Runge-Kutta and Collocation Methods

Florian Landis

Overview

- Define Runge-Kutta methods.
- Introduce collocation methods.
- Identify collocation methods as Runge-Kutta methods.
- Find conditions to determine, of what order collocation methods are.

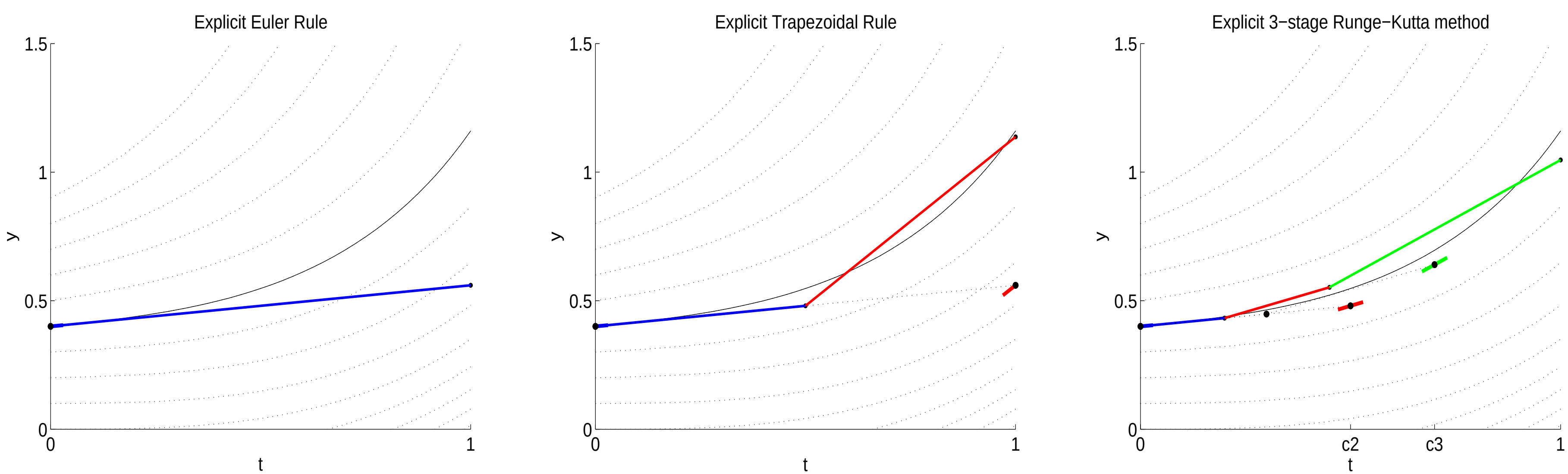
Introduction

General Goal: Find approximation to the solutions of

$$\dot{y}(t) = f(t, y), \qquad y(t_0) = y_0$$

using one step methods.

3 Examples of one step methods (step size $h = 1$) for the Riccati equation $\partial_t y = y^2 + t^2$:



Runga-Kutta Method

Definition 1 (Runge-Kutta) *Let* b_i, a_{ij} *(*$i, j = 1, \ldots, s$*) be real numbers and let* $c_i = \sum_{j=1}^{s} a_{ij}$. *An* s-stage Runge-Kutta method *is given by*

$$
\begin{aligned}
k_i &= f\Big(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j\Big), \quad i = 1, \ldots, s \qquad (1) \\
y_1 &= y_0 + h \sum_{j=1}^{s} b_i k_i.
\end{aligned}
$$

Distinguish:

explicit Runge-Kutta $a_{ij} = 0$ for $j \geq i$

implicit Runge-Kutta full matrix (a_{ij}) of non-zero coefficients allowed

Implicit function theorem: for h small enough, (1) has a locally unique solution close to $k_i \approx f(t_0, y_0)$.

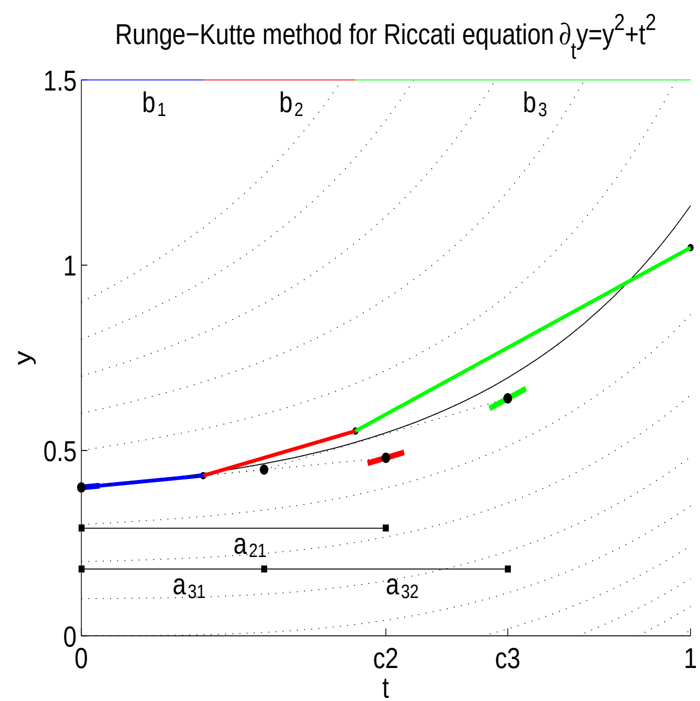
Butcher Diagram

The coefficients of the Runge-Kutta method are usually displayed in a Butcher diagram:

$$
\begin{array}{c|ccc}
c_1 & a_{11} & \dots & a_{1s} \\
\vdots & \vdots & & \vdots \\
c_s & a_{s1} & \dots & a_{ss} \\
\hline
 & b_1 & \dots & b_s
\end{array}
$$

Example for explicit Runge-Kutta:

$$
\begin{array}{c|ccc}
0 & & & \\
0.5 & 0.5 & & \\
0.7 & 0.3 & 0.4 & \\
\hline
 & 0.2 & 0.25 & 0.55
\end{array}
\quad \longrightarrow
$$



Order of the Runge-Kutta method

A general one-step method has *order* p, if

$$y_1 - y(t_0 + h) = \mathcal{O}(h^{p+1}) \quad \text{as } h \to 0.$$

By the Taylor expansions

$$y(t_0 + h) = y(t_0) + h \cdot f(t_0, y(t_0)) + \frac{1}{2}h^2 \cdot \frac{d}{dt} f(t, y(t))\Big|_{t=t_0} + \dots$$

$$y_1 = y_0 + h \sum_{i=1}^{s} b_i \left[f(t_0, y_0) + h \cdot \frac{d}{dh} f\Big(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j\Big)\Big|_{h=0} + \dots \right]$$

of y and y_1 of the Runge-Kutta method, one obtains the following conditions for the coefficients:

$$\begin{aligned} \textstyle\sum_i b_i &= 1 && \text{for order 1,} \\ \textstyle\sum_i b_i c_i &= 1/2 && \text{for order 2,} \\ \textstyle\sum_i b_i c_i^2 &= 1/3 && \\ \text{and } \textstyle\sum_{i,j} b_i a_{ij} c_j &= 1/6 && \text{for order 3.} \end{aligned}$$

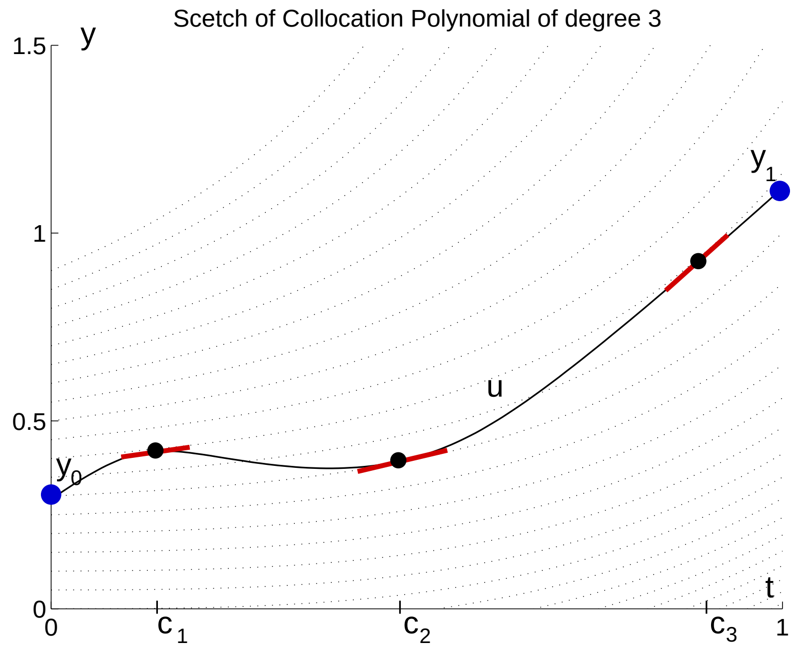
The Collocation Method

Definition 2 (Collocation Method) *Let* $c_1, \ldots, c_s$ *be distinct real numbers (usually* $0 \leq c_i \leq 1$*). The* collocation polynomial $u(t)$ *is a polynomial of degree* s *satisfying*

$$u(t_0) = y_0 \tag{2}$$

$$\dot{u}(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h)), \quad i = 1, \ldots, s, \tag{3}$$

and the numerical solution of the collocation method *is defined by* $y_1 = u(t_0 + h)$.



The Collocation Method

Theorem 1 (Guillou & Soulé 1969, Wright 1970) *The collocation method for $c_1, \ldots, c_s$ is equivalent to the s-stage Runge-Kutta method with coefficients*

$$a_{ij} = \int_0^{c_i} \ell_j(\tau)\,\mathrm{d}\tau, \qquad b_i = \int_0^1 \ell_i(\tau)\,\mathrm{d}\tau,$$

where $\ell_i(\tau)$ is the Lagrange polynomial $\ell_i(\tau) = \prod_{l \neq i}(\tau - c_l)/(c_i - c_l)$.

Moreover:

$$u(t_0 + \tau h) = y_0 + h \sum_{j=1}^{s} k_j \int_0^{\tau} \ell_j(\sigma)\,\mathrm{d}\sigma.$$

Thus, the existence of the collocation polynomial depends on the existence of the k_i (given for $h \to 0$).

Proof of Theorem 1

Proof. Let $u(t)$ be the collocation polynomial and define $k_i := \dot{u}(t_0 + c_i h)$. By the Lagrange interpolation formula we have $\dot{u}(t_0 + \tau h) = \sum_{j=1}^{s} k_j \cdot \ell_j(\tau)$, and by integration we get

$$u(t_0 + c_i h) = y_0 + h \sum_{j=1}^{s} k_j \int_0^{c_i} \ell_j(\tau)\, \mathrm{d}\tau.$$

Inserted into the definition of the collocation polynomial

$$\dot{u}(t_0 + c_i h) = f(t_0 + c_i h, u(t_0 + c_i h)),$$

this gives the first formula of the Runge-Kutta equation

$$k_i = f\Big(t_0 + c_i h, y_0 + h \sum_{j=1}^{s} a_{ij} k_j\Big).$$

Integration from 0 to 1 yields $y_1 = y_0 + h \sum_{j=1}^{s} b_i k_i$. □

Collocation Coefficients

If a Runge-Kutta methods corresponds to a collocation method of order s,

$$a_{ij} = \int_0^{c_i} \ell_j(\tau)\,\mathrm{d}\tau, \qquad b_i = \int_0^1 \ell_i(\tau)\,\mathrm{d}\tau,$$

leads to:

$$C(q=s) \quad : \qquad \sum_{j=1}^{s} a_{ij} c_j^{k-1} = \frac{c_i^k}{k}, \quad \forall i,\ k = 1, \ldots, q$$

$$B(p=s) \quad : \qquad \sum_{j=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \quad k = 1, \ldots, p$$

since $\tau^{k-1} = \sum_{j-1}^{s} c_j^{k-1} \ell_j(\tau)$ for $k = 1, \ldots, s$.

Note: $B(p) \Rightarrow y_0 + \sum_{i=1}^{s} b_i f(t_0 + hc_i)$ approximates the solution to $\dot{y} = f(t),\ y(t_0) = y_0$ with order p.

Order of The Collocation Method

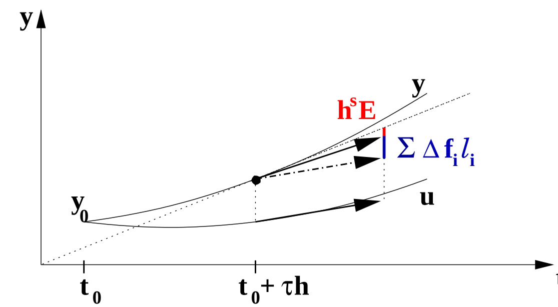
Lemma 2 *The collocation polynomial $u(t)$ is an approximation of order s to the exact solution of $\dot{y} = f(t, y)$, $y(t_0) = y_0$ on the whole interval, i.e.,*

$$||u(t) - y(t)|| \leq C \cdot h^{s+1} \qquad \text{for } t \in [t_0, t_0 + h]$$

and for sufficiently small h.
Moreover, the derivatives of $u(t)$ satisfy for $t \in [t_0, t_0 + h]$

$$||u^{(k)}(t) - y^{(k)}(t)|| \leq C \cdot h^{s+1-k} \qquad \text{for } k = 0, \ldots, s.$$

Proof of Lemma 2



$$\dot{u}(t_0+\tau h) = \sum_{j=1}^{s} f\big(t_0+c_i h, u(t_0+c_i h)\big)\ell_j(\tau),$$

$$\dot{y}(t_0+\tau h) = \sum_{j=1}^{s} f\big(t_0+c_i h, y(t_0+c_i h)\big)\ell_j(\tau) + h^s E(\tau,h)$$

$$||E(\tau,h)|| \le 2 \max_{t\in[t_0,t_0+h]} \frac{||y^{(s+1)}(t)||}{s!}$$

Integrating the difference of the above two equations gives

$$y(t_0+\tau h) - u(t_0+\tau h) = h\sum_{i=1}^{s} \Delta f_i \int_0^{\tau} \ell_i(\sigma)\,\mathrm{d}\sigma + h^{s+1}\int_0^{\tau} E(\sigma,h)\,\mathrm{d}\sigma$$

with $\Delta f_i = f\big(t_0+c_i h, y(t_0+c_i h)\big) - f\big(t_0+c_i h, u(t_0+c_i h)\big)$.

Proof of Lemma 2

Using a Lipschitz condition for $f(t, y)$ on the Integral

$$y(t_0 + \tau h) - u(t_0 + \tau h) = h \sum_{i=1}^{s} \Delta f_i \int_0^{\tau} \ell_i(\sigma)\,\mathrm{d}\tau + h^{s+1} \int_0^{\tau} E(\sigma, h)\,\mathrm{d}\sigma$$

yields

$$\max_{t \in [t_0, t_0+h]} ||y(t) - u(t)|| \leq h\,C\,L \max_{t \in [t_0, t_0+h]} ||y(t) - u(t)|| + \text{Const.} \cdot h^{s+1},$$

implying $||u(t) - y(t)|| \leq C \cdot h^{s+1}$ for sufficiently small $h > 0$.

Superconvergence

Theorem 3 (Superconvergence) *If the condition $B(p)$ holds for some $p \geq s$, then the collocation method has order p. This means that the collocation method has the same order as the underlying quadrature formula.*

$$B(p) \quad : \qquad \sum_{j=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \quad k = 1, \ldots, p$$

Note: $B(p)$ cannot be met for $p > 2s$.

Proof of Superconvergence

Proof. We consider the collocation polynomial $u(t)$ as the solution of a perturbed differential equation

$$\dot{u} = f(t, u) + \delta(t)$$

with defect $\delta(t) := \dot{u}(t) - f\big(t, u(t)\big)$. Subtracting $\dot{y}(t) = f(t, y)$ from the above we get after linearization that

$$\underbrace{\dot{u}(t) - \dot{y}(t)}_{\dot{\mathcal{E}}(t)} = \frac{\partial f}{\partial y}\Big(t, y(t)\Big) \underbrace{\Big(u(t) - y(t)\Big)}_{\mathcal{E}(t)} + \delta(t) + r(t),$$

where, for $t_0 \leq t \leq t_0 + h$, the remainder $r(t)$ is of size $\mathcal{O}\big(||u(t) - y(t)||^2\big) = \mathcal{O}(h^{2s+2})$ by lemma 2.

Conclusion

- Collocation methods with polynomials of degree s are equivalent to s-stage Runge-Kutta methods:

$$a_{ij} = \int_0^{c_i} \ell_j(\tau)\,\mathrm{d}\tau, \qquad b_i = \int_0^1 \ell_i(\tau)\,\mathrm{d}\tau,$$

- Collocation polynomials of degrees s lead to collocation methods of order s or better:
- If $B(p)$ is met for $p > s$, the corresponding collocation method is of order p.

$$B(p) \quad : \qquad \sum_{j=1}^{s} b_i c_i^{k-1} = \frac{1}{k}, \quad k = 1, \ldots, p.$$

Proof of Lemma 2

The second statement follows from the first one:

Taking the kth derivative of

$$y(t_0 + \tau h) - u(t_0 + \tau h) = h \sum_{i=1}^{s} \Delta f_i \int_0^\tau \ell_i(\sigma)\,\mathrm{d}\tau + h^{s+1} \int_0^\tau E(\sigma, h)\,\mathrm{d}\sigma$$

gives

$$h^k \big(y^{(k)}(t_0 + \tau h) - u^{(k)}(t_0 + \tau h)\big) = h \sum_{i=1}^{s} \Delta f_i \ell_i^{(k-1)}(\tau) + h^{s+1} E^{(k-1)}(\tau, h).$$

With

$$||E^{(k-1)}(\tau, h)|| \leq \max_{t \in [t_0, t_0+h]} \frac{||y^{(s+1)}(t)||}{(s-k+1)!}$$

and a Lipschitz condition for $f(t, y)$, $||u^{(k)} - y^{(k)}|| \leq C \cdot h^{s+1-k}$ follows.

Variation of Constants Formula

For homogeneous systems of linear equations

$$\dot{y}(t) = A(t)y(t)$$

with initial condition $y(t_0) = y_0$, the solution can be written as

$$y(t) = R(t, t_0)y_0 \Leftrightarrow \dot{R}(t, s) = A(t)R(t, s).$$

Using this *resolvent* of the homogeneous differential system, the solution to inhomogeneous problems

$$\dot{y}(t) = A(t)y(t) + f(t)$$

can be found with the *variation of constants formula*:

$$y(t) = R(t, t_0)y_0 + \int_{t_0}^{t} R(t, s)f(s)\,\mathrm{d}s.$$

Proof of Superconvergence

With

$$\dot{R}(t,s) = \frac{\partial}{\partial y} f(t, y(t)) R(t,s)$$

The variation of constants formula then yields

$$y_1 - y(t_0 + h) = \mathcal{E}(t_0 + h) = \int_{t_0}^{t_0+h} R(t_0 + h, s)\Big(\delta(s) + r(s)\Big)\,\mathrm{d}s$$

as the solution of

$$\dot{\mathcal{E}}(t_0 + h) = \frac{\partial f}{\partial y}\Big(t, y(t)\Big)\mathcal{E}(t) + \delta(t) + r(t).$$

The contribution of $r(t)$:

$$r(t) \sim \mathcal{O}(h^{2s+2}) \Rightarrow \int_{t_0}^{t_0+h} R(t_0 + h, s) r(s)\,\mathrm{d}s \sim \mathcal{O}(h^{2s+3})$$

Proof of Superconvergence

The main idea now is to apply the quadrature formula $(b_i, c_i)_{i=1}^s$ to the integral of $g(s) = R(t_0 + h, s)\delta(s)$:

$$\int_{t_0}^{t_0+h} g(s)\,\mathrm{d}s = \sum_{i=1}^{s} b_i g(t_0 + hc_i) + \text{quadrature Error.}$$

From $\delta(s)|_{t_0+c_ih} = 0$ follows $\sum_{i=1}^s b_i g(t_0 + hc_i) = 0$. Thus,

$$\int_{t_0}^{t_0+h} g(s)\,\mathrm{d}s = \text{quadrature Error} \leq h^{p+1} \frac{\partial^p}{\partial s^p} g(s).$$

$\frac{\partial^p}{\partial s^p} g(s)$ is bounded independently of h by Lemma 2. Therefore

$$\mathcal{E}(t_0 + h) = \int_{t_0}^{t_0+h} R(t_0 + h, s)\Big(\delta(s) + r(s)\Big)\,\mathrm{d}s \sim \mathcal{O}(h^{p+1}).$$