Runge 2nd Order Method

Major: All Engineering Majors

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Runge-Kutta 2nd Order Method

Runge-Kutta 2ⁿᵈ Order Method

For $\dfrac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

Heun's Method

Heun's method

Here $a_2 = 1/2$ is chosen
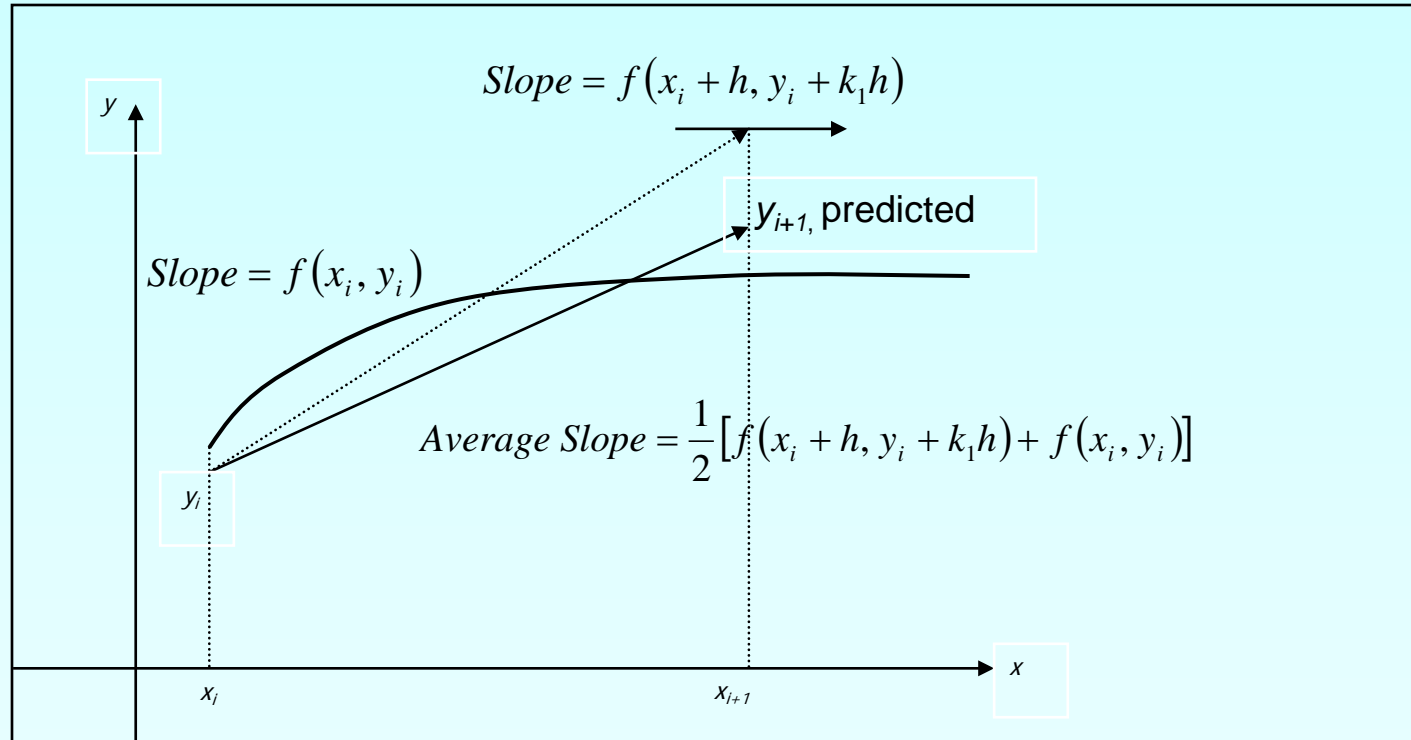
$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$Slope = f(x_i + h, y_i + k_1 h)$$

$y_{i+1,}$ predicted

$$Slope = f(x_i, y_i)$$

$$Average\ Slope = \frac{1}{2}[f(x_i + h, y_i + k_1 h) + f(x_i, y_i)]$$

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, \, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \dfrac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2 \right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, \, y_i + \frac{3}{4}k_1 h \right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \; y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \; y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left(\theta^4 - 81 \times 10^8 \right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Heun's method. Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12} \left(\theta^4 - 81 \times 10^8 \right)$$

$$f(t,\theta) = -2.2067 \times 10^{-12} \left(\theta^4 - 81 \times 10^8 \right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2} k_1 + \frac{1}{2} k_2 \right) h$$

Solution

Step 1:
$$i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200K$$

$$k_1 = f(t_0, \theta_o)$$
$$= f(0, 1200)$$
$$= -2.2067 \times 10^{-12}(1200^4 - 81 \times 10^8)$$
$$= -4.5579$$

$$k_2 = f(t_0 + h, \theta_0 + k_1 h)$$
$$= f(0 + 240, 1200 + (-4.5579)240)$$
$$= f(240, 106.09)$$
$$= -2.2067 \times 10^{-12}(106.09^4 - 81 \times 10^8)$$
$$= 0.017595$$

$$\theta_1 = \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240$$

$$= 1200 + (-2.2702)240$$

$$= 655.16K$$

Solution Cont

Step 2: $i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$

$$k_1 = f(t_1, \theta_1)$$
$$= f(240, 655.16)$$
$$= -2.2067 \times 10^{-12}(655.16^4 - 81 \times 10^8)$$
$$= -0.38869$$

$$k_2 = f(t_1 + h, \theta_1 + k_1 h)$$
$$= f(240 + 240, 655.16 + (-0.38869)240)$$
$$= f(480, 561.87)$$
$$= -2.2067 \times 10^{-12}(561.87^4 - 81 \times 10^8)$$
$$= -0.20206$$

$$\theta_2 = \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240$$

$$= 655.16 + (-0.29538)240$$
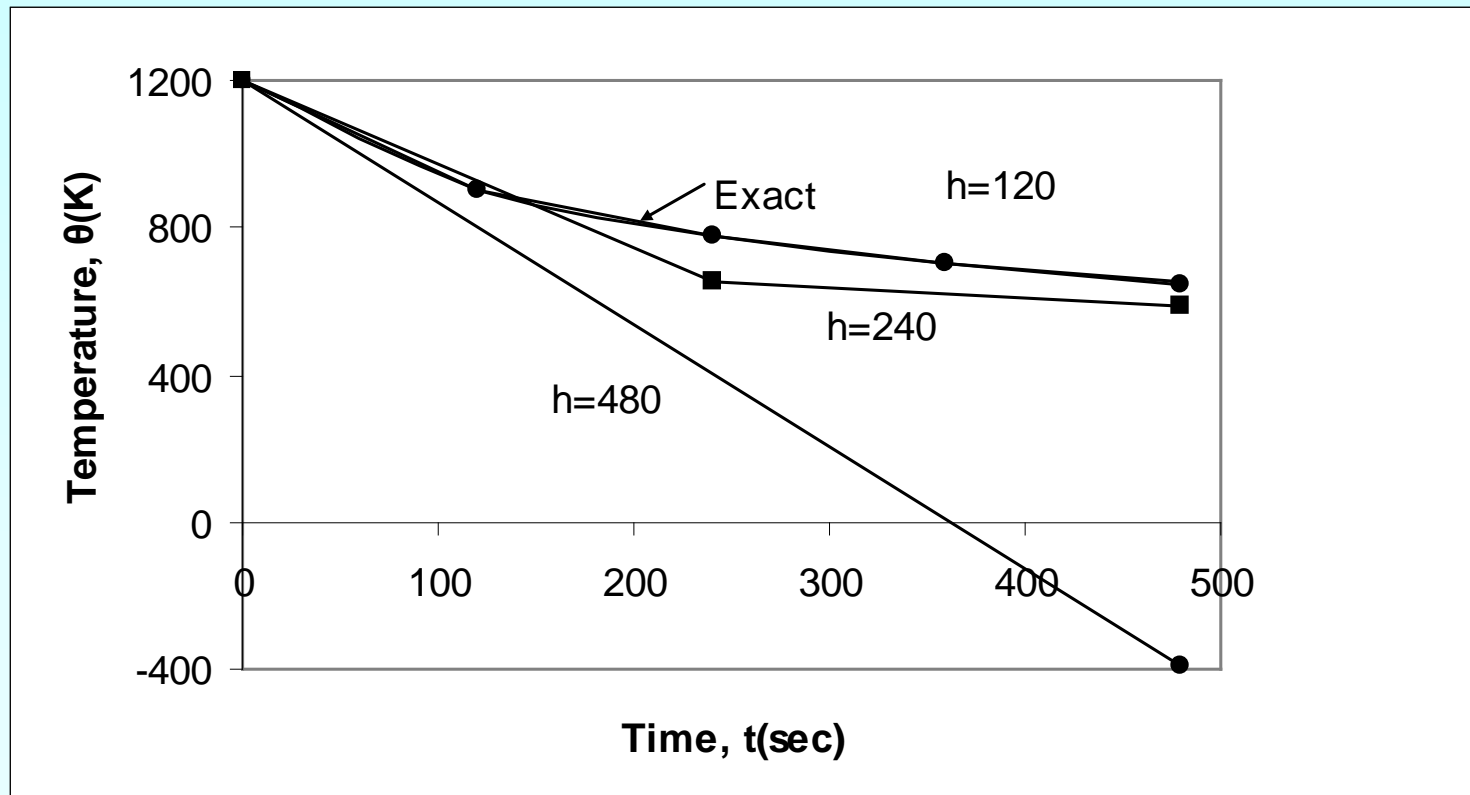
$$= 584.27K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57 K$$

Comparison with exact results

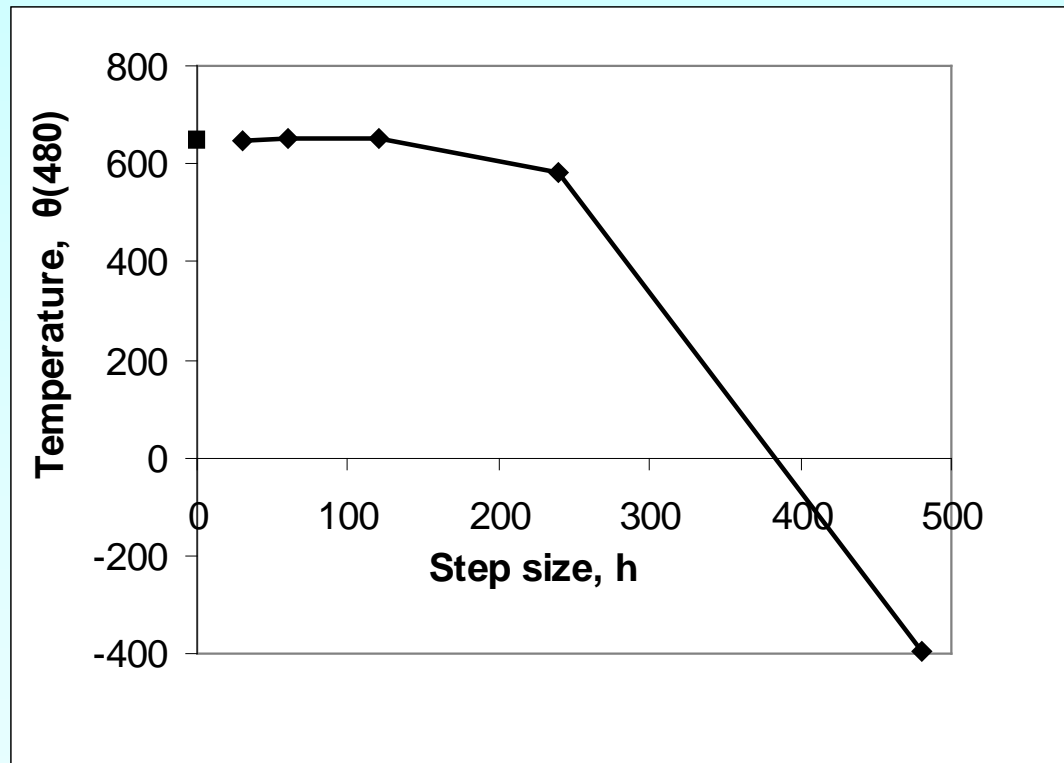**Figure 2.** Heun's method results for different step sizes

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	$\theta(480)$	E_t	$|\epsilon_t|\%$
480	−393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	−3.7762	0.58313
60	649.91	−2.3406	0.36145
30	648.21	−0.63219	0.097625

$$\theta(480) = 647.57 K \quad \text{(exact)}$$

Effects of step size on Heun's Method

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2ⁿᵈ Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\theta(480)$			
	Euler	Heun	Midpoint	Ralston
480	−987.84	−393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

$$\theta(480) = 647.57K \quad \text{(exact)}$$

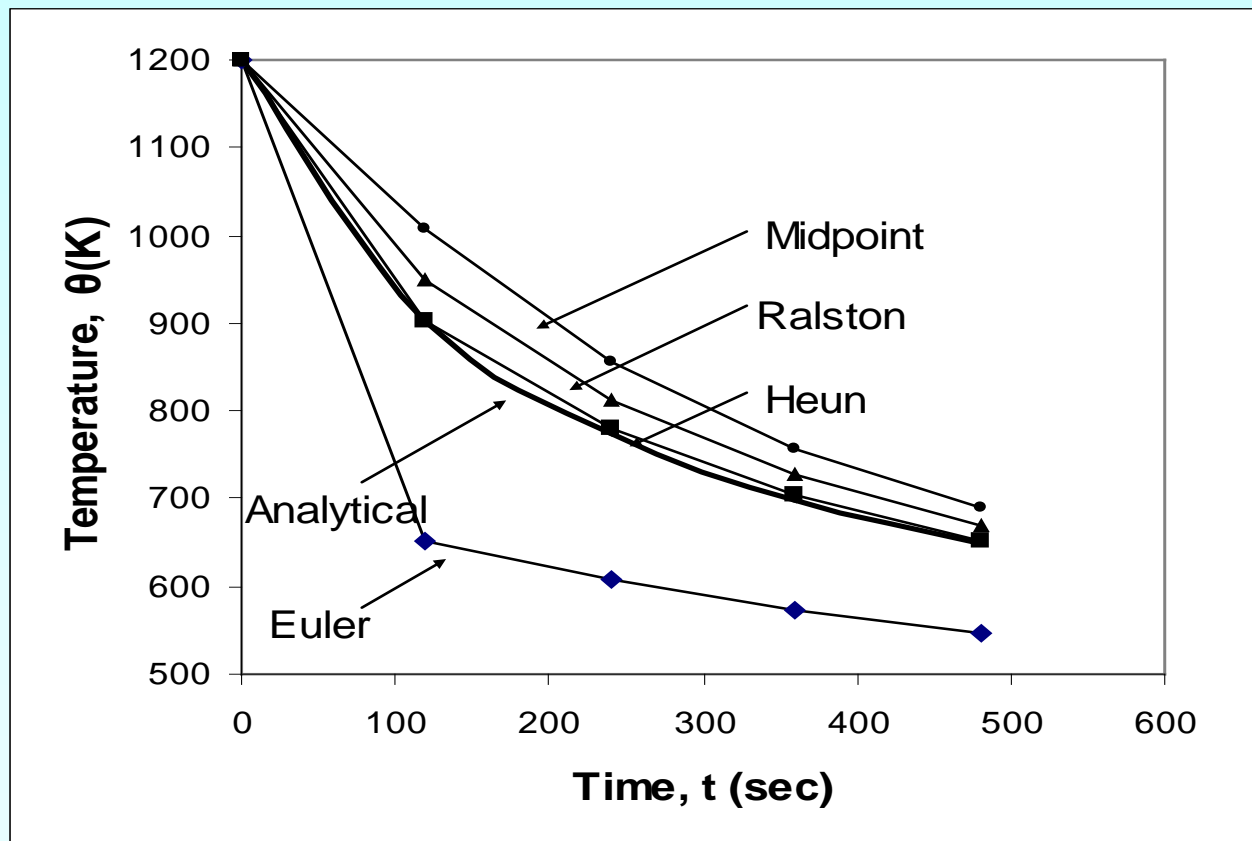# Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2.   Comparison of Euler and the Runge-Kutta methods

Step size, h	$\left\|\in_t\right\|\%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Comparison of Euler and Runge-Kutta 2nd Order Methods

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

THE END