of the Finite Element Method for Heat and Fluid Flow, 2004 – R.W. Lewis, P. Nithiarasu, K.N. Seetharamu
5) The Method of Weighted Residuals and Variational Principles, 1972 – B. Finlayson

Thank You