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A COMPARISON OF NUMERICAL METHODS FOR ANALYZING THE DYNAMIC RESPONSE OF STRUCTURES

By

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SYNOPSIS

A comparative study of step-by-step methods which are commonly used in the numerical analysis of the dynamic response of structures is presented. The method of analysis is based on the general theory of the calculus of difference equations and the algebra of matrices. The available step-by-step techniques discussed are classified into three groups:

1. Acceleration methods,
2. Difference equation methods,
3. Numerical solutions of differential equations.

Comparisons have been made between the available techniques with respect to the accuracy of a single step, propagation of errors after a length of time, limitations imposed by instability and lack of convergence, time consumption, and self-checking provisions of the procedures. The purpose of the work has been to determine the range of applicability of the various techniques.

COMPARISON OF STEP-BY-STEP METHODS FOR ANALYZING THE DYNAMIC RESPONSE OF STRUCTURES

I. INTRODUCTION

1.1 Summary

This dissertation is concerned with the analysis of step-by-step methods commonly used in numerical solutions of the dynamic response of structures. Rigorous solutions are not always possible for structures with non-linear characteristics under dynamic loads such as wind, impact, blast, earthquake or vibratory motions, particularly in the case of multi-degree-of-freedom systems with plastic resistance or with a varying elastic behavior as a function of time. Consequently a numerical approach is indispensable for such conditions and step-by-step studies of motion with respect to time are extremely useful.

The purpose of this dissertation is to study the accuracy and range of applicability of various step-by-step techniques now available and frequently used in problems of dynamic response of structures. These step-by-step procedures may be classified for convenience of discussion into three groups: 1. Acceleration methods; 2. Difference equation methods; and 3. Numerical solution of differential equations. In the acceleration methods the displacement and velocity at the end of a time interval are each expressed in terms of the displacement and velocity at the beginning of the time interval, together with the accelerations which occur at the ends of the interval, a law of variation of the acceleration within this time interval being assumed. The acceleration is in turn governed by the differential equation of motion and the problem may then be solved by an algebraic solution of simultaneous equations or by cut-and-try iterations.

The second group of methods involves the application of finite difference equations which are obtained from the given differential equations of motion.

Displacement at each successive step during the motion can be derived from the displacements previously obtained by means of finite difference equations. For multi-degree-of-freedom structures the solution may be accomplished by solving a set of linear simultaneous equations or by inverting a matrix.

The third group of methods includes conventional devices developed by mathematicians for the numerical solution of various differential equations and which are quite general in application. They may be adopted even in more complicated problems than those involved in the equation of motion which we usually encounter.

The analyse the characteristics of each of the available techniques it is best to obtain beforehand an algebraic equation representing each of the various techniques of step-by-step procedures, then to compare it with the rigorous solution of the differential equation of motion and investigate its propagation of errors. This can be done in one of the following ways. First, it is possible to express the approximate solution in the form of a finite difference equation in terms of displacements and find its complementary and particular solutions by means of the calculus of finite difference. If the approximate procedure is readily given in a finite difference form, no work is necessary in transforming the original procedure into finite difference equations. Secondly, the set of linear equations used in the approximate technique may also be expressed in a matrix form such that a column matrix consisting of displacement, velocity and acceleration at the end of the time interval is equal to the product of a square matrix into a column matrix consisting of displacement, velocity and acceleration at the beginning of the time interval. When the procedure is successively carried out n times, the square matrix multiplies itself to the nth power and shows the relation between the initial and final conditions. The former way is more simple as far as mathematics is concerned but reveals directly the dynamic response only in displacement, while the latter, though involving more algebra, gives not only

displacement, but velocity and acceleration as well if desired.

The dynamic analysis of a structure is usually based on the following umptions:

1. The mass of the structure may be represented by a number of separate ıcentrated masses supported by a flexible and weightless framework.

2. The resistance-deflection relationship of the structure can be deter- ned beforehand over the whole range of action, and the time history of displace- ıt or external forces is known.

Without loss of generalization, the present analysis has been confined to single-degree-of-freedom system. Nevertheless the motion of more complicated ulti-degree-of-freedom systems can be considered as being made up of the motion n several modes, each mode acting as a single-degree-of-freedom.

Generally speaking, accuracy may be attained if the time interval is sufficiently small while too large an interval may produce very misleading results. However, since different degrees of accuracy can result from different methods of application, the choice of time interval depends upon the accuracy desired and the amount of work required.

Acceleration methods need no special training for their application since they are based on fundamental concepts, but these methods are always handicapped by the criteria of convergence and stability. The constant acceleration method[1]* is objectionable because of its rapid divergence of amplitude. Timoshenko's modified acceleration method[1] gives better results than that of constant acceleration, yet the frequency error is still appreciable. It is, however, free from stability difficulties and has no enlarging or diminishing effect of the velocity response. Newmark's linear acceleration method [2] has better agreement in frequency, but overshoots a little in amplitude due to the enlarged velocity response. Newmark's parabolic acceleration method[2] has even better agreement

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* Numbers in parentheses refer to items in the Bibliography.

.n frequency, but unfortunately its amplitude diverges exponentially, and it is :herefore of less value for a long lapse of time in spite of its accuracy in the :irst cycle of vibration. Newmark's β-method[3][4] may be regarded as a generalized acceleration method, introducing a new parameter β in the displacement equation so as to control the effect of acceleration. With $\beta=1/4$, it is identical with Timoshenko's modified method. With $\beta=1/6$, it is the same as the linear acceleration method. It corresponds to the difference equation method adopted by Levy[5] when $\beta=0$, and to that given by Salvadori[6] when $\beta=1/12$. The great advantage of this generalization is that it permits a convenient choice of the time interval determined by the convergence criterion during the operation.

Difference equation methods also have criteria for stability. These procedures are not self-checking. A little more time economy may be gained since only the displacement is necessary for the computation and the velocity may be disregarded in each step thus saving time in calculations. As stated before, the difference equations adopted by Levy and Salvadori may be considered as identical to Newmark's β-method when $\beta=0$ and $\beta=1/12$ respectively, except that the initial conditions are treated differently. Houbolt's method[7] is said to be an improvement over Levy's method, since it employs a cubic curve of displacement for the difference equation, yet it suffers from the converging characteristic of the amplitude and from a large error in period. The computed amplitude of an undamped system as computed by this method will decay rapidly after a few cycles of vibration even when a small time interval is used.

The accuracy of the numerical solutions of differential equations developed by Euler, Runge and Kutta [8][9] is discussed in many books and papers. The application of these methods to linear vibration problems is somewhat time-consuming in comparison with the methods above mentioned particularly in multi-degree-of-freedom systems. Runge-Kutta's method has an advantage for general applicability in that it is always stable and it has the proper criterion for

ritical damping in viscous damping conditions.

Comparisons of true amplitude and period with 'pseudo' or computed amplitude and period in each method of numerical solution are made to investigate the effects of length of time interval, natural frequency of the structure and other parameters. Additional discussion of these factors is presented in later chapters.

II. GENERAL METHODS OF ANALYSIS

.1 Calculus of Finite Difference Equations

Analysis may be made for each method by expressing the given differential quation of motion, combined with the procedure of operation, into a difference quation. Then the properties of this difference equation represent the character-stics of the corresponding numerical method. In the second group of available echniques described in the last chapter, finite difference equations are readily 'ormed from the differential equation of motion by replacing the higher orders of lerivatives by central difference patterns. In the first and third groups of available techniques more algebraic work is required to convert the equations of motion into a difference equation. However, the equations of operation prescribing the given motion can always be expressed in terms of displacements and velocities in a linear relation, and can easily be put in a difference equation form. In acceleration methods the equations of operation may at first contain some acceleration terms but one can soon eliminate them since the final acceleration itself can be expressed in terms of displacement, velocity and initial acceleration. Thus if the equation of motion is given in the form

$$\ddot{y} + 2rp\dot{y} + p^2y = F(t) \tag{2.1.1}$$

where p is the natural frequency of vibration and r the coefficient of viscous damping in terms of p, it is possible to represent the numerical procedure by a finite difference equation in the form

$$a_1 y_{n+1} + a_2 y_n + a_3 y_{n-1} = b_1 F(t_{n+1}) + b_2 F(t_n) + b_3 F(t_{n-1}) \tag{2.1.2}$$

or, in the case of the parabolic acceleration method or Houbolt's method,

$$a_1 y_{n+1} + a_2 y_n + a_3 y_{n-1} + a_4 y_{n-2} = b_1 F(t_{n+1}) + b_2 F(t_n) + b_3 F(t_{n-1}) + b_4 F(t_{n-2}) \tag{2.1.3}$$

The solutions of these difference equations are

$$y_n = c_1 x_1^n + c_2 x_2^n \tag{2.1.4}$$

$$y_n = c_1 x_1^n + c_2 x_2^n + c_3 x_3^n \tag{2.1.5}$$

pectively, where x_1, x_2, and x_3 are the roots of the equation

$$a_1 x^2 + a_2 x + a_3 = 0 \tag{2.1.6}$$

$$a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0 \tag{2.1.7}$$

responding to Eqs. (2.1.2) and (2.1.3), and c_1, c_2, and c_3 are constants
ermined from the initial conditions.

If the roots x_1 and x_2 are conjugate complex roots, the response of the merical procedure is periodic although there may exist errors in both amplitude d frequency. One the other hand when all the x roots are real, the solution comes aperiodic and unstable. By 'stable' we mean that the response of the merical solution remains periodic and without fluctuation or rapid divergence in plitude. As far as time period is concerned, the observation of these roots erves therefore as a criterion of stability. Divergence of amplitude may also be egarded as kind of instability and it will be shown in later chapters that it is lue to the presence of a factor with an exponential power of time which occurs in the general equation of response. If the factor equals one, the amplitude will neither diverge nor converge and is therefore stable. When the factor is larger than one, the amplitude diverges with a rate which depends on the magnitude of the factor. Slow divergence is generally acceptable in some problems, it is determined y the allowable error in amplitude and not by the criterion of stability.

Particular solutions of these difference equations may be obtained by the calculus of finite differences though sometimes this may involve difficulties in more complicated forcing functions. However, an approximation can generally be made by expressing the forcing function in a power series or a Fourier expansion which is always solvable in this kind of finite difference equations.

The finite difference equations (2.1.2) and (2.1.3) consist entirely of

displacement terms and therefore the general solution shows only the response in displacement of the structure at the end of the time interval due to the displacement and velocity at the beginning of the time interval and also due to the exterior forces if there are any. In order to bring out the response in velocity of the structure, another set of difference equation containing all velocity terms must be formed from the fundamental equations of the numerical solution, such as

$$a_1 \dot{y}_{n+1} + a_2 \dot{y}_n + a_3 \dot{y}_{n-1} = b_1 F(t_{n+1}) + b_2 F(t_n) + b_3 F(t_{n-1}) \tag{2.1.8}$$

or

$$a_1 \dot{y}_{n+1} + a_2 \dot{y}_n + a_3 \dot{y}_{n-1} + a_4 \dot{y}_{n-2} = b_1 F(t_{n+1}) + b_2 F(t_n) + b_3 F(t_{n-1}) + b_4 F(t_{n-2}) \tag{2.1.9}$$

Similarly, the finite difference equations may also contain only acceleration terms in the form of

$$a_1 \ddot{y}_{n+1} + a_2 \ddot{y}_n + a_3 \ddot{y}_{n-1} = b_1 F(t_{n+1}) + b_2 F(t_n) + b_3 F(t_{n-1}) \tag{2.1.10}$$

or

$$a_1 \ddot{y}_{n+1} + a_2 \ddot{y}_n + a_3 \ddot{y}_{n-1} + a_4 \ddot{y}_{n-2} = b_1 F(t_{n+1}) + b_2 F(t_n) + b_3 F(t_{n-1}) + b_4 F(t_{n-2}) \tag{2.1.11}$$

if the response in acceleration is required.

All the results of numerical solutions are henceforth to be compared with the exact solution. In the present analysis the motion of a structure which is assumed to be of elastic behavior is prescribed by the well-known differential equation (2.1.1) whose general solution is known to be

$$\bar{y} = e^{-rpt}\left(A \cos\sqrt{1-r^2}\, pt + B \sin\sqrt{1-r^2}\, pt\right) + \bar{y}_p \tag{2.1.12}$$

where $\bar{y}_p$ is the particular solution, and A and B are constants determined from the initial conditions.

For free vibration, $F(t) = 0$, $\bar{y}_p = 0$, and the equation of motion becomes

$$\ddot{y} + 2rp\dot{y} + p^2 y = 0 \ . \tag{2.1.13}$$

The solution is

$$\bar{y} = e^{-rpt}\left(y_0 \cos\sqrt{1-r^2}\, pt + \frac{ry_0 + \frac{\dot{y}_0}{p}}{\sqrt{1-r^2}} \sin\sqrt{1-r^2}\, pt\right) \tag{2.1.14}$$

and

$$\bar{\dot{y}} = e^{-rpt}\left(\dot{y}_0 \cos\sqrt{1-r^2}\, pt - \frac{r\dot{y}_0 + py_0}{\sqrt{1-r^2}} \sin\sqrt{1-r^2}\, pt\right) \tag{2.1.15}$$

: free vibration without damping, the equation of motion can further be

to

$$+ p^2 y = 0 \tag{2.1.16}$$

lution is

$$\bar{y} = y_0 \cos pt + \frac{\dot{y}_0}{p} \sin pt \tag{2.1.17}$$

$$\bar{\dot{y}} = \dot{y}_0 \cos pt - p y_0 \sin pt \tag{2.1.18}$$

ebra of Matrices

This is applicable to the first and third groups of methods provided that placement, velocity and acceleration at the end of any time interval can be ed in a linear form in terms of the displacement, velocity, acceleration ne other parameters at the beginning of the time interval. For example,

$$\left.\begin{aligned} y_1 &= a_{11} y_0 + a_{12} \dot{y}_0 + a_{13} \ddot{y}_0 + a_{14} \\ \dot{y}_1 &= a_{21} y_0 + a_{22} \dot{y}_0 + a_{23} \ddot{y}_0 + a_{24} \\ \ddot{y}_1 &= a_{31} y_0 + a_{32} \dot{y}_0 + a_{33} \ddot{y}_0 + a_{34} \end{aligned}\right\}, \tag{2.2.1}$$

rix form representing these linear simultaneous equations can be written as

$$\begin{bmatrix} y_1 \\ \dot{y}_1 \\ \ddot{y}_1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ \dot{y}_0 \\ \ddot{y}_0 \\ 1 \end{bmatrix} \tag{2.2.2}$$

, or, in more abbreviated

$$[y_1] = [A][y_0]. \tag{2.2.3}$$

the numerical solution is carried through n successive steps of equal time ion, with the final displacement function of a previous time interval be- g the initial condition of a new time interval, the matrix [A] operates on f n times, so that

$$[y_n] = [A]^n [y_0]. \tag{2.2.4}$$

: $[A]^n$ can be expanded by means of the Cayley-Hamilton theorem and
's theorem as soon as the characteristic roots are obtained.

The characteristic roots of the matrix $[A]$ not only gives the expansion but also determines the criterion of stability exactly as in the finite ce method described in the preceding article. The presence of a pair of :e complex characteristic roots signifies stability and periodic response numerical solution while all real roots indicate that the response is a-
.c.

The method may become very tedious in the case of forced vibrations since esence of more than two characteristic roots in the matrix will add too much raic work to the simplification process. The method of analysis by finite rence equations is preferrable in this case.

III. ANALYSIS OF AVAILABLE TECHNIQUES

3.1 Acceleration Methods

3.1.1 Constant Acceleration Method

The basic assumption of this method is that the acceleration of the mass in motion remains unchanged throughout a small time interval and is equal in direction and magnitude to the acceleration at the starting point of the concerning time interval. The assumption is a rough one, and provides a rapid but inaccurate procedure. The error in this method is so large that it is seldom used. A slight modification and a little more work improve the results considerably. The advantage of speed of operation cannot compensate for the loss in accuracy.

Let us consider first a single mass in free vibration without damping. Then from elementary mechanics we obtain the following expressions:

$$\dot{y}_{n+1} = \dot{y}_n + \ddot{y}_n h \tag{3.1.1.1}$$

$$y_{n+1} = y_n + (\dot{y}_n + \dot{y}_{n+1})\frac{h}{2}$$

$$= y_n + \dot{y}_n h + \ddot{y}_n \frac{h^2}{2} \tag{3.1.1.2}$$

where h denotes the time interval.

Now the differential equation of motion for a body in free vibration without damping is

$$\ddot{y} + p^2 y = 0 \tag{2.1.16}$$

from which the relation

$$\ddot{y}_n = -p^2 y_n \tag{3.1.1.3}$$

is obtained. Substituting in Eqs. (3.1.1.1) and (3.1.1.2), we get

$$\dot{y}_{n+1} = -p^2 h y_n + \dot{y}_n \tag{3.1.1.4}$$

and

$$y_{n+1} = \left(1 - \frac{p^2 h^2}{2}\right) y_n + h\dot{y}_n \quad . \tag{3.1.1.5}$$

From these relations of displacement and velocity, one gets a finite difference

ation in terms of displacements corresponding to the computed results from the stant acceleration method:

$$y_{n+1} - 2\left(1 - \frac{p^2h^2}{4}\right)y_n + \left(1 + \frac{p^2h^2}{2}\right)y_{n-1} = 0 \tag{3.1.1.6}$$

The solution of this finite difference equation, together with Eq. .1.1.5), yields the general equation for the displacement predicted by the pproximate solution:

$$y_n = \left(1 + \frac{p^2h^2}{2}\right)^{\frac{n}{2}}\left(y_o \cos n\mu + \frac{\frac{\dot{y}_o}{p} - \frac{phy_o}{4}}{\sqrt{1 - \frac{p^2h^2}{16}}} \sin n\mu\right) \tag{3.1.1.7}$$

here $$\mu = \arcsin ph\sqrt{\frac{1 - \frac{p^2h^2}{16}}{1 + \frac{p^2h^2}{2}}} = \arccos \frac{1 - \frac{p^2h^2}{4}}{\sqrt{1 + \frac{p^2h^2}{2}}} \tag{3.1.1.8}$$

omparing this with Eq. (2.1.17) of the exact solution, it is obvious that when the time interval h is very small, this approximate method approaches the exact solution as a limit. Since the time interval is not zero, there is an error in the procedure. We split the resulting error into two parts, one is the error in frequency or period, the other is in amplitude. The phase angle nμ should be equal to pt if the solution is exact. In other words, the exact value of μ should be pt/n or ph. Hence we obtain a relationship between the pseudo period of the numerical solution and the true period of the exact solution, i.e.

$$\frac{T_s}{T} = \frac{ph}{\mu} \tag{3.1.1.9}$$

The frequency has an error of 3 percent for ph = 0.5 and of 10 percent for ph = 1.0.

The error in amplitude is objectionably large since the computed displacement is subjected to a magnifying factor $\left(1 + \frac{p^2h^2}{2}\right)^{\frac{n}{2}}$ which diverges rapidly with the number of steps of operations n. This source of error is dominant although there also exist some other errors in the coefficients of y_o and $\dot{y}_o$. The coefficient of y_o becomes $\left(1 + \frac{p^2h^2}{2}\right)^{\frac{n}{2}}\left(1 - \frac{ph}{4\sqrt{1 - \frac{p^2h^2}{16}}} \tan n\mu\right)$ instead of 1 and varies as a function of n and ph. The coefficient of $\dot{y}_o$ becomes $\frac{\left(1 + \frac{p^2h^2}{2}\right)^{\frac{n}{2}}}{p\sqrt{1 - \frac{p^2h^2}{16}}}$ instead of 1/p, which also shows a rapid divergence of amplitude. Fig. 1

lustrates the rapid divergence of the envelope of amplitude for a single moving ass subject to $y_o=0$ and $\dot{y}_o=p$.

The criterion of stability shows that ph should be less than 4. Any alue of ph greater than 4 yields aperiodic response of displacement and velocity.

No criterion of convergence exists for this method since one operation is sufficient for each step since no repetition or trial necessary.

For free vibration with viscous damping, the analysis is more complicated since it involves one more parameter r, the damping factor of the motion. The difference equation now becomes

$$y_{n+1} - 2\left(1-\frac{rph}{2}-\frac{p^2h^2}{4}\right)y_n + (1-rph)\left(1+\frac{p^2h^2}{2}\right)y_{n-1} = 0 \qquad (3.1.1.10)$$

and its solution is

$$y = (1-rph)^{\frac{n}{2}}\left(1+\frac{p^2h^2}{2}\right)^{\frac{n}{2}}\left[y_o\cos n\mu + \frac{y_o\left(\frac{r}{2}-\frac{ph}{4}\right)+\frac{\dot{y}_o}{p}(1-rph)}{\sqrt{1-\frac{r^2}{4}-\frac{3r}{4}ph-\frac{p^2h^2}{16}}}\sin n\mu\right] \qquad (3.1.1.11)$$

where $$\mu = \arcsin ph\sqrt{\frac{1-\frac{r^2}{4}-\frac{3r}{4}ph-\frac{p^2h^2}{16}}{(1-rph)\left(1+\frac{p^2h^2}{2}\right)}} = \arccos\frac{1-\frac{rph}{2}-\frac{p^2h^2}{4}}{\sqrt{(1-rph)\left(1+\frac{p^2h^2}{2}\right)}}$$

The ratio of pseudo period to true period becomes

$$\frac{T_s}{T} = \frac{ph\sqrt{1-r^2}}{\mu} \qquad (3.1.1.12)$$

Comparison of amplitudes may be made from Eqs. (3.1.1.11) and (2.1.14). Figs. 2 and 3 shows the comparisons of period and amplitude for different damping factor r.

Two kinds of comparison in amplitude are made for all techniques described in this thesis. The first one deals with the magnitude of the envelope of vibration at pt = 1, regardless of the magnifying effect of the sine term in the general expression. In other words, this is a comparison of the exponential factor which multiplies the solution. The purpose of this comparison is to demonstrate the rate of divergence or convergence and then to judge its applicability. This ratio varies exponentially with pt, and therefore the amplitude ratio at any instant of time may be found by its exponential relation with the

io at pt = 1.

Another comparison deals with the peak amplitudes in the first cycle of ration due to an initial velocity $\dot{y}_o = p$. The first peak amplitude occurs at = $\pi/2$ theoretically, but it may deviate from the true value in numerical solu- ons due to the error in period which therefore plays an important role in the eudo peak amplitudes. This kind of comparison may give a better picture both of e actual oscillatory motion and of the pseudo motion derived from the approxi- ate techniques.

The criterion for stability of the constant acceleration method is xpressed by the following equation

$$p^2h^2 + 12\,rph - 16\left(1 - \frac{r^2}{4}\right) = 0 \,. \qquad (3.1.1.13)$$

his shows that when $ph > (4\sqrt{2r^2+1} - 6r)$, the computed displacement of notion is aperiodic which is not true for r less than 1. For the critical damping condition, i.e. r = 1, ph should be made equal to or greater than 0.9282 in order to procure an aperiodic response.

The constant acceleration method is too crude in accuracy and therefore not much used in practice. It is only accurace to the second order of h and errors may arise from the third power of the time interval since

$$y_1 = \left(1 - \frac{p^2h^2}{2}\right) y_o + (1 - rph)h\dot{y}_o \qquad (3.1.1.14)$$

3.1.2 Timoshenko's Modified Acceleration Method

This is an improved method obtained from the last one by modifying the acceleration of the moving body. The acceleration here is assumed constant throughout the time interval and equal to the average of its initial and final values. The elementary equations of motion are therefore

$$\dot{y}_{n+1} = \dot{y}_n + \frac{1}{2}(\ddot{y}_n + \ddot{y}_{n+1})h \qquad (3.1.2.1)$$

$$= y_n + \frac{1}{2}(\dot{y}_n + \dot{y}_{n+1})h = y_n + \dot{y}_n h + \frac{1}{4}\ddot{y}_n h^2 + \frac{1}{4}\ddot{y}_{n+1} h^2 \qquad (3.1.2.2)$$

be noted that the above equations may not be consistent. When comtined differential equation of free vibration without damping, these equations

$$+ \frac{p^2h^2}{4})\dot{y}_{n+1} = -p^2 h y_n + (1 - \frac{p^2h^2}{4})\dot{y}_n \qquad (3.1.2.3)$$

$$1 + \frac{p^2h^2}{4})y_{n+1} = (1 - \frac{p^2h^2}{4})y_n + h\dot{y}_n \qquad (3.1.2.4)$$

ference equation in terms of displacement now becomes

$$y_{n+1} - (2 - \frac{p^2h^2}{1 + \frac{p^2h^2}{4}})y_n + y_{n-1} = 0 \qquad (3.1.2.5)$$

s solution is

$$y_n = y_0 \cos n\mu + \frac{\dot{y}_0}{p} \sin n\mu \qquad (3.1.2.6)$$

$$\mu = \arcsin \frac{ph}{1 + \frac{p^2h^2}{4}} = \arccos \frac{1 - \frac{p^2h^2}{4}}{1 + \frac{p^2h^2}{4}} \qquad (3.1.2.7)$$

Similarly it can also be shown that

$$\dot{y}_n = \dot{y}_0 \cos n\mu - p y_0 \sin n\mu \qquad (3.1.2.8)$$

This approximate solution has no error in amplitude; neither the initial placement nor the initial velocity produces errors which would affect the itude of displacement or velocity thereafter computed. (See Fig. 4) The error arising in this method is due to the difference of phase-angles or discrepancy in period or frequency which can be expressed by the equation

$$\frac{T_s}{T} = \frac{ph}{\arcsin \frac{ph}{1 + \frac{p^2h^2}{4}}} \qquad (3.1.2.9)$$

is plotted in Fig. 5.

Another advantage of this method is that no criterion for stability need mposed. The length of the time interval can be chosen corresponding to the racy desired. Unfortunately, the error in period is so large that even a interval of about 1/6 of the natural period will result an 8 percent error requency.

If viscous damping is considered, one can obtain the following equations

$$+\left(1-rph-\frac{p^2h^2}{4}\right)\dot{y}_n \qquad (3.1.2.10)$$

$$h-\frac{p^2h^2}{4}\Big)y_n+h\dot{y}_n \qquad (3.1.2.11)$$

splacement

$$^2_{\frac{p^2h^2}{4}}\Big)y_n+\left(1-\frac{rph}{1+\frac{p^2h^2}{4}}\right)y_{n-1}=0 \qquad (3.1.2.12)$$

$$y_0\cos n\mu+\frac{ry_0+\frac{\dot{y}_0}{p}}{\sqrt{1-r^2}}\sin n\mu\Big) \qquad (3.1.2.13)$$

$$\frac{}{-r^2p^2h^2}}=\arccos\frac{1-\frac{p^2h^2}{4}}{\sqrt{\left(1+\frac{p^2h^2}{4}\right)^2-r^2p^2h^2}} \qquad (3.1.2.14)$$

e stability may be expressed as $1-r^2>0$. This
nt of the natural frequency and time interval, i.e. there
h affects the stability of response. The stability criterion
criterion of critical damping.
g Eq. (3.1.2.13) with the exact solution, Eq. (2.1.14), one
th period and amplitude errors exist in the approximate solu-
of pseudo to actual values of periods and amplitudes are as

$$\frac{\arcsin\frac{ph\sqrt{1-r^2}}{\sqrt{\left(1+\frac{p^2h^2}{4}\right)^2-r^2p^2h^2}}}{ph\sqrt{1-r^2}} \qquad (3.1.2.15)$$

$$=\left(\frac{1-rph+\frac{p^2h^2}{4}}{1+rph+\frac{p^2h^2}{4}}\right)^{\frac{r}{2ph}}\cdot e^{r} \qquad (3.1.2.16)$$

t pt=1

plotted for these equations in Figs. 5 and 6.

ewmark's Linear and Parabolic Acceleration Methods

These methods are based on the assumption that the variation of the
ration of the mass in motion is, respectively, linear or parabolic. The

city is determined from a definite integral of acceleration over the range of , and the displacement is, in turn, an integral of the velocity. The elemen- equations of motion are therefore consistent.

The linear acceleration method is first considered. The equations of ion are

$$\dot{y}_{n+1} = \dot{y}_n + \frac{h}{2}\ddot{y}_n + \frac{h}{2}\ddot{y}_{n+1} \qquad (3.1.3.1)$$

$$y_{n+1} = y_n + h\dot{y}_n + \frac{h^2}{3}\ddot{y}_n + \frac{h^2}{6}\ddot{y}_{n+1} \qquad (3.1.3.2)$$

In case of free vibration without damping, $\ddot{y} = -p^2 y$. The uations can be simplified to the form

$$\left(1+\frac{p^2h^2}{6}\right)y_{n+1} = \left(1-\frac{p^2h^2}{3}\right)y_n + h\dot{y}_n \qquad (3.1.3.3)$$

$$\left(1+\frac{p^2h^2}{6}\right)\dot{y}_{n+1} = -p^2h\left(1-\frac{p^2h^2}{12}\right)y_n + \left(1-\frac{p^2h^2}{3}\right)\dot{y}_n \ , \qquad (3.1.3.4)$$

rom which the difference equation

$$y_{n+1} - \left(2-\frac{p^2h^2}{1+\frac{p^2h^2}{6}}\right)y_n + y_{n-1} = 0 \qquad (3.1.3.5)$$

is derived. The solution of this difference equation is

$$y_n = y_0 \cos n\mu + \frac{\dot{y}_0}{p\sqrt{1-\frac{p^2h^2}{12}}}\sin n\mu \ , \qquad (3.1.3.6)$$

where $$\mu = \arcsin \frac{ph\sqrt{1-\frac{p^2h^2}{12}}}{1+\frac{p^2h^2}{6}} = \arccos \frac{1-\frac{p^2h^2}{3}}{1+\frac{p^2h^2}{6}} \ . \qquad (3.1.3.7)$$

Similarly, $$\dot{y}_n = -y_0 p\sqrt{1-\frac{p^2h^2}{12}}\cdot\sin n\mu + \dot{y}_0 \cos n\mu \qquad (3.1.3.8)$$

Making comparisons with the exact solution as before, the error in the approximate solution consists of two parts: the error in period and the error in amplitude. The error in period is expressed by the ratio

$$\frac{T_s}{T} = \frac{ph}{\arcsin \frac{ph\sqrt{1-\frac{p^2h^2}{12}}}{1+\frac{p^2h^2}{6}}} \qquad (3.1.3.9)$$

The amplitude error in this case is constant for a given time interval, it does not vary with the lapse of time and is solely a function of the initial velocity of the mass. If the mass starts from rest no amplitude error will occur. Hence the amplitude error depends on the proportion of initial displacement and velocity.

large for a motion due to a small initial displacement and large
and will be small for a motion due to a small initial velocity.
response of a single mass due to $y_o = 0$, and $\dot{y}_o = p$.
terion of stability is found to be $ph < \sqrt{12}$. Any value of ph
$\sqrt{12}$ will result in aperiodic response by the approximate solution.
case of free vibration with damping, the difference equation of
becomes

$$\left(\frac{h}{\frac{p^2h^2}{6}}\right)y_{n+1} - \left(2 - \frac{p^2h^2}{1+\frac{p^2h^2}{6}}\right)y_n + \left(1 - \frac{rph}{1+\frac{p^2h^2}{6}}\right)y_{n-1} = 0 \qquad (3.1.3.10)$$

is

$$\frac{1-rph+\frac{p^2h^2}{6}}{1+rph+\frac{p^2h^2}{6}}\Bigg)^{\frac{n}{2}}\left(y_o \cos n\mu + \frac{(1-\frac{p^2h^2}{6})ry_o + (1-\frac{r^2p^2h^2}{3})\frac{\dot{y}_o}{p}}{\sqrt{1-r^2-\frac{p^2h^2}{12}}}\sin n\mu\right) \qquad (3.1.3.11)$$

$$= \arcsin \frac{ph\sqrt{1-r^2-\frac{p^2h^2}{12}}}{\sqrt{(1+\frac{p^2h^2}{6})^2 - r^2p^2h^2}} = \arccos \frac{1-\frac{p^2h^2}{3}}{\sqrt{(1+\frac{p^2h^2}{6})^2 - r^2p^2h^2}} . \qquad (3.1.3.12)$$

rion for stability is now

$$1 - r^2 - \frac{p^2h^2}{12} = 0 \qquad \text{for } r < 1 \qquad (3.1.3.13)$$

Characteristics of the pseudo period and pseudo amplitude are shown on
nd 9.

The parabolic acceleration method differs from the above method in assum-
the acceleration has a parabolic variation; thus $\ddot{y}_{n+1} = \ddot{y}_n + k_1h + k_2h^2$.
dure of operation is similar to the linear acceleration method except
more initial condition is required to start with. That is, one needs
ously known steps to carry out a new step. The procedure may be started
the following ways:

1. Use linear acceleration for the starting interval.
2. Use linear acceleration for a starting interval which is only half
s the regular interval, then get a special parabolic acceleration in-
lf as long as the regular interval and proceed.

The equations of motion from elementary mechanics are

$$\dot{y}_n = \dot{y}_{n-1} + \frac{h}{12}(-\ddot{y}_{n-2} + 8\ddot{y}_{n-1} + 5\ddot{y}_n) \qquad (3.1.3.14)$$

$$y_n = y_{n-1} + \dot{y}_{n-1}h + \frac{h^2}{24}(-\ddot{y}_{n-2} + 10\ddot{y}_{n-1} + 3\ddot{y}_n) \qquad (3.1.3.15)$$

and the differential equation of an oscillatory motion without damping is $\ddot{y}+p^2y=0$. The difference equation corresponding to the above equations is

$$\left(1+\frac{p^2h^2}{8}\right)y_{n+2} - \left(2-\frac{17}{24}p^2h^2\right)y_{n+1} + \left(1+\frac{5}{24}p^2h^2\right)y_n - \frac{p^2h^2}{24}y_{n-1} = 0 \ . \qquad (3.1.3.16)$$

The solution of this equation can be expressed in the form of Eq. (2.1.5) which can also be written as

$$y_n = A\rho^n + R^n(B\cos n\mu + C\sin n\mu), \qquad (3.1.3.17)$$

where A, B and C are constants determined from initial conditions, ρ, R and μ are functions of ph. The value of ρ is small and terms containing it are of less importance than other terms of the expression. R is dominant since it is greater than 1 and of exponential power n as steps of time proceed. Therefore, the procedure is of a divergent nature and is not desirable over a long lapse of time. Furthermore, the criterion of stability $ph < 3.357758$ limits the applicability of long time intervals. Although the parabolic acceleration method is more accurate than the linear one as far as the first cycle of oscillation is concerned, the errors in amplitude enlarge rapidly from the second cycle onward. (See Fig. 10.) Figs. 11 and 12 show how T_s/T and A_s/A vary with ph, neglecting the first term of Eq. (3.1.3.17).

An attempt has been made to improve the accuracy of the linear acceleration method and to lessen the work of computation of the parabolic acceleration method by applying linear and parabolic accelerations alternately in successive steps, i.e., using linear acceleration in the first step, parabolic in the second, linear in the third, parabolic in the fourth, and so on. The result, as one may expect, turns out to be intermediate between the two methods. The difference equation for the displacements at an even number of steps in the case of free

vibration with no damping is written as

$$y_{2(n+1)} - \frac{2\left(1-\frac{41}{24}p^2h^2+\frac{5}{24}p^4h^4\right)}{\left(1+\frac{p^2h^2}{8}\right)\left(1+\frac{p^2h^2}{6}\right)} y_{2n} + \frac{1+\frac{7}{12}p^2h^2+\frac{97}{576}p^4h^4+\frac{7}{288}p^6h^6+\frac{1}{768}p^8h^8}{\left(1+\frac{p^2h^2}{8}\right)^2\left(1+\frac{p^2h^2}{6}\right)^2} y_{2(n-1)} = 0 \qquad (3.1.3.18)$$

and its solution is

$$y_{2n} = R^n\left[y_0 \cos n\mu + \frac{p^3h^3y_0 + 12(8-3p^2h^2)\frac{\dot{y}_0}{p}}{\sqrt{9216-7296p^2h^2+1696p^4h^4-97p^6h^6}} \sin n\mu\right] \qquad (3.1.3.19)$$

where
$$\mu = \arcsin ph\sqrt{\frac{9216-7296p^2h^2+1696p^4h^4-97p^6h^6}{2304+1344p^2h^2+388p^4h^4+56p^6h^6+3p^8h^8}}$$

$$= \arccos \frac{2(24-41p^2h^2+5p^4h^4)}{\sqrt{2304+1344p^2h^2+388p^4h^4+56p^6h^6+3p^8h^8}} \qquad (3.1.3.20)$$

and
$$R = \frac{\sqrt{2304+1344p^2h^2+388p^4h^4+56p^6h^6+3p^8h^8}}{(8+p^2h^2)(6+p^2h^2)} \qquad (3.1.3.21)$$

The values of T_s/T and A_s/A are plotted against ph in Figs. 11 and 12. It is evident that the propagation of error is divergent. The criterion of stability of this alternate linear-parabolic acceleration method for free vibration without damping is found to be ph = 1.6171 which is quite unfavorable for long time intervals.

3.1.4 Newmark's β-Method

This is a generalization of first degree acceleration methods obtained by introducing a parameter β into the elementary equations of kinematics. Thus

$$\dot{y}_{n+1} = \dot{y}_n + \frac{h}{2}\ddot{y}_n + \frac{h}{2}\ddot{y}_{n+1} \qquad (3.1.4.1)$$

$$y_{n+1} = y_n + \dot{y}_n h + \left(\frac{1}{2}-\beta\right)\ddot{y}_n h^2 + \beta\ddot{y}_{n+1}h^2 \qquad (3.1.4.2)$$

It is obvious here that this is equivalent to Timoshenko's modified method when β = 1/4 and to Newmark's linear acceleration method when β = 1/6.

In the case of free vibration without damping, the difference equation of displacements appears as

$$y_{n+1} - \left(2-\frac{p^2h^2}{1+\beta p^2h^2}\right)y_n + y_{n-1} = 0. \qquad (3.1.4.3)$$

ifference equation becomes

$$\frac{1}{\sqrt{1-(\frac{1}{4}-\beta)p^2h^2}}\sin\eta\mu \qquad (3.1.4.4)$$

$$\frac{(\frac{1}{4}-\beta)p^2h^2}{\beta p^2h^2} = \arccos\frac{1-(\frac{1}{2}-\beta)p^2h^2}{1+\beta p^2h^2} \qquad (3.1.4.5)$$

pseudo period with the true period, it can be seen that

pressed in a series form of

$$3-1)p^2h^2-\frac{1}{5760}(720\beta^2-120\beta+17)p^4h^4+---- \qquad (3.1.4.6)$$

1/12, the ratio T_s/T will be closest to unity for any

1. This means that $\beta = 1/12$ will give the best result in

ed amplitude is neither divergent nor convergent as time pro-

some error is involved in the response to an initial velocity.

ws that the term containing y_o does not contain any error in

e one with $\dot{y}_o$ is amplified by the factor $\frac{1}{\sqrt{1-(\frac{1}{4}-\beta)p^2h^2}}$

endent on β and ph and is free from influence of the proceeding

lustrates the variation of the velocity amplification factor

$\sqrt{p^2h^2}$ with ph for different values of .

erion for stability is

$$\frac{p^2h^2}{\beta p^2h^2} < 4 \qquad (3.1.4.7)$$

s condition is always fulfilled for any value of ph and therefore

erion need be imposed. With $\beta > 1/4$, no ph can possibly satisfy

With $\beta < 1/4$, the larger β, the longer the time interval which can

= 0, the critical ph is equal to 2.

terion for convergence is

$$ph < \frac{1}{\sqrt{\beta}} \qquad (3.1.4.8)$$

a larger β permits a smaller applicable time interval. For $\beta = 0$

ocedure immediately converges regardless of the time interval used.

For problems of free vibration with viscous damping, the difference equa-

of the procedure may be written as

$$\left(1+\frac{rph}{1+\beta p^2h^2}\right)y_{n+1}-\left(2-\frac{p^2h^2}{1+\beta p^2h^2}\right)y_n+\left(1-\frac{rph}{1+\beta p^2h^2}\right)y_{n-1}=0. \qquad (3.1.4.9)$$

solution is

$$y=\left(\frac{1-rph+\beta p^2h^2}{1+rph+\beta p^2h^2}\right)^{\frac{n}{2}}\left(y_0\cos n\mu+\frac{\left[1-\frac{1}{2}(1-4\beta)p^2h^2\right]ry_0+\left[1-(1-4\beta)r^2p^2h^2\right]\frac{\dot{y}_0}{p}}{\sqrt{1-r^2-(\frac{1}{4}-\beta)p^2h^2}}\sin n\mu\right) \qquad (3.1.4.10)$$

e

$$\mu=\arcsin ph\sqrt{\frac{1-r^2-(\frac{1}{4}-\beta)p^2h^2}{(1+\beta p^2h^2)^2-r^2p^2h^2}}$$

$$=\arccos\frac{1-(\frac{1}{2}-\beta)p^2h^2}{\sqrt{(1+\beta p^2h^2)^2-r^2p^2h^2}} \qquad (3.1.4.11)$$

Now the error in period is not only dependent on β and ph, but also on r.

ain, if the ratio T_s/T is expressed in a series form as

$$\frac{T_s}{T}=\frac{ph\sqrt{1-r^2}}{\mu}=1+\frac{(12\beta-1)-4(6\beta+1)r^2+8r^3}{24(1-r^2)}p^2h^2+\cdots\cdots \qquad (3.1.4.12)$$

t will be found that $\beta=(1+4r^2-8r^4)/12(1-2r^2)$ (3.1.4.13)

s the best value as far as period is concerned.

The error in amplitude is dominated by the exponential factor multiplying

he whole expression in Eq. (3.1.4.10) particularly after a long sequence of time.

eglecting the coefficients which combine with y_0 and $\dot{y}_0$ in the expression, one

ay compare amplitudes by taking the ratio

$$\left(\frac{A_s}{A}\right)_{at\ pt=1}=e^r\left(\frac{1-rph+\beta p^2h^2}{1+rph+\beta p^2h^2}\right)^{\frac{1}{2ph}} \qquad (3.1.4.14)$$

or best agreement in amplitude, $\beta=\frac{r^2}{3}$.

The criterion for stability is

$$p^2h^2<\frac{4(1-r^2)}{1-4\beta} \qquad (3.1.4.15)$$

or $r<1$. Except when $\beta = 1/4$, the numerical procedure does not present an agree-

ent on the criterion of critical damping, greater discrepancy usually occurs for

arger time intervals. Critical damping may occur in the numerical procedure even or low values of r if too long a time interval is taken.

The criterion for convergence is

$$rph + \beta p^2h^2 < 1 \qquad (3.1.4.16)$$

There is one more restriction of this method in the viscous damping problem. A degenerate case of the difference equation will occur if $\frac{rph}{1+\beta p^2h^2}$ is equal to +1 or -1. Under this condition the method is not workable. However, when

$$\frac{rph}{1+\beta p^2h^2} = 1$$

i.e.

$$rph + \beta p^2h^2 = 1 + 2\beta p^2h^2 ,$$

we see that the convergence criterion is violated. On the other hand, when

$$\frac{rph}{1+\beta p^2h^2} = -1$$

the spring constant is negative.

The β-method is also applicable to forced vibrations as represented by Eq. (2.1.1) with good accuracy. The error due to the presence of a forcing function F(t) may be seen by comparison with the exact particular solution. Now, let $\bar{y}_p$ be the particular solution of the given differential equation of motion, Eq. (2.1.1), and let y_p be the corresponding solution obtained by the numerical procedure. Analysis is made for an undamped system of single mass subject to an applied force F(t) as follows:

Given the equation of motion

$$\ddot{y} + p^2y = F(t) \qquad (3.1.4.17)$$

the exact solution is

$$\bar{y} = (y_o - y_{po})\cos pt + \frac{\dot{y}_o - \dot{y}_{po}}{p}\sin pt + \bar{y}_p \qquad (3,1.4.18)$$

where $\bar{y}_p$ is the particular solution.

The corresponding difference equation when using the β-method is found to be

$$y_{n+1} - \left(2 - \frac{p^2h^2}{1+\beta p^2h^2}\right)y_n + y_{n-1} = \frac{\beta h^2}{1+\beta p^2h^2}\left[F(t_{n+1}) - \left(2 - \frac{1}{\beta}\right)F(t_n) + F(t_{n-1})\right] \qquad (3.1.4.19)$$

and its solution is

$$y_n = A\cos n\mu + B\sin n\mu + y_p \qquad (3.1.4.20)$$

where y_p is the particular solution of the difference equation and

$$\mu = \arcsin \frac{ph\sqrt{1-(\frac{1}{4}-\beta)p^2h^2}}{1+\beta p^2h^2} = \arccos \frac{1-(\frac{1}{2}-\beta)p^2h^2}{1+\beta p^2h^2}$$

$$A = y_0 - y_{p0}$$

and $$B = \frac{1}{p\sqrt{1+(\beta-\frac{1}{4})p^2h^2}}\left\{\dot{y}_0 + \frac{h}{2}\left[F(t_0)-p^2y_{p0}\right]+\beta h\left[F(t_1)-F(t_0)\right]-\frac{(1+\beta p^2h^2)}{h}\left[y_{p1}-y_{p0}\right]\right\}$$

Comparing Eqs. (3.1.4.18) and (3.1.4.20), we see that when $h\to 0$, if $y_p \to \bar{y}_p$, the numerical solution will approach the exact solution. The following comparisons are made for different kind of forcing functions:

(a) The forcing function is a polynomial in time, i.e.

$$F(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots\cdots \qquad (3.1.4.21)$$

then
$$\begin{aligned}\bar{y}_p = \frac{1}{p^2}\Big\{&\left[a_0 - \frac{2\cdot1\,a_2}{p^2} + \frac{4\cdot3\cdot2\cdot1\,a_4}{p^4} - \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1\,a_6}{p^6} + \cdots\right]\\ &+\left[a_1 - \frac{3\cdot2\,a_3}{p^2} + \frac{5\cdot4\cdot3\cdot2\,a_5}{p^4} - \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\,a_7}{p^6} + \cdots\right]t\\ &+\left[a_2 - \frac{4\cdot3\,a_4}{p^2} + \frac{6\cdot5\cdot4\cdot3\,a_6}{p^4} - \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\,a_8}{p^6} + \cdots\right]t^2\\ &+\left[a_3 - \frac{5\cdot4\,a_5}{p^2} + \frac{7\cdot6\cdot5\cdot4\,a_7}{p^4} - \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\,a_9}{p^6} + \cdots\right]t^3\\ &+ \text{----------}\Big\}\end{aligned} \qquad (3.1.4.22)$$

$$\begin{aligned}y_p = \frac{1}{p^2}\Big\{&\left[a_0 - \frac{2\cdot1\,k_1a_2}{p^2} + \frac{4\cdot3\cdot2\cdot1\,k_2a_4}{p^4} - \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1\,k_3a_6}{p^6} + \cdots\right]\\ &+\left[a_1 - \frac{3\cdot2\,k_1a_3}{p^2} + \frac{5\cdot4\cdot3\cdot2\,k_2a_5}{p^4} - \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\,k_3a_7}{p^6} + \cdots\right]t\\ &+\left[a_2 - \frac{4\cdot3\,k_1a_4}{p^2} + \frac{6\cdot5\cdot4\cdot3\,k_2a_6}{p^4} - \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\,k_3a_8}{p^6} + \cdots\right]t^2\\ &+\left[a_3 - \frac{5\cdot4\,k_1a_5}{p^2} + \frac{7\cdot6\cdot5\cdot4\,k_2a_6}{p^4} - \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\,k_3a_9}{p^6} + \cdots\right]t^3\\ &+ \text{----------}\Big\}\end{aligned} \qquad (3.1.4.23)$$

$$k_1 = 1$$

$$k_2 = 1 + \frac{1}{12}(12\beta - 1)p^2h^2$$

$$k_3 = 1 + \frac{1}{6}(12\beta - 1)p^2h^2 + \frac{1}{360}(360\beta^2 - 60\beta + 1)p^4h^4$$

$$k_4 = 1 + \frac{1}{4}(12\beta - 1)p^2h^2 + \frac{1}{80}(80\beta^2 - 40\beta + 1)p^4h^4 + \frac{1}{20{,}160}(20{,}160\beta^3 - 5{,}040\beta^2 + 252\beta - 1)p^6h^6$$

$$k_5 = 1 + \frac{1}{3}(12\beta - 1)p^2h^2 + \frac{1}{740}(4{,}440\beta^2 - 740\beta + 1)p^4h^4 + \frac{1}{30{,}240}(120{,}960\beta^3 - 30{,}240\beta^2 + 1{,}764\beta - 17)p^6h^6$$
$$+ \frac{1}{1{,}814{,}400}(1{,}814{,}400\beta^4 - 604{,}800\beta^3 + 52{,}920\beta^2 - 1{,}020\beta + 1)p^8h^8$$

From the above we see that the numerical solution is exact for a third

e polynomial at any value of β, and for a fifth degree polynomial when β =

(b) The forcing function is trigonometric, i.e.

$$F(t) = A \sin at + B \cos bt \tag{3.1.4.24}$$

n $$\bar{y}_p = \frac{A}{p^2 - a^2} \sin at + \frac{B}{p^2 - b^2} \cos bt \tag{3.1.4.25}$$

$$y_p = \frac{A \sin at}{p^2 - \frac{2\sin^2 ah}{(1 + \cos ah - 2\beta \sin^2 ah)h^2}} + \frac{B \cos bt}{p^2 - \frac{2\sin^2 bh}{(1 + \cos bh - 2\beta \sin^2 bh)h^2}} \tag{3.1.4.26}$$

(c) Exponential forcing functions, as

$$F(t) = a^t \tag{3.1.4.27}$$

en $$\bar{y}_p = \frac{a^t}{p^2 + (\log a)^2} \tag{3.1.4.28}$$

$$y_p = \frac{a^t}{p^2 + \frac{a^{2h} - 2a^h + 1}{\beta h^2 [a^{2h} + (\frac{1}{\beta} - 2)a^h + 1]}} \tag{3.1.4.29}$$

In all cases above, when $h \to 0$, $y_p \to \bar{y}_p$, and therefore the numerical solu-

on approaches the exact solution as a limit.

The β-method may also be applied to other form of motion than the one re-

esented by Eq. (2.1.1). Consider the motion prescribed by the linear differen-

al equation

$$\dot{y} - y = t \tag{3.1.4.30}$$

n of which is known to be

$$t - (t+1) \qquad (3.1.4.31)$$

stant determined from the initial conditions. Applying the β-

problem, one finds the difference equation of displacement to be

$$)y_n - \tfrac{1}{4}[2+2(1-2\beta)h+(1-4\beta)h^2]y_{n-1} = \frac{h^2}{4}+\frac{h}{4}[2+(1-4\beta)h]t_{n-1} \qquad (3.1.4.32)$$

al solution is

$$: \left(1+h+\frac{h^2}{2-4\beta h}\right)^n - (t+1) \qquad (3.1.4.33)$$

ue of β for this case is 1/6.

s of Finite Differences

y's Method

. method proposed by Levy replaces the second derivative $\ddot{y}_n$ in the equa-

ption by finite differences, $\frac{1}{h^2}(y_{n+1} - 2y_n + y_{n-1})$. For a free

with no damping prescribed by the given differential equation

$$\ddot{y} + p^2 y = 0 \qquad (2.1.16)$$

tuting $\ddot{y}_n = \frac{1}{h^2}(y_{n+1} - 2y_n + y_{n-1})$, one obtains

$$y_{n+1} - (2 - p^2h^2)y_n + y_{n-1} = 0 \qquad (3.2.1.1)$$

viously identical with Newmark's generalized acceleration method for β=0,

at the treatment of initial velocity is different. The general solution

.2.1) is

$$y_n = A\cos n\mu + B\sin n\mu \qquad (3.2.1.2)$$

$$\mu = \arcsin ph\sqrt{1-\frac{p^2h^2}{4}} = \arccos\left(1-\frac{p^2h^2}{2}\right) \qquad (3.2.1.3)$$

B are constants determined from initial conditions, with $ph < 2$ as

criterion. There is no difficulty in finding A which is always equal

t trouble arises in the determination of B which depends on the

erpretation of the initial velocity, or in other words, on the way in which y_1 obtained from the known values of y_o and $\dot{y}_o$. When $\dot{y}_o = 0$, one may assume that $= y_{-1}$, then solve simultaneously with Eq. (3.2.1.1) to get y_1. Otherwise, if $\neq 0$, some other assumption must be used.

If one begins with Newmark's β-method of β = 0 for the first step in order obtain y_1, then

$$y_1 = \left(1 - \frac{p^2h^2}{2}\right)y_o + h\dot{y}_o \tag{3.2.1.4}$$

d the result will be the same as Newmark's β-method for β = 0, i.e.,

$$B = \frac{\dot{y}_o}{p\sqrt{1 - \frac{p^2h^2}{4}}} \tag{3.2.1.5}$$

If, taking the formula of elementary mechanics

$$y_1 = y_o + h\dot{y}_o \tag{3.2.1.6}$$

e obtain

$$B = \frac{\frac{\dot{y}_o}{p} + \frac{ph}{2}y_o}{\sqrt{1 - \frac{p^2h^2}{4}}} \tag{3.2.1.7}$$

The result is, of course, less accurate.

On taking $y_{\frac{1}{2}} = y_o + \frac{h}{2}\dot{y}_o$ for the first half time interval and getting y_1 from the difference equation of the half time interval $y_1 - \left(2 - \frac{p^2h^2}{4}\right)y_{\frac{1}{2}} + y_o = 0$, we obtain

$$B = \frac{\left(1 - \frac{p^2h^2}{8}\right)\frac{\dot{y}_o}{p} + \frac{ph}{4}y_o}{\sqrt{1 - \frac{p^2h^2}{4}}} \tag{3.2.1.8}$$

This result is generally better than that of Eq. (3.2.1.7).

On assuming $y_{-1} = 0$ and solving for y_1,

$$B = \frac{\left(1 - \frac{p^2h^2}{2}\right)y_o}{ph\sqrt{1 - \frac{p^2h^2}{4}}} . \tag{3.2.1.9}$$

This is only true when $\dot{y}_o = \frac{\left(1 - \frac{p^2h^2}{2}\right)y_o}{h\sqrt{1 - \frac{p^2h^2}{4}}}$, and is only used when $\dot{y}_o$ is uncertain.

In viscous damping problems, by replacing $\dot{y}_n$ by $\frac{1}{2h}\left(y_{n+1} - y_{n-1}\right)$ and $\ddot{y}_n$ by $\frac{1}{h^2}\left(y_{n+1} - 2y_n + y_{n-1}\right)$ in the differential equation of motion,

$$\ddot{y} + 2rp\dot{y} + p^2y = 0 , \tag{2.1.13}$$

one may obtain the difference equation

$$(1 + rph)y_{n+1} - (2 - p^2h^2)y_n + (1 - rph)y_{n-1} = 0 \tag{3.2.1.10}$$

ch is idential to Eq. (3.1.4.9) for $\beta = 0$. The result will be the same as that Newmark's generalized acceleration method for $\beta = 0$ with exception of the treat-t of initial velocity. The discussion is therefore not repeated here.

For comparison of pseudo and true periods and amplitudes of this method, e Figs. 14 and 15.

2.2 Salvadori's Method

This is an application of a procedure due to Fox vibration problems. The cond derivative $\ddot{y}$ is replaced by the first two terms of the central difference pansion,

$$\ddot{y} = \frac{1}{h^2}\left(\Delta^2 - \frac{\Delta^4}{12}\right) y \qquad (3.2.2.1)$$

here Δ^2 and Δ^4 are the second and fourth central differences of y

$$\Delta^2 y_n = y_{n+1} - 2y_n + y_{n-1}$$

$$\Delta^4 y_n = y_{n+2} - 4y_{n+1} + 6y_n - 4y_{n-1} + y_{n-2}$$

perating then with $\left(1 + \frac{\Delta^2}{12}\right)$ on each term of the equation

$$\ddot{y} + p^2 y = 0$$

Dropping the sixth-difference terms, the equation may be simplified to

$$y_{n+1} - \left(2 - \frac{p^2h^2}{1 + \frac{p^2h^2}{12}}\right) y_n + y_{n-1} = 0 \qquad (3.2.2.2)$$

which is obviously the same as Newmark's generalized acceleration method for $\beta = 1/12$. (See Eq. (3.1.4.3).) The solution is in the form

$$y_n = A \cos n\mu + B \sin n\mu \qquad (3.2.2.3)$$

where

$$\mu = \arcsin \frac{ph\sqrt{1 - \frac{p^2h^2}{6}}}{1 + \frac{p^2h^2}{12}} = \arccos \frac{1 - \frac{5p^2h^2}{12}}{1 + \frac{p^2h^2}{12}} \qquad (3.2.2.4)$$

The constants A and B are determined from the initial conditions. For free vibrations, A is always equal to y_0, while B depends on the way in which y_1 is obtained from the initial velocity $\dot{y}_0$.

If the procedure is started with Newmark's β-method for $\beta = 1/12$ until y_1

is obtained, that is if

$$y_1 = \frac{1}{1+\frac{p^2h^2}{12}}\left[\left(1-\frac{5}{12}p^2h^2\right)y_0 + h\dot{y}_0\right] , \qquad (3.2.2.5)$$

the result will be the same as that given by Newmark's β-method for β = 1/12.

$$\beta = \frac{\dot{y}_0}{p\sqrt{1-\frac{p^2h^2}{6}}} \qquad (3.2.2.6)$$

If we take the formula of elementary mechanics

$$y_1 = y_0 + h\dot{y}_0 , \qquad (3.2.2.7)$$

then

$$\beta = \frac{\left(1+\frac{p^2h^2}{12}\right)\frac{\dot{y}_0}{p} + \frac{ph}{2}y_0}{\sqrt{1-\frac{p^2h^2}{6}}} \qquad (3.2.2.8)$$

which involves more error than the previous result. However, the accuracy of the velocity response can be improved if the initial velocity is properly treated.

In the case of forced vibrations with a forcing function F(t), the difference equation of Salvadori's method becomes

$$y_{n+1} - \left(2-\frac{p^2h^2}{1+\frac{p^2h^2}{12}}\right)y_n + y_{n-1} = \frac{h^2}{12\left(1+\frac{p^2h^2}{12}\right)}\left\{F(t_{n+1}) + 10F(t_n) + F(t_{n-1})\right\} \qquad (3.2.2.9)$$

which is again the same as Newmark's method for β = 1/12. (See Eq. 3.1.4.19).) This method is accurate to the order of h^5 provided that the motion starts from rest. If the motion begins with a finite velocity, the treatment of the initial velocity for the difference equation determines the accuracy. The discussion of this method is included in Newmark's method in previous and later chapters, and is not restated here.

Salvadori treats the damped motion problem in the same way as Levy does by transforming the damped motion equation into an algebraic equation by the substitution of central averaged differences for the derivatives. The difference equation thus formed is

$$(1+rph)y_{n+1} - (2-p^2h^2)y_n + (1-rph)y_{n-1} = 0 \qquad (3.2.2.10)$$

which is the same as Eq. (3.2.1.9). There is a difference in procedure when applied to a multi-degree-of-freedom structure but it does not affect the nature of the errors.

3 Houbolt's Method

Houbolt's method is based on the assumption of a cubic curve for the dis-
:ement of the moving body, considering that four successive ordinates can be
sed through by a cubuc curve. With this assumption the following difference
ations for the final derivative may be obtained.

$$\ddot{y}_n = \frac{1}{h^2}(2y_n - 5y_{n-1} + 4y_{n-2} - y_{n-3}) \qquad (3.2.3.1)$$

$$\dot{y}_n = \frac{1}{6h}(11y_n - 18y_{n-1} + 9y_{n-2} - 2y_{n-3}) \qquad (3.2.3.2)$$

The derivatives at the third of the four ordinates are sometimes of use
d are as follows:

$$\ddot{y}_n = \frac{1}{h^2}(y_{n+1} - 2y_n + y_{n-1}) \qquad (3.2.3.3)$$

$$\dot{y}_n = \frac{1}{6h}(2y_{n+1} + 3y_n - 6y_{n-1} + y_{n-2}) \qquad (3.2.3.4)$$

For free vibration without damping, substitute Eq. (3.2.3.1) into the
differential equation of motion

$$\ddot{y} + p^2y = 0 \qquad (2.1.16)$$

The following difference equation is obtained.

$$(2 + p^2h^2)y_n - 5y_{n-1} + 4y_{n-2} - y_{n-3} = 0 \qquad (3.2.3.5)$$

Its solution is

$$y_n = c_1x_1^n + c_2x_2^n + c_3x_3^n \qquad (3.2.3.6)$$

where x_1, x_2, and x_3 are the roots of the equation

$$(2 + p^2h^2)x^3 - 5x^2 + 4x - 1 = 0 \qquad (3.2.3.7)$$

It can be shown by the theory of equations that this equation contains one real root and two conjugate complex roots for any value of ph. Therefore the solution has always an oscillating nature and no criterion of stability governs the choice of the time interval, although the amplitude may be damped out very rapidly as time proceeds (see Fig. 18). Eq. (3.2.3.6) may also be written in the form

$$y_n = Ae^{-apt} + e^{-bpt}(B\cos cpt + C\sin cpt) \qquad (3.2.3.8)$$

where a, b, and c are all functions of ph. Here A, B and C are constants deter-
mined from the initial conditions. The first term of the equation is negligibly

small while the second term multiplied by e^{-bpt} is dominant. Table (3.2.3.1) lists the numerical values of a, b and c for various ph, Fig. 19 shows the ratio of pseudo period to true period T_s/T, and Fig. 20 shows the ratio of amplitudes A_s/A.

The disadvantages of this method are two-fold. First, it needs one more initial condition to start with. Although this can be found by taking account of the initial acceleration it also involves effort to trace the back differences including the simultaneous solution of equations Eqs. (3.2.3.1) to (3.2.3.4). If the initial conditions are awkwardly treated, e.g. by making the assumption that the fictitious displacements at t = -h and t = -2h are zero, the error introduced by these erroneous assumptions would be greater than that of the method itself. The step-by-step evaluation of succeeding displacements cannot proceed in a straightforward manner until three initial displacements have been established. Secondly, the amplitude of a slightly damped system decreases so rapidly even for a time interval of about one-sixth of the natural period, that the amplitude is reduced 50 percent after one and one-half cycles of vibration. (See Fig. 18.) Finer time intervals and thus more computational effort must be used to reduce the damping effect of the procedure.

For free vibration with viscous damping, the difference equation of this method becomes

$$(2+\tfrac{11}{3}rph+p^2h^2)y_n-(5+6rph)y_{n-1}+(4+3rph)y_{n-2}-(1-\tfrac{2}{3}rph)y_{n-3}=0 \qquad (3.2.3.9)$$

The solution is in the form of Eq. (3.2.3.8) with a, b, and c functions of r and ph. The criterion for stability becomes

$$4(1-r^2)-24r(1-r^2)ph+(27-48r^2+\tfrac{52}{3}r^4)p^2h^2+4r(9-5r^2)p^3h^3+12r^2p^4h^4>0 \qquad (3.2.3.10)$$

Values of a, b and c are listed in Table 3.2.3.1 for various ph. The ratios of T_s/T and A_s/A are also plotted against ph in Figs. 19 and 20. It can be seen that the error in period increases with time interval h and the damping factor r. The amplitude ratio is less than 1 for systems with slight damping and greater than 1 for system with higher damping factor r.

2.3.1 -- Values of a, b and c of Eq. (3.2.3.8).

	ph	a	b	c
	0.5	1.5587	0.0318	0.9208
	1.0	0.9024	0.0981	0.8016
	1.5	0.6724	0.1461	0.6940
	2.0	0.5494	0.1733	0.6047
	2.5	0.4706	0.1868	0.5325
	3.0	0.4149	0.1922	0.4740
0.25	0.5	1.4176	0.2074	0.8808
	1.0	0.8185	0.1963	0.7357
	1.5	0.6108	0.1959	0.6213
	2.0	0.4987	0.1941	0.5350
	2.5	0.4262	0.1882	0.4687
	3.0	0.3746	0.1820	0.4165
0.50	0.5	1.2436	0.3766	0.8312
	1.0	0.7304	0.2787	0.6836
	1.5	0.5528	0.2371	0.5694
	2.0	0.4547	0.2122	0.4871
	2.5	0.3900	0.1942	0.4258
	3.0	0.3435	0.1799	0.3785
= 0.75	0.5	1.0018	0.5638	0.7848
	1.0	0.6349	0.3544	0.6439
	1.5	0.4960	0.2739	0.5307
	2.0	0.4142	0.2302	0.4519
	2.5	0.3584	0.2019	0.3945
	3.0	0.3173	0.1815	0.3507

Numerical Solution of Differential Equations

.1 Euler's and Modified Euler Method

Euler's approximation is based on the assumption that if y is expressed a function of x by the equation $\frac{dy}{dx} = f(x,y)$, the increment in y corresponding to an increment, Δx, in x is given approximately by the equation $= f(x,y)\Delta x$, the value of f(x,y) being that at the beginning of interval Δx. Applied to the problem of free vibration, with damping, erned by the differential equation $\ddot{y} + 2rp\dot{y} + p^2y = 0$, the formulas for new placement and velocity at end of a time interval are found to be

$$y_1 = y_0 + h\dot{y}_0 \qquad (3.3.1.1)$$

and $$\dot{y}_1 = (1 - 2rph)\dot{y}_0 - p^2 h y_0 \qquad (3.3.1.2)$$

When this procedure is carried on in a step-by-step manner, using the displacement and velocity found from the previous time interval as the initial condition of the new time interval, the difference equation becomes

$$y_{n+1} - 2(1 - rph)y_n + (1 - 2rph + p^2h^2)y_{n-1} = 0 \qquad (3.3.1.3)$$

The solution of this equation is

$$y_n = R^n \left(y_0 \cos n\mu + \frac{r y_0 + \frac{\dot{y}_0}{p}}{\sqrt{1 - r^2}} \sin n\mu \right) \qquad (3.3.1.4)$$

where $$R = \sqrt{1 - 2rph + p^2h^2} \qquad (3.3.1.5)$$

and $$\mu = \arcsin \frac{ph\sqrt{1 - r^2}}{R} = \arccos \frac{1 - rph}{R} \qquad (3.3.1.6)$$

The error is of the second order and the amplitude damps out gradually with time. There is no limiting criterion for stability in case of free vibration with no damping. Any time interval will obtain oscillatory response. For free vibrations with damping, the criterion for critical damping is the same in the numerical solution as in exact solution, i.e. aperiodic motion at critical damping occurs when r = 1.

Figs. 21 and 22 show the errors in period and amplitude at pt = 1 for various ph.

The modified method of Euler takes the true average value of dy/dx over an interval instead of dy/dx at the beginning of the interval for the equation $\frac{dy}{dx} = f(x, y)$. This method gives a more accurate value of the increment of y due to increment of x than Euler's original method and the error is of third order of $(\Delta x)^3$. This method may be represented by the formula

$$\Delta y = \frac{1}{2}(\Delta' + \Delta'') \qquad (3.3.1.7)$$

where $$\Delta' = f(x_0, y_0)\Delta x$$

$$\Delta'' = f(x_0 + \Delta x, y_0 + \Delta y)\Delta x$$

When applied to the problem of free vibration with damping, the displacement and

velocity at the end of an interval now become

$$y_{n+1} = (1 - \frac{p^2h^2}{2}) y_n + (1 - rph) h\dot{y}_n \qquad (3.3.1.8)$$

$$\dot{y}_{n+1} = [1 - 2rph - (\tfrac{1}{2} - 2r^2)p^2h^2]\dot{y}_n + (-1 + rph)p^2h\, y_n \qquad (3.3.1.9)$$

This procedure may be represented by the difference equation

$$y_{n+1} - 2[1 - rph - (\tfrac{1}{2} - r^2)p^2h^2]y_n + [1 - 2rph + 2r^2p^2h^2 - rp^3h^3 + \tfrac{1}{4}p^4h^4]y_{n-1} = 0 \qquad (3.3.1.10)$$

the solution of which is found to be

$$y_n = R^n\left(y_0 \cos n\mu + \frac{ry_0 + \frac{\dot{y}_0}{p}}{\sqrt{1 - r^2}} \sin n\mu\right) \qquad (3.3.1.11)$$

where

$$R = \sqrt{1 - 2rph + 2r^2p^2h^2 - rp^3h^3 + \tfrac{1}{4}p^4h^4} \qquad (3.3.1.12)$$

$$\mu = \arcsin \frac{ph(1 - rph)\sqrt{1 - r^2}}{R} = \arccos \frac{1 - rph - (\frac{1}{2} - r^2)p^2h^2}{R} \qquad (3.3.1.13)$$

Again, there is no limiting criterion for stability in the case of free vibration without damping, but the amplitude of vibration seems to damp out gradually if the range is carried too far. In damped vibration problems, the numerical method has the same criterion for critical damping as the exact solution.

Figs. 23 and 24 show the errors in period and amplitude for various ph of this method.

3.3.2 Runge, Heun and Kutta's Third Order Rule

Various formulas have been devised for numerical integration of the differential equation $\frac{dy}{dx} = f(x,y)$ by Runge, Heun and Kutta. These methods, accurate to the third order of Δx, are summarized as follows:

1. Runge's Original Formula:

$$\Delta y = \Delta'''' + \tfrac{1}{3}\left\{\tfrac{1}{2}(\Delta' + \Delta''') - \Delta''''\right\} \qquad (3.3.2.1)$$

where

$$\Delta' = f(x, y)\cdot\Delta x$$

$$\Delta'' = f(x + \Delta x, y + \Delta')\cdot\Delta x$$

$$\Delta''' = f(x + \Delta x, y + \Delta'')\cdot\Delta x$$

$$\Delta'''' = f(x + \tfrac{1}{2}\Delta x, y + \tfrac{1}{2}\Delta')\cdot\Delta x$$

2. Heun:

$$\Delta y = \tfrac{1}{4}(\Delta' + 3\Delta''') \qquad (3.3.2.2)$$

where

$$\Delta' = f(x, y)\cdot\Delta x$$
$$\Delta'' = f(x + \tfrac{1}{3}\Delta x,\ y + \tfrac{1}{3}\Delta')\cdot\Delta x$$
$$\Delta''' = f(x + \tfrac{2}{3}\Delta x,\ y + \tfrac{2}{3}\Delta'')\cdot\Delta x$$

3. Kutta's Third Order Rule:

$$\Delta y = \tfrac{1}{6}(\Delta' - 4\Delta'' + \Delta''') \qquad (3.3.2.3)$$

where

$$\Delta' = f(x, y)\cdot\Delta x$$
$$\Delta'' = f(x + \tfrac{1}{2}\Delta x,\ y + \tfrac{1}{2}\Delta')\cdot\Delta x$$
$$\Delta''' = f(x + \Delta x,\ y + 2\Delta'' - \Delta')\cdot\Delta x$$

All these formulas when applied to free vibrations with or without damping yield the same results. When applied to forced vibrations, their results have slight discrepancies but all contain errors of the fourth order. It is hard to say which of the above formulas is best because the agreement with the exact solution depends on the type of forcing function, damping coefficient and time interval in a very complicated manner.

Considering the free vibration problem with damping, the following discussion is common to all three of the methods above mentioned.

The displacement and velocity at end of a time interval are found to be

$$y_1 = (1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{3}p^3h^3)y_0 + [1 - rph - \tfrac{1}{6}(1 - 4r^2)p^2h^2]\,h\dot{y}_0 \qquad (3.3.2.4)$$

$$\dot{y}_1 = -[1 - rph - \tfrac{1}{6}(1 - 4r^2)p^2h^2]\,p^2hy_0 + [1 - 2rph - \tfrac{1}{2}(1 - 4r^2)p^2h^2 + \tfrac{2}{3}r(1 - 2r^2)p^3h^3]\dot{y}_0 \qquad (3.3.2.5)$$

These equations, if applied successively by using the final value of previous step as the initial value of the new step, lead to the difference equation

$$y_{n+1} - [2 - 2rph - (1 - 2r^2)p^2h^2 + \tfrac{r}{3}(3 - 4r^2)p^3h^3]\,y_n$$
$$+ [1 - 2rph + 2r^2p^2h^2 - \tfrac{4}{3}r^3p^3h^3 - \tfrac{1}{12}(1 - 8r^2)p^4h^4 - \tfrac{r}{6}p^5h^5 + \tfrac{1}{36}p^6h^6]\,y_{n-1} = 0$$

(3.3.2.6)

The solution of this equation is

$$y_n = R^n\left(y_0 \cos n\mu + \frac{r y_0 + \frac{\dot{y}_0}{p}}{\sqrt{1-r^2}} \sin n\mu\right) \tag{3.3.2.7}$$

where $$R = \sqrt{1 - 2rph + 2r^2p^2h^2 - \frac{4}{3}r^3p^3h^3 - \frac{1}{12}(1-8r^2)p^4h^4 - \frac{r}{6}p^5h^5 + \frac{1}{36}p^6h^6}$$

$$\mu = \arcsin \frac{ph\left[1 - rph - \frac{1}{6}(1-4r^2)p^2h^2\sqrt{1-r^2}\right]}{R} \tag{3.3.2.8}$$

$$= \arccos \frac{1 - rph - \frac{1}{2}(1-2r^2)p^2h^2 + \frac{r}{6}(3-4r^2)p^3h^3}{R} \tag{3.3.2.9}$$

As before, these methods have no limiting criteria of time interval for stability in problems of free vibration without damping, nor there is any discrepancy in the damping coefficient r at the point of critical damping.

3.3.3 Kutta's Fourth Order Rules

The formulas with error of order $(\Delta x)^5$ derived by Kutta are:

1. Kutta's Simpson's Rule

$$\Delta y = \frac{1}{6}\left(\Delta' + 2\Delta'' + 2\Delta''' + \Delta''''\right) \tag{3.3.3.1}$$

where $$\Delta' = f(x_0, y_0)\cdot\Delta x$$

$$\Delta'' = f\left(x_0 + \tfrac{1}{2}\Delta x,\ y_0 + \tfrac{1}{2}\Delta'\right)\cdot\Delta x$$

$$\Delta''' = f\left(x_0 + \tfrac{1}{2}\Delta x,\ y_0 + \tfrac{1}{2}\Delta''\right)\cdot\Delta x$$

$$\Delta'''' = f\left(x_0 + \Delta x,\ y_0 + \Delta'''\right)\cdot\Delta x$$

2. Kutta's Three-eighth Rule

$$\Delta y = \frac{1}{8}\left(\Delta' + 3\Delta'' + 3\Delta''' + \Delta''''\right) \tag{3.3.3.2}$$

where $$\Delta' = f(x_0, y_0)\cdot\Delta x$$

$$\Delta'' = f\left(x_0 + \tfrac{1}{3}\Delta x,\ y_0 + \tfrac{1}{3}\Delta'\right)\cdot\Delta x$$

$$\Delta''' = f\left(x_0 + \tfrac{2}{3}\Delta x,\ y_0 + \Delta'' - \tfrac{1}{3}\Delta'\right)\cdot\Delta x$$

$$\Delta'''' = f\left(x_0 + \Delta x,\ y_0 + \Delta''' - \Delta'' + \Delta'\right)\cdot\Delta x$$

No difference in results between these two rules occurs when applied to free vibration problems. For forced vibrations the results will differ slightly but both are of the same order of error. The agreement with the exact solution

varies with r, ph and t.

The displacement and velocity at end of a time interval in the problem of free vibration with damping are found by either of the two rules to be

$$y_1 = \left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{1}{3}rp^3h^3 + \tfrac{1}{24}(1-4r^2)p^4h^4\right]y_0 + \left[1 - rph - \tfrac{1}{6}(1-4r^2)p^2h^2 + \tfrac{r}{6}(1-2r^2)p^3h^3\right]h\dot{y}_0 \qquad (3.3.3.3)$$

$$\dot{y}_1 = \left[-1 + rph + \tfrac{1}{6}(1-4r^2)p^2h^2 - \tfrac{r}{6}(1-2r^2)p^3h^3\right]p^2hy_0 + \left[1 - 2rph - \tfrac{1}{2}(1-4r^2)p^2h^2 + \tfrac{2r}{3}(1-2r^2)p^3h^3 + \tfrac{1}{24}(1-12r^2+16r^4)p^4h^4\right]\dot{y}_0 \qquad (3.3.3.4)$$

This may be put in difference equation form for the step-by-step method, as follows

$$y_{n+1} - \left[2 - 2rph - (1-2r^2)p^2h^2 + \tfrac{r}{3}(3-4r^2)p^3h^3 + \tfrac{1}{12}(1-8r^2+8r^4)p^4h^4\right]y_n + \left[1 - 2rph + 2r^2p^2h^2 - \tfrac{4}{3}r^3p^3h^3 + \tfrac{2}{3}r^4p^4h^4 + \tfrac{r}{12}(1-4r^2)p^5h^5 - \tfrac{1}{72}(1-6r^2)p^6h^6 - \tfrac{1}{72}rp^7h^7 + \tfrac{1}{576}p^8h^8\right]y_{n-1} = 0 \qquad (3.3.3.5)$$

The solution is found to be

$$y_n = R^n\left(y_0\cos n\mu + \frac{ry_0 + \frac{\dot{y}_0}{p}}{\sqrt{1-r^2}}\sin n\mu\right) \qquad (3.3.3.6)$$

where

$$R = \left[1 - 2rph + 2r^2p^2h^2 - \tfrac{4}{3}r^3p^3h^3 + \tfrac{2}{3}r^4p^4h^4 + \tfrac{r}{12}(1-4r^2)p^5h^5 - \tfrac{1}{72}(1-6r^2)p^6h^6 - \tfrac{1}{72}rp^7h^7 + \tfrac{1}{578}p^8h^8\right]^{\frac{1}{2}} \qquad (3.3.3.7)$$

$$\mu = \arcsin\frac{ph\left[1 - rph - \frac{1}{6}(1-4r^2)p^2h^2 + \frac{r}{6}(1-2r^2)p^3h^3\right]\sqrt{1-r^2}}{R} = \arccos\frac{1 - rph - \frac{1}{2}(1-2r^2)p^2h^2 + \frac{r}{6}(3-4r^2)p^3h^3 + \frac{1}{24}(1-8r^2+8r^4)p^4h^4}{R} \qquad (3.3.3.8)$$

Like all other methods of this kind, the fourth order rules have no limiting criteria for stability. Any time interval can be used according to the accuracy desired. When the system reaches its critical damping, i.e., r = 1, the numerical solution becomes aperiodic. A difficulty associated with the method is that the amplitude will gradually damp itself out even in the case of an undamped system, and the method is therefore not desirable for a long period of time.

IV. DISCUSSION

4.1 Accuracy

It is usually convenient to compare the accuracy of various methods by the order of time interval h involved in the error produced. The final displacement over a time range computed by numerical methods can be expanded into a polynomial of ph and then compared term by term with a standard power series derived from the exact solution. The error occurring in each step of the step-by-step method may thus be observed, especially as to the order of ph involved. Considering first the case of free vibration, with damping, which is governed by the differential equation

$$\ddot{y} + 2rp\dot{y} + p^2y = 0 \tag{2.1.3}$$

its solution

$$y = e^{-rpt}\left(y_0 \cos\sqrt{1-r^2}\,pt + \frac{ry_0 + \frac{\dot{y}_0}{p}}{\sqrt{1-r^2}} \sin\sqrt{1-r^2}\,pt\right) \tag{2.1.4}$$

may be expressed in a power series for a time interval h

$$\begin{aligned} y_{n+1} = {} & y_n\left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{3}p^3h^3 + \tfrac{1}{24}(1-4r^2)p^4h^4 - \tfrac{r}{30}(1-2r^2)p^5h^5 + \cdots\right] \\ & + \dot{y}_n h\left[1 - rph - \tfrac{1}{6}(1-4r^2)p^2h^2 + \tfrac{1}{6}(1-2r^2)p^3h^3 + \tfrac{1}{120}(1-12r^2+16r^4)p^4h^4 + \cdots\right] \end{aligned} \tag{4.1.1}$$

The following is a collection of equations which show the accuracy of different numerical methods:

Constant acceleration method:

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2\right]y_n + \left[1 - rph\right]h\dot{y}_n \tag{3.1.1.13}$$

Newmark's β-method:

$$\begin{aligned} y_{n+1} = {} & \left[1 - \tfrac{1}{2}p^2h^2 + 2\beta rp^3h^3 - \tfrac{1}{2}(4\beta r^2 - \beta)p^4h^4\right. \\ & \left. + \tfrac{1}{2}(4\beta r^3 - \beta r - 4\beta^2 r)p^5h^5 - \tfrac{1}{2}(4\beta r^4 - \beta r^2 - 8\beta^2 r^2 + \beta^2)p^6h^6 + \cdots\right]y_n \\ & + \left[1 - rph + (4\beta r^2 - \beta)p^2h^2 - 4(r^3\beta - 2r\beta)p^3h^3\right. \\ & \left. + (4r^4\beta - 2r^2\beta - 4r^2\beta^2 + \beta^2)p^4h^4 + \cdots\right]h\dot{y}_n \end{aligned} \tag{4.1.2}$$

For $\beta = 1/4$ (Timoshenko's modified method.)

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{2}p^3h^3 - \cdots\right]y_n + \left[1 - rph - \tfrac{1}{4}(1-4r^2)p^2h^2 + \cdots\right]h\dot{y}_n \qquad (4.1.3)$$

For $\beta = 1/6$ (Linear acceleration method.)

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{3}p^3h^3 + \tfrac{1}{12}(1-4r^2)p^4h^4 + \cdots\right]y_n + \left[1 - rph - \tfrac{1}{6}(1-4r^2)p^2h^2 + \tfrac{2r}{3}(2-r^2)p^3h^3 + \cdots\right]h\dot{y}_n \qquad (4.1.4)$$

For $\beta = 1/12$ (Salvadori's method.)

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{6}p^3h^3 + \tfrac{1}{24}(1-4r^2)p^4h^4 - \tfrac{r}{18}(1-3r^2)p^5h^5 + \cdots\right]y_n + \left[1 - rph - \tfrac{1}{12}(1-4r^2)p^2h^2 + \cdots\right]h\dot{y}_n \qquad (4.1.5)$$

For $\beta = 0$ (Levy's method.)

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2\right]y_n + \left[1 - rph\right]h\dot{y}_n \qquad (4.1.6)$$

Parabolic acceleration method:

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{3}p^3h^3 + \tfrac{1}{24}(1-4r^2)p^4h^4 + \tfrac{r}{1728}(563-192r^2)p^5h^5 + \cdots\right]y_n + \left[1 - rph - \tfrac{1}{6}(1-4r^2)p^2h^2 + \tfrac{r}{6}(1-2r^2)p^3h^3 - \tfrac{1}{72}(1-16r^2+96r^4)p^4h^4 + \cdots\right]h\dot{y}_n \qquad (4.1.7)$$

Houbolt's method:

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{3}p^3h^3 + \tfrac{1}{12}(1-4r^2)p^4h^4 + \cdots\right]y_n + \left[1 - rph - \tfrac{1}{6}(1-4r^2)p^2h^2 + \tfrac{r}{3}(1-2r^2)p^3h^3 + \cdots\right]h\dot{y}_n \qquad (4.1.8)$$

Euler's method:

$$y_{n+1} = y_n + h\dot{y}_n \qquad (3.3.1.1)$$

Modified Euler Method:

$$y_{n+1} = \left(1 - \frac{p^2h^2}{2}\right)y_n + \left(1 - rph\right)h\dot{y}_n \qquad (3.3.1.6)$$

Runge, Heun and Kutta's third order rule:

$$y_{n+1} = \left(1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{3}p^3h^3\right)y_n + \left[1 - rph - \tfrac{1}{6}(1-4r^2)p^2h^2\right]h\dot{y}_n \qquad (3.3.2.4)$$

Kutta's method:

$$y_{n+1} = \left[1 - \tfrac{1}{2}p^2h^2 + \tfrac{r}{3}p^3h^3 + \tfrac{1}{24}(1-4r^2)p^4h^4\right]y_n + \left[1 - rph - \tfrac{1}{6}(1-4r^2)p^2h^2 + \tfrac{r}{6}(1-2r^2)p^3h^3\right]h\dot{y}_o \qquad (3.3.3.3)$$

From the above listings of equations for various methods, one may observe that:

1. The constant acceleration, linear acceleration and parabolic acceleration methods have, respectively, an accuracy of the order of h^2, h^3 and h^4 in both displacement and velocity. The accuracy of Newmark's β-method depends on

the choice of β.

2. Levy's and Houbolt's methods have, respectively, an accuracy of the order of h^2 and h^3 in both displacement and velocity responses.

3. Salvadori's method is of order h^5 only in displacement response and only for undamped systems. If r and $\dot{y}_o$ are not zero, this method is only of h^2 accuracy unless a good interpretation of $\dot{y}_o$ is made.

4. Euler's method is of the first order; modified Euler method, second order; Runge and Heun, third order; and Kutta's Simpson's rule or three-eighth rule, fourth order generally.

5. In some special cases the accuracy of the above methods may be promoted one more order. These will be listed in Chapter 5.

In the case of forced vibrations, the error which enters the particular solution also governs the accuracy of the method since it determines the constants for the initial conditions. As before, Newmark's linear acceleration method is of third order accuracy for any system with or without damping. For an undamped system starting at rest, Salvadori's method has an accuracy of fifth order. Euler's, modified Euler, Runge-Heun and Kutta's methods are still of first, second, third and fourth order respectively.

The comparison of errors by polynomials is only good for ph less than 1, because the error would otherwise be dominated by the higher powers of ph which would vitiate the analysis. Nevertheless, usual practice indicates that a large time interval, say from 1/4 to 1/2 of the natural period of vibration (ph=1.5708 to 3.1416) is highly desirable in rapid and rough estimations. The effect of large time intervals is shown in the graphs of Appendix 3.

4.2 Propagation of Errors

The preceding article concerns the accuracy of displacement response of various methods in one single step of operation, the error indicated in the

expanded polynomial being compared with a common initial condition. But if step-by-step evaluations are set up, with a common initial condition for the first step, the initial condition of the second step will contain error which is different in various methods. After a chain of steps is completed, the error propagates in different ways, divergent or convergent, accumulating or self-eliminating, as a function of time, length of time interval, natural frequency and damping factor.

Shown in previous chapters are the equations for y_n and in Appendix 3 are graphs of T_s/T and A_s/A for various methods. These may serve for an estimate of the propagation of errors.

Errors are of two fundamental types, error in period and error in amplitude. The error in period or in frequency is solely due to the discrepancy in phase angle which is directly proportional to the lapse of time. The error in amplitude, on the other hand, chiefly depending on the factor R^n in the equations of y_n, is an exponential function of time. From the equations for y_n derived and listed in Chapter 3, it is evident that the condition for no error in period is $ph = \mu$ and that for no divergent error in amplitude is $Re^{rph} = 1$. All pseudo periods and pseudo amplitudes have been compared with true periods and true amplitudes in the preceding chapter. The ratio of periods is given by the relation

$$\frac{T_s}{T} = \frac{ph\sqrt{1-r^2}}{\mu} \qquad (4.2.1)$$

and the amplitudes by A_s/A. There are two ways of comparing the amplitudes. The first one concerns the ordinate of the envelope which prescribes the periodic response, while the second way takes account of the peak amplitude in the first cycle of vibration, subject to a certain initial velocity.

In the first way of comparison, one finds that

$$\frac{A_s}{A} = \frac{R^n}{e^{-rpt}} \qquad (4.2.2)$$

neglecting the magnifying effect of initial velocity in some cases.

Since the ratio of amplitudes is not constant with time, it is reasonable

to compare the methods at a certain designated time, say, at t = 1/p, i.e. pt = 1. Then

$$\left(\frac{A_s}{A}\right)_{pt=1} = R^{\frac{1}{ph}} \cdot e^{r} \qquad (4.2.3)$$

Note that the ratio A_s/A at pt=1 here does not mean the comparison made with the pseudo amplitude actually computed which is not only affected by the factor R, but also usually by the change in velocity response. The velocity response depends on ph as stated in Art. 3.1.4, and sometimes on the initial displacement as in Eq. (3.1.3.18). However, when the time interval is not close to the criterion of stability, this error in amplitude due to initial velocity response is of a constant nature, doing much less harm than the exponential factor R after a considerable lapse of time, and is not taken into consideration. Therefore the ratic A_s/A for the envelopes of the periodic curves is still useful for judging the convergence and divergence of errors.

Newmark's β-method for all values of β from 0 to 1/4, together with Timoshenko's modified, Newmark's linear acceleration, Levy's and Salvadori's methods, have $A_s/A = 1$ in free vibration of an undamped system. Constant and parabolic acceleration methods are the ones which have divergeng amplitudes while the others, including Houbolt's, Euler's, modified Euler, Runge's, Heun's and Kutta's methods, have convergent amplitude although the rapidity of divergence or convergence is different.

The presence of damping may add complication to the analysis. Plottings of A_s/A at pt = 1 shown in Appendix 3 are self-explanatory. The relation between A_s/A at any time and A_s/A at pt = 1 is shown in Fig. 29.

In the second method of comparison, peak amplitudes in the first cycle of vibration due to an initial velocity $\dot{y}_o = p$ are compared and shown in Appendix 3. This comparison may be of more interest in practical problems of vibration since it gives actual amplitudes of periodic motion. The magnifying effect on the sine term of the general equation is generally taken into consideration except

in the parabolic acceleration method and Houbolt's method where the magnification factor depends also on the treatment of initial conditions. Note that the magnitude of the peak amplitude does not depend only on the value of R, but also on the error in period as well.

Error in period is, as a rule, constant for a given ph. Generally speaking, the ratio T_s/T increases or decreases with the broadening of time interval. $T_s/T > 1$ indicates a larger pseudo period or a retardation of phase angle and vice versa.

4.3 Stability and Convergence

The applicability of different available techniques places some limitations on the time interval used, not only as regards accuracy, but also as regards stability and convergence. All acceleration methods have a limiting criterion of convergence because of their iterative procedure. All acceleration methods, except Timoshenko's modified method, also have a limiting criterion of stability beyond which aperiodic response will occur. This has been discussed in Art. 3.1.4. (See Eq. (3.1.4.7) and (3.1.4.8).) Larger values of β provide a wider range of time interval for stability, but a shorter range for convergence. On the other hand, when $\beta = 0$, freedom from the convergence criterion is obtained at the loss of range for stability. The presence of damping will also affect both criteria; the greater the damping factor r, the shorter the range of time interval available.

The difference equation methods have generally no limiting criterion for convergence because of the nature of the procedure. However, a criterion for stability still governs those techniques which have been discussed in previous chapters. Levy's and Salvadori's methods have the same criterion for stability as that of Newmark's β-method when $\beta = 0$ and $\beta = 1/12$ respectively. The Houbolt method criterion for stability has been given in Eq. (3.2.3.10).

The methods of numerical integration described in Art. 3.3 have the

avoiding limiting criteria both of stability and convergence. The
ıe interval may therefore be made according to the accuracy desired.
visable to use any time interval greater than 1/p, that is, ph $>$ 1,
:ry misleading results may be obtained because of the fact that higher
n will dominate the solution.

he criterion of critical damping in the exact solution is r = 1, while
of the numerical solutions have the same criterion. Those which have
terion of critical damping are: Timoshenko's modified method or Newmark's
for $\beta = 1/4$, Euler's method, modified Euler method, Runge's, Heun's and
ıethods. In most of other methods critical damping occurs even when $r<1$.
ar acceleration method, Salvadori's method, Levy's method and Newmark's
1 (with the exception of $\beta = 1/4$) are all of this group. The criterion of
stant acceleration method may be higher or lower than the actual criterion,
ng on the product of the natural frequency of the structure and the time
al used. Houbolt's method generally exhibits periodic response for all
of r and ph except in some cases when r is greater than 0.94 and ph very
Fig. 30 illustrates the criteria of critical damping for various techni-
The region above a curve is that of aperiodic response, while the region
a curve is that of periodic response.

rocedures of Operation

Acceleration methods require an iterative procedure starting from an
d value of acceleration and arriving at a derived acceleration by use of
uations of motion until a close agreement is obtained between the assumed
rived values. With a proper choice of time interval, three of four trials
ch interval of time will usually be sufficient to reach convergence in a
degree-of-freedom system. The time consumed in completing a step by an
ic desk computer is about nine minutes for a two-degree-of-freedom system

wenty minutes for a five-degree-of-freedom one after the equations are a routine form is made. The result of each step is self-checking except cial case of $\beta = 0$ in Newmark's β-method. Both displacement and velocity as a supplementary help for giving a clearer picture of motion and for

Difference equation methods are faster because displacements are directly from the difference equations and no extra work to obtain velocities is However, they suffer from the absence of self-checking procedures unless ctional device is provided. For problems of multi-degree-of-freedom systems, suggested a recurrence-matrix solution in which the equations of motion pressed in a recurrence matrix equation and solved by inverting the matrix. ori expressed the equations of motion for every three adjacent masses so each equation contains only three unknowns and may be solved by relaxation, and error, or successive approximations. The evaluation of displacements six significant figures for a five-degree-of-freedom took approximately ten tes after the computations had been standardized.(6)

Runge's and Kutta's methods are the most time-consuming as far as the of an electric desk computing machine is concerned. It takes more than thirty tes to complete a step by Kutta's fourth order formula for a two-degree-of-dom system. Furthermore, since there is no self-checking of calculations, akes may easily be introduced into the computations due to the intricate work ross-substitution in the procedure.

V. CONCLUSIONS

The general results of this study are tabulated on the next page (Table in which the advantages and disadvantages of each of the available techniques .isted. Graphs showing the errors in period and amplitude for a range of time :val from 0 to about half of the natural period in various methods are given he end of this dissertation. It is therefore possible to choose a suitable nique for a specific problem according to the accuracy and amount of work re- :ed. In general, the larger the time interval, the cruder the results. Values ph less than 1 always give reliable results for all techniques, but variations .l be great when ph$>$1, and these graphs may be found useful for judgement when ing large intervals.

In ordinary problems of vibratory motion, Newmark's β-method is most luable because of its flexibility in application. The choice of time interval y be made for the desired rate of convergence and accuracy by adjustment of the - parameter. The linear acceleration method, a special case of the β-method for 1/6, is most consistent in degree of error when the motion is that of forced bration with damping, with initial displacement and velocity. Timoshenko's thod is best applied to an undamped system when the response in amplitude is portant. The constant acceleration method and Euler's method are not advisable ing to their inaccuracy. If the masses in motion are not damped and have no itial velocity, Salvadori's method is most rapid and accurate. For rapid and ss accurate work, Levy's method may prove useful, but care should be taken in e treatment of initial velocity. Runge and Kutta's methods are noted for their curacy and generality in application, having no restrictions with respect to ability and convergence, but they are handicapped by the tedious procedure which not generally desirable for use as an ordinary engineering design technique.

Table 5.1 -- Summarized Result of Analysis

Item No.	Techniques	Order of Accuracy			Amplitude when r=0
		Displacement Response	Velocity Response	Forced Vibration	
1.	Constant Acceleration	2nd	2nd		divergent
2.	Timo. Mod. Accel. Newmark $\beta = 1/4$	2nd (3rd when r = 0)	2nd (3rd when r = 1/2)	2nd (3rd when r = 0)	constant
3.	Linear Acceleration Newmark $\beta = 1/6$	3rd (4th when r = 1/2)	3rd (4th when r = 0)	3rd	constant
4.	Parabolic Acceleration	4th	4th	4th	divergent
5.	Newmark's β-method	Depends on β			constant
6.	Levy Newmark $\beta = 0$	2nd (3rd when r = 0)	2nd (3rd when r = 1/2)	2nd (3rd when r = 0)	constant
7.	Salvadori Newmark $\beta = 1/12$	2nd (5th when r = 0)	2nd (3rd when r = 1/2)	2nd (5th when r = 0)	constant
8.	Houbolt	3rd (4th when r = 1/2)	3rd (4th when r = 0)	3rd	conv.
9.	Euler	1st	1st (2nd when r = 0)	1st	conv.
10.	Modified Euler	2nd (3rd when r = 0)	2nd	2nd	conv.
11.	Runge, Heun	3rd	3rd (4th when r = 0)	3rd	conv.
12.	Kutta 4th Order	4th (5th when r = 0)	4th	4th	conv.

Table 5.1 -- Summarized Result of Analysis. (Concluded)

Item No.	Criterion of Stability	Criterion of Convergence	Self-Checking	Time Consumption
1.	Eq. (3.1.1.13)	no	no	less
2.	no	$rph + \frac{1}{4}p^2h^2 < 1$	yes	fair
3.	Eq. (3.1.3.13)	$rph + \frac{1}{2}p^2h^2 < 1$	yes	fair
4.	$ph < 3.357758$ for $r = 0$	$ph < 2.828427$ for $r = 0$	yes	more
5.	Eq. (3.1.4.15)	Eq. (3.1.4.16)	yes	fair
6.	$ph < 4(1-r^2)$	no	no	less
7.	$ph < 6(1-r^2)$	no	no	less
8.	Eq. (3.2.3.10)	no	no	fair
9.	no	no	no	less
10.	no	no	no	fair
11.	no	no	no	more
12.	no	no	no	more

APPENDIX 1. - - NOMENCLATURE

The letter symbols and notations used in this thesis are defined as follows except otherwise noted.

A = amplitude of the moving mass.

A_s = pseudo amplitude obtained by numerical method.

$A, B, C,$ = constants determined from initial conditions.

$a, b, c,$ = coefficients used in equations.

e = 2.718.... base of 'natural' logarithms

F = forcing function.

h = time intervals.

k = constants.

n = number of time intervals taken in the step-by-step evaluation, or subscript to designate displacement function at a particular time, as y_o, y_n or $\dot{y}_o$, $\dot{y}_n$.

p = circular frequency of natural vibration.

r = factor of viscous damping in terms of p.

T = natural period of a general system, in general $T = 2\pi/p$.

T_s = pseudo period of vibration obtained by numerical method.

t = time.

t_n = time corresponding to end of nth time interval.

x = variable in general equations.

y = displacement of mass

$\dot{y}$ = first derivative of displacement with respect to time, i.e. velocity of mass.

$\ddot{y}$ = second derivative of displacement with respect to time, i.e. acceleration of mass.

$\bar{y}$ = exact solution of differential equation.

y_p = particular solution of difference equation for numerical methods.

APPENDIX 1. - - NOMENCLATURE (Concluded)

$\bar{y}_p$ = exact particular solution of differential equation.

β = coefficient measuring proportion of acceleration at end of interval in determination of displacement.

μ = phase angle at end of the first time interval obtained by numerical solution.

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APPENDIX 3. -- FIGURES

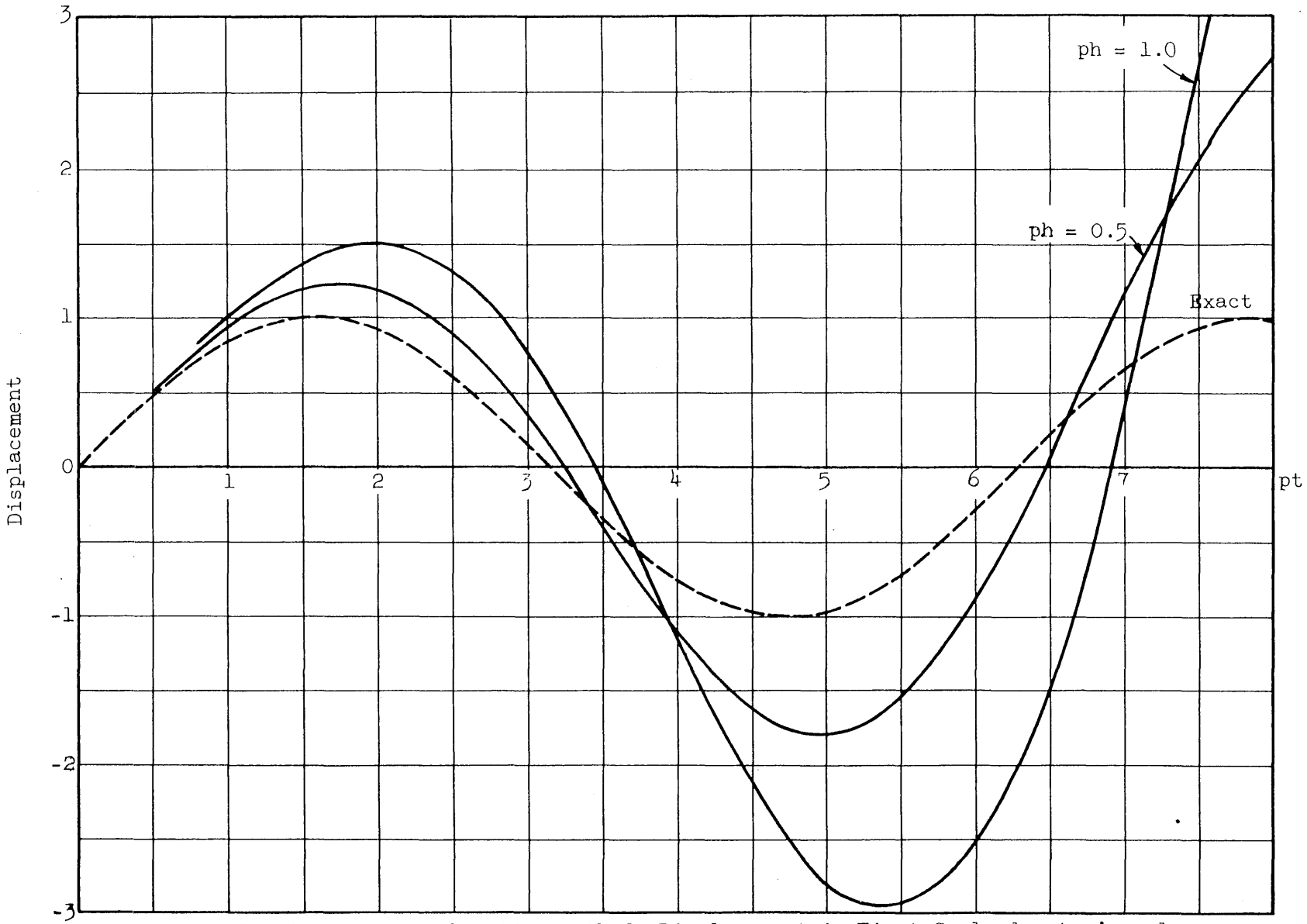

Fig. 1 - Constant Acceleration Method: Displacement in First Cycle due to $\dot{y}_o = 1$.
Free Vibration. No Damping.

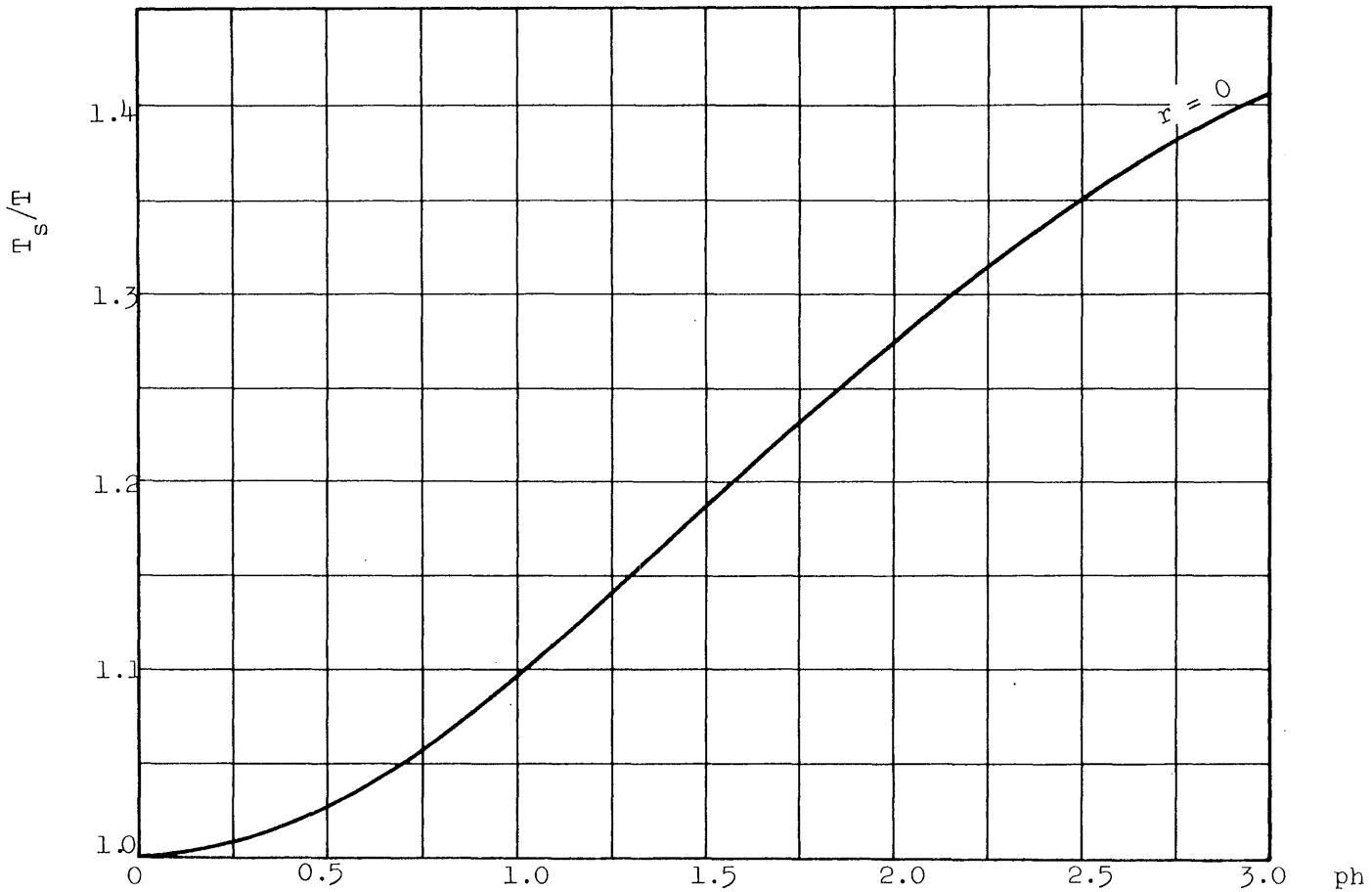

Fig. 2 - Constant Acceleration Method: Error in Period

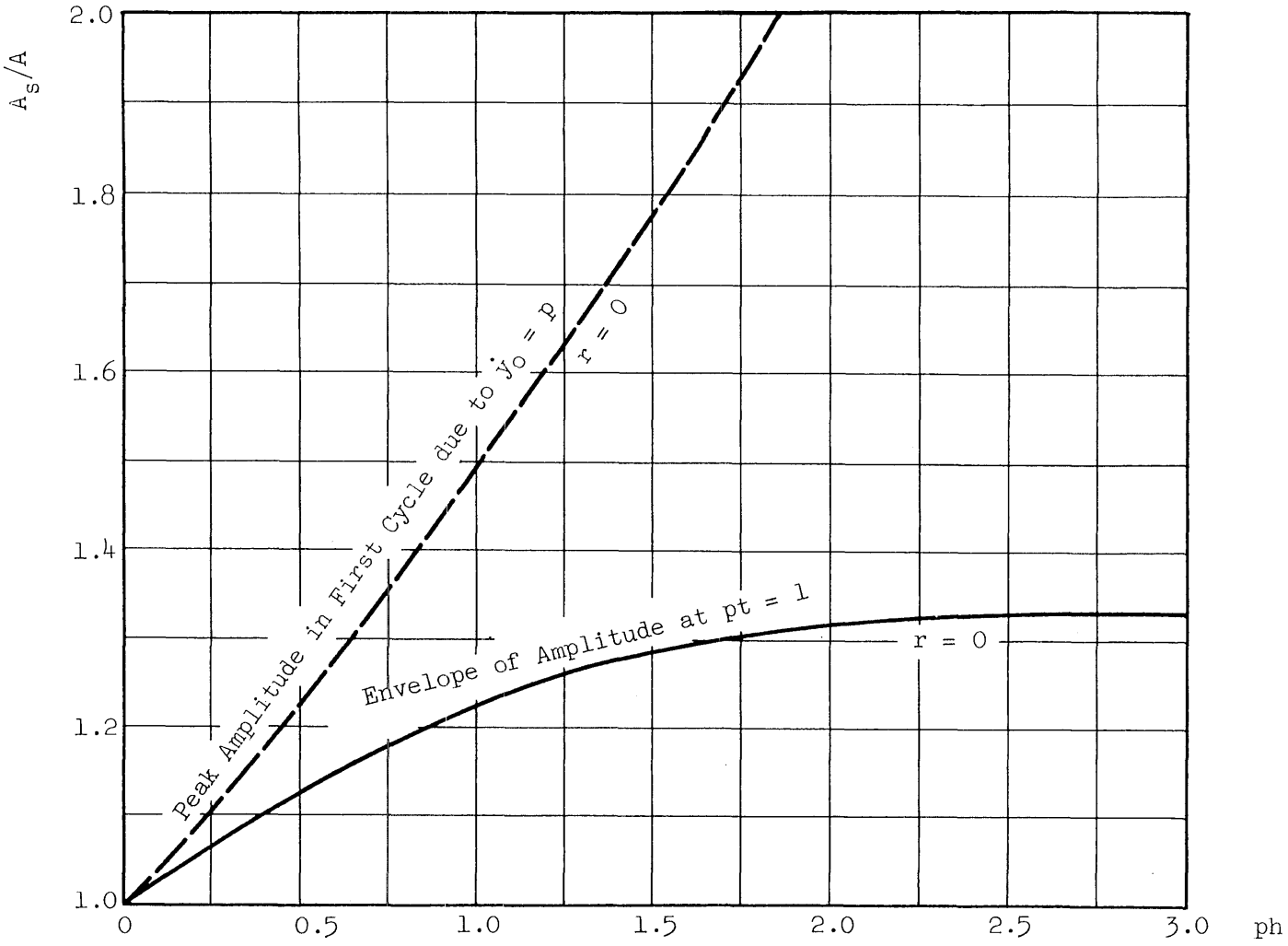

Fig. 3 - Constant Acceleration Method: Error in Amplitude

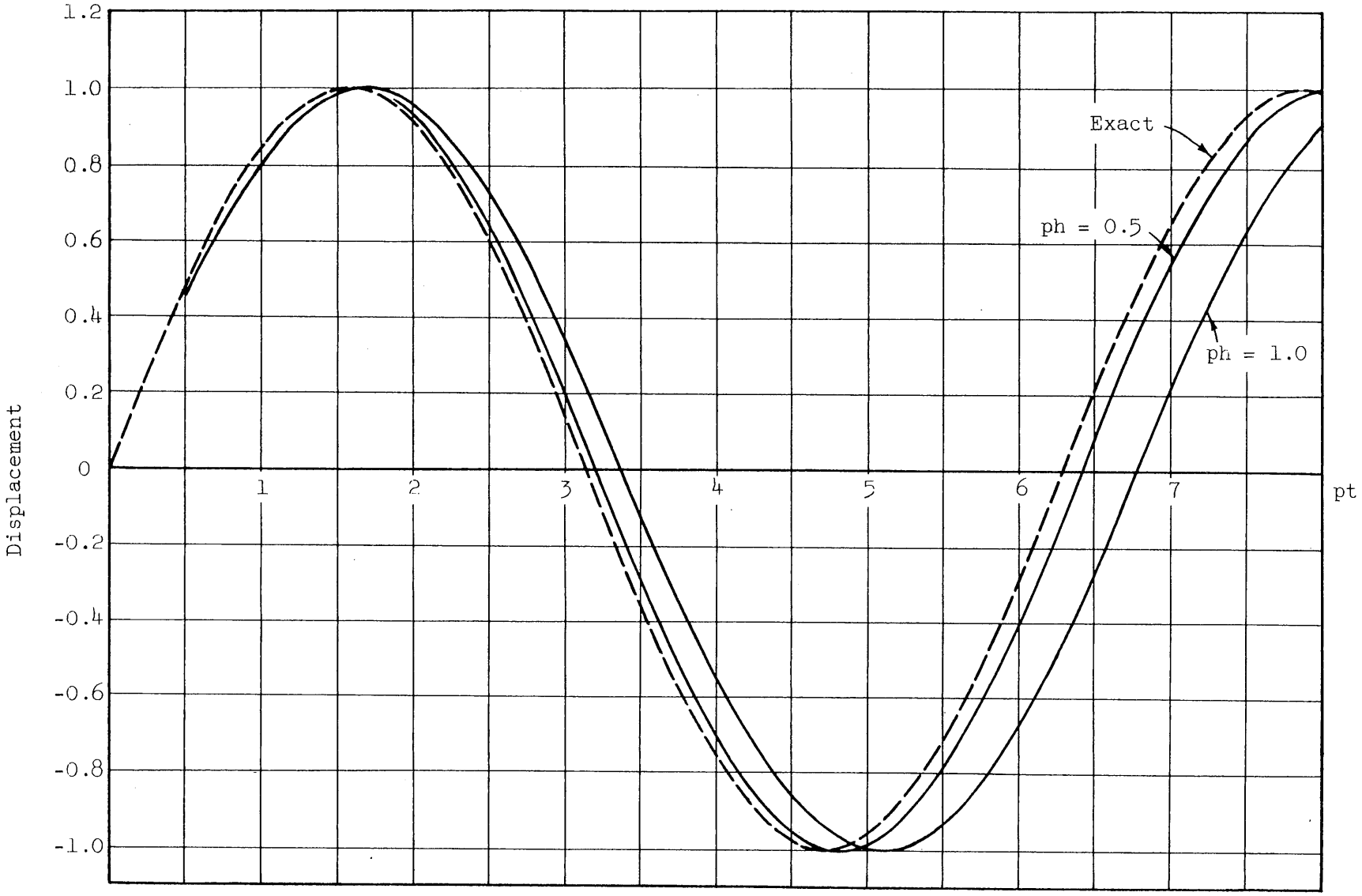

Fig. 4 - Timoshenko's Modified Method (or Newmark's Method for $\beta = 1/4$): Displacement in First Cycle due to $\dot{y}_o = p$. Free Vibration. No Damping.

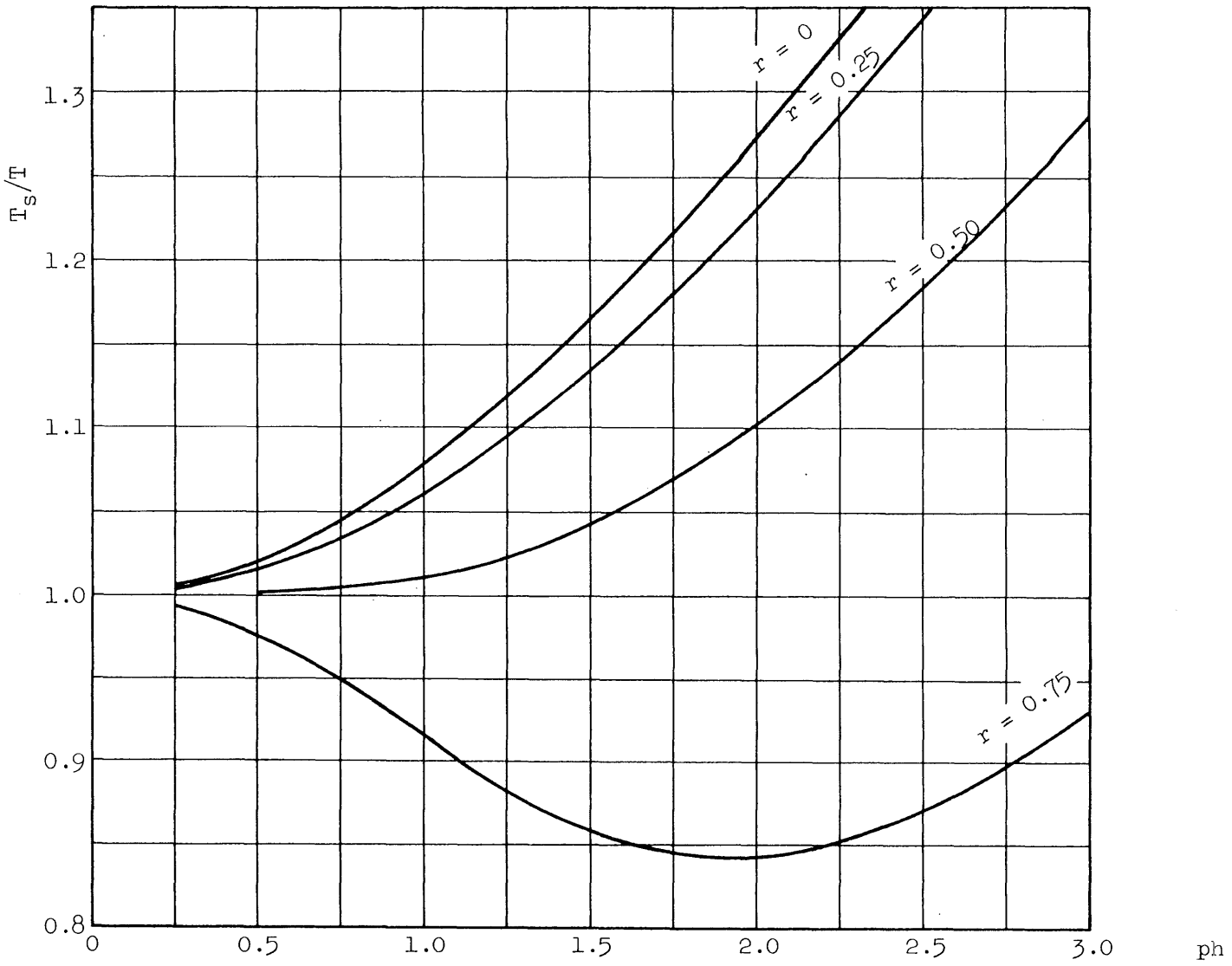

Fig. 5 - Timoshenko's Modified Method (or Newmark's Method for $\beta = 1/4$): Error in Period.

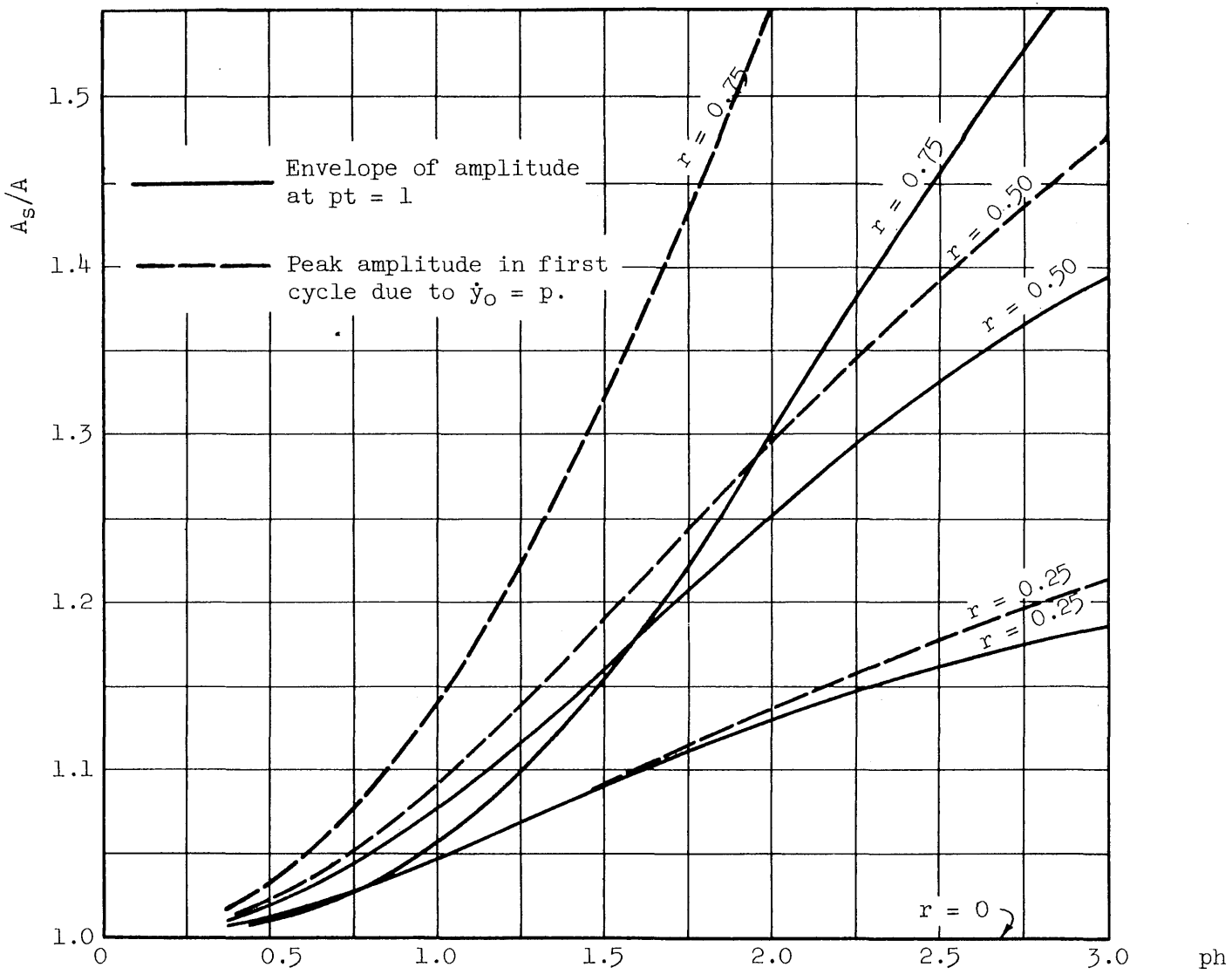

Fig. 6 - Timoshenko's Modified Method (or Newmark's Method for $\beta = 1/4$): Error in Amplitude.

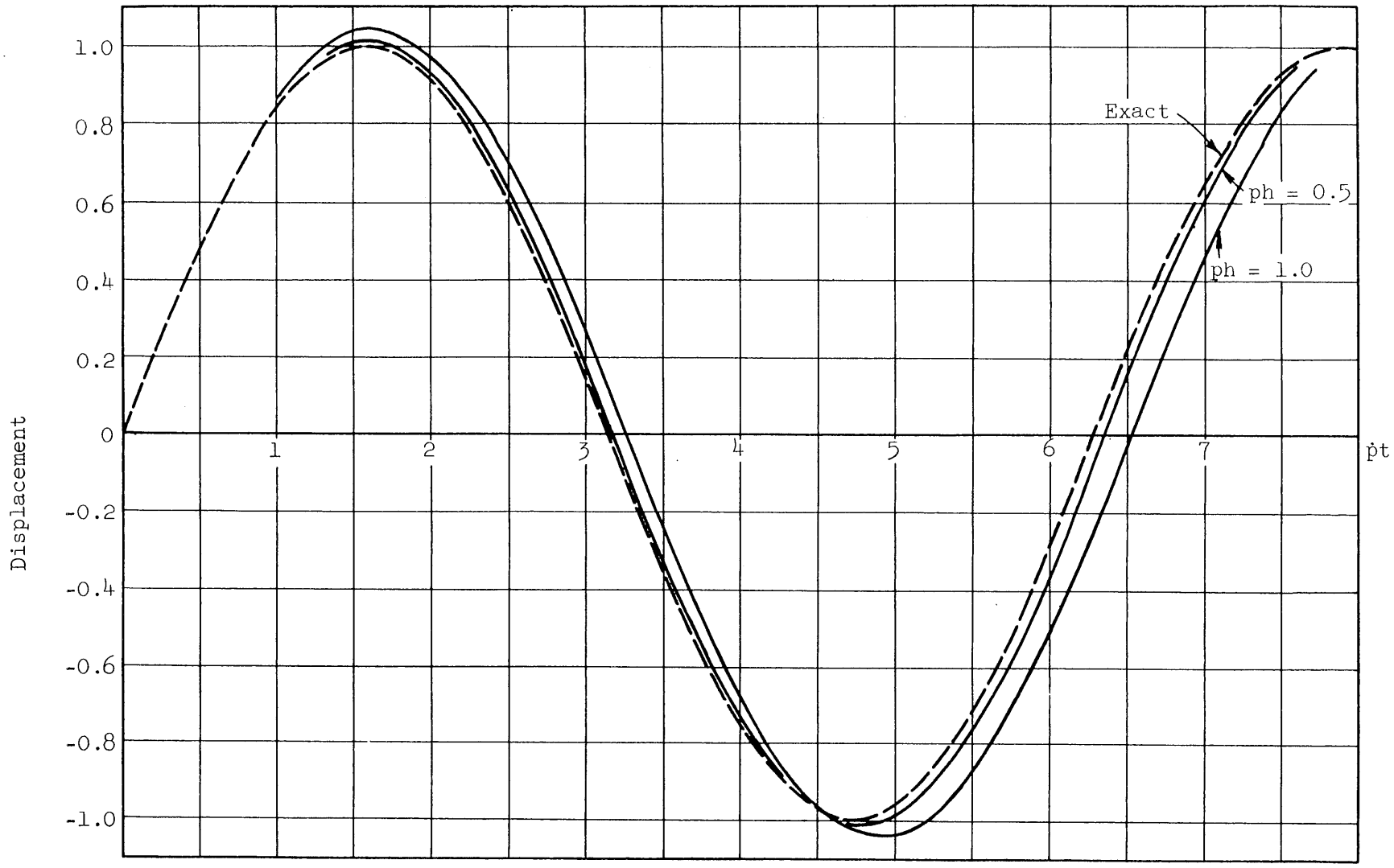

Fig. 7 - Linear Acceleration Method (or Newmark's Method for $\beta = 1/6$): Displacement in First Cycle due to $\dot{y}_o = p$. Free Vibration. No Damping.

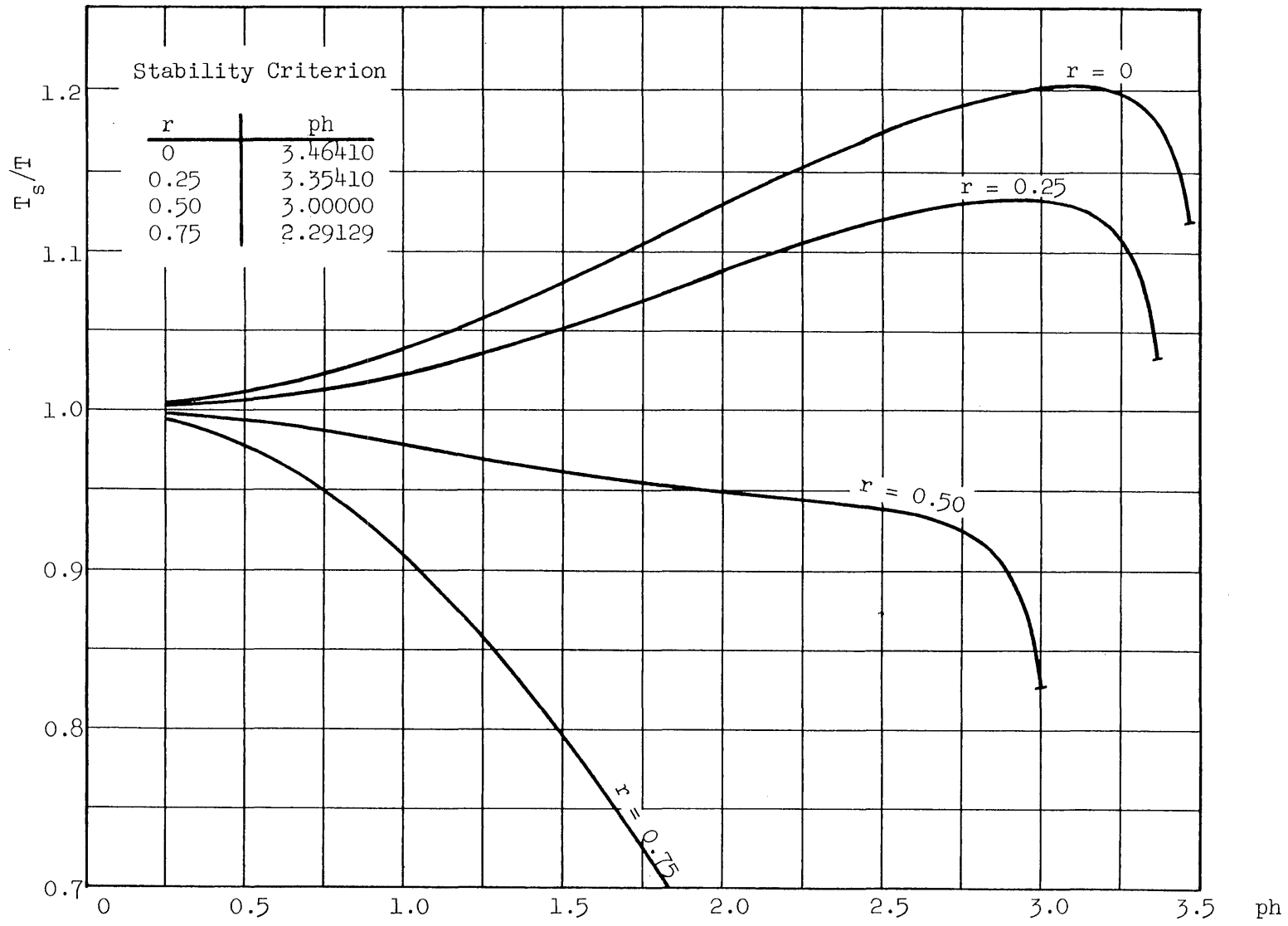

Fig. 8 - Linear Acceleration Method (or Newmark's Method for $\beta = 1/6$): Error in Period.

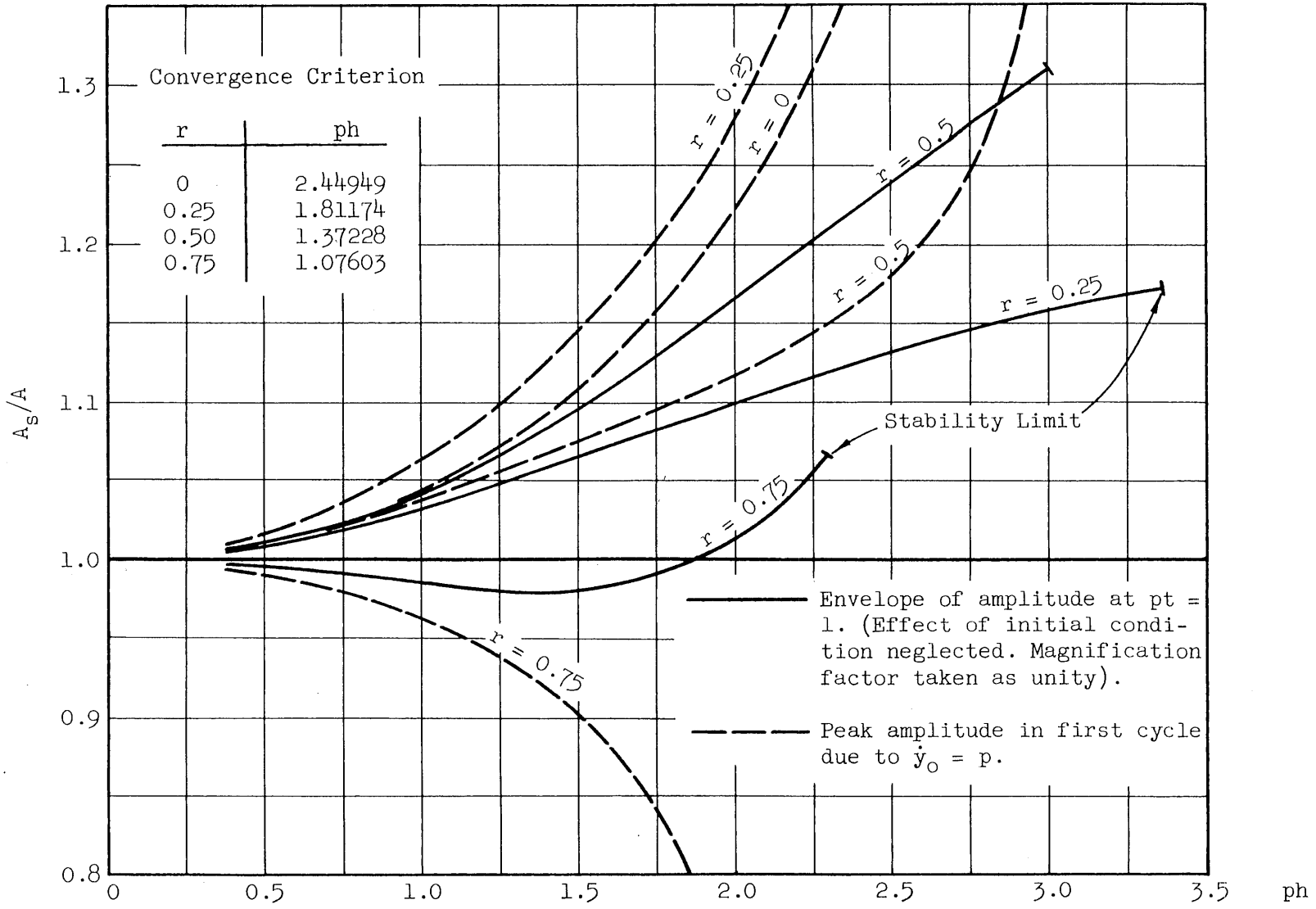

Fig. 9 - Linear Acceleration Method (or Newmark's Method for $\beta = 1/6$): Error in Amplitude.

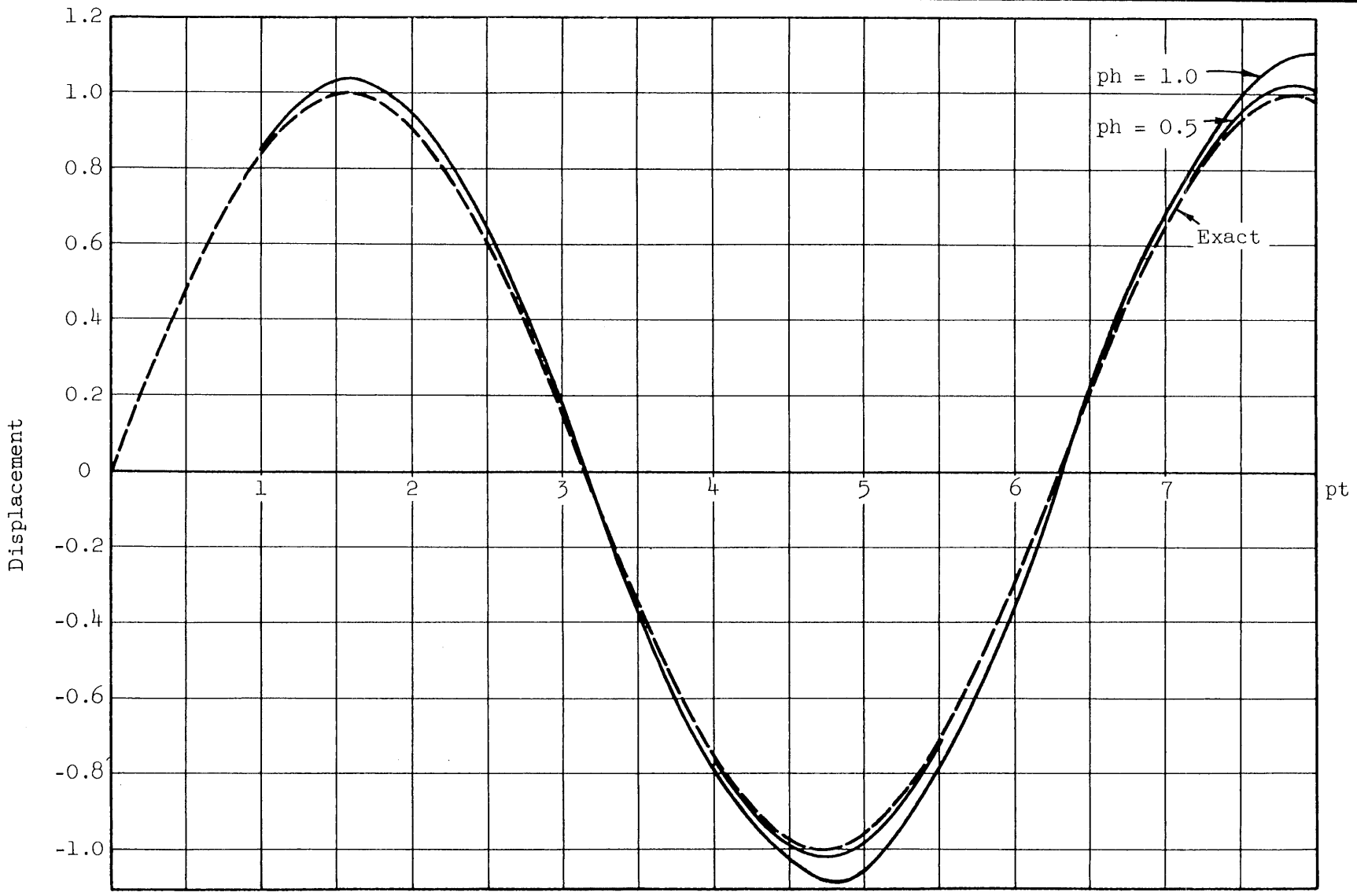

Fig. 10 - Parabolic Acceleration Method: Displacement in the First Cycle due to $\dot{y}_o = p$. Free Vibration. No Damping.

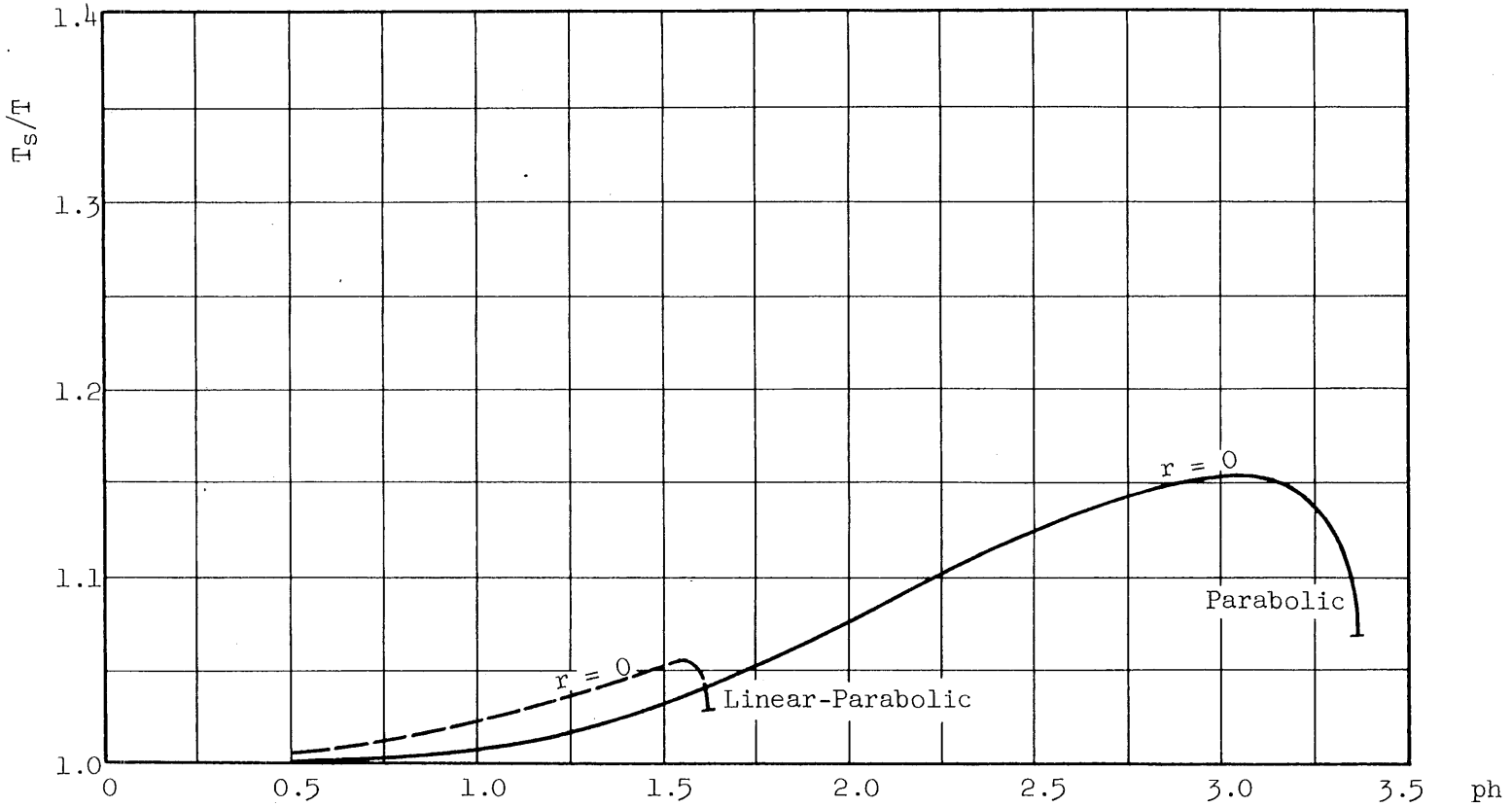

Fig. 11 - Parabolic Acceleration Method: Error in Period

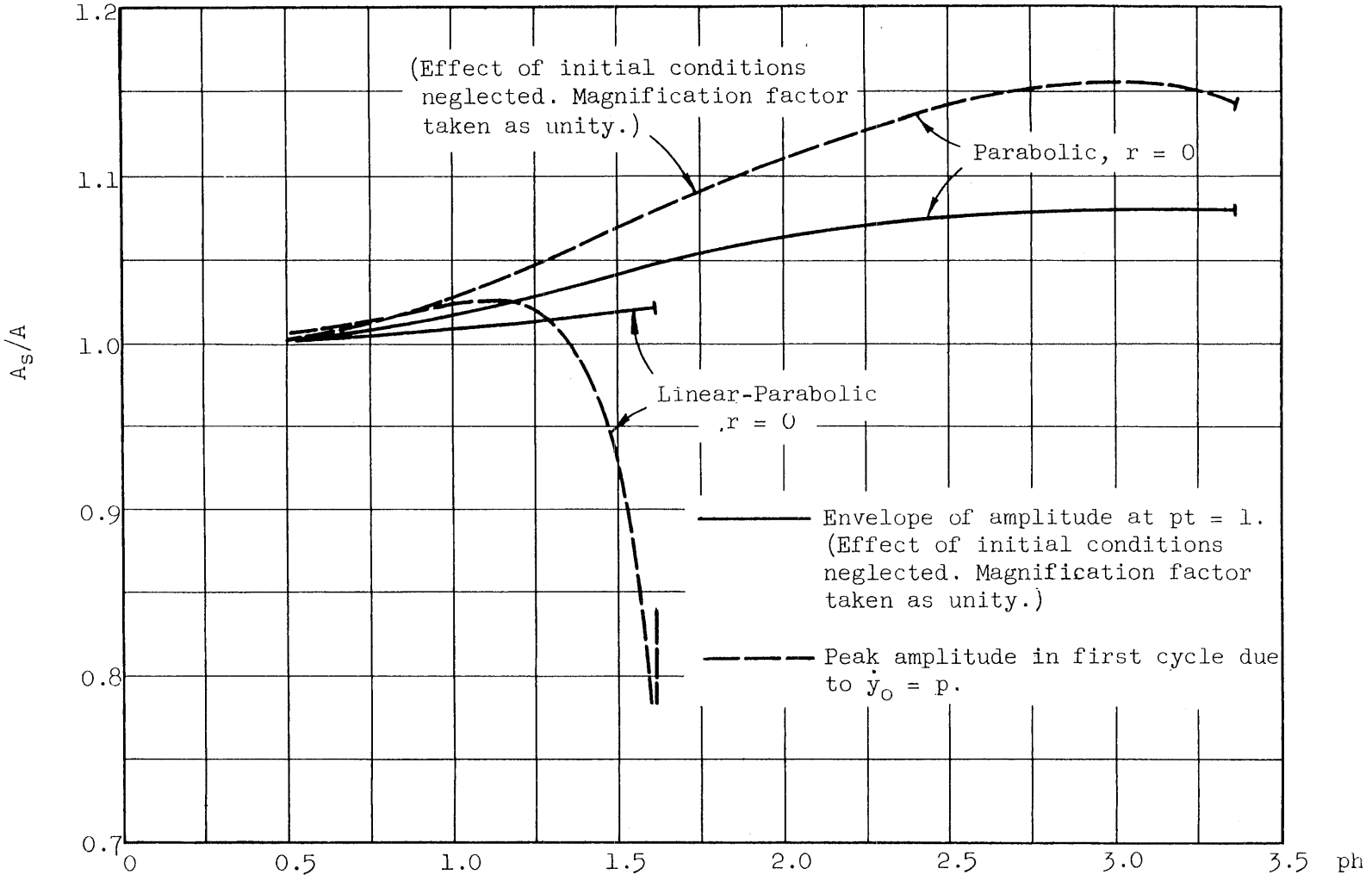

Fig. 12 - Parabolic Acceleration Method: Error in Amplitude

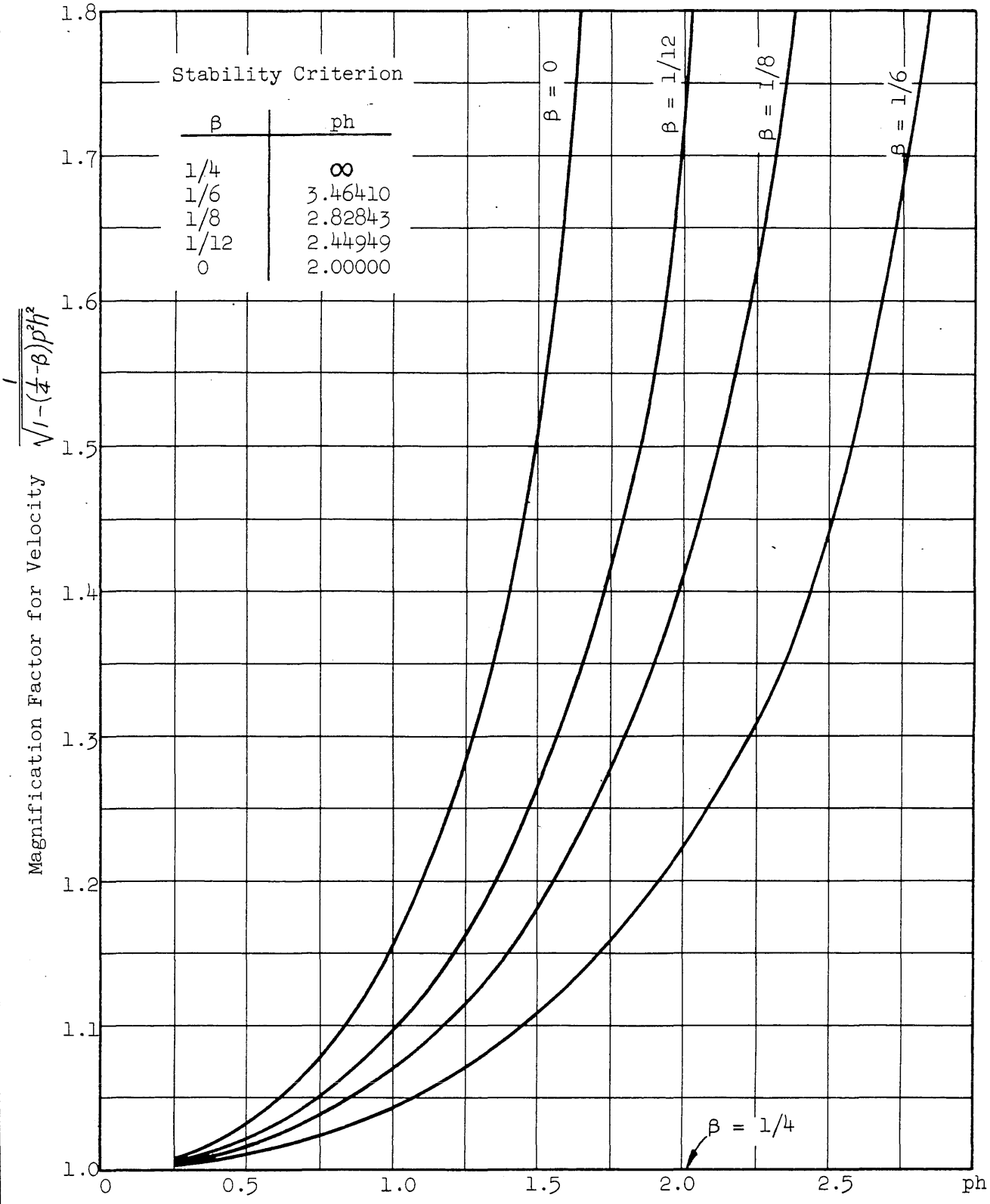

Fig. 13 - Newmark's β-Method: Magnification Factor for Velocity

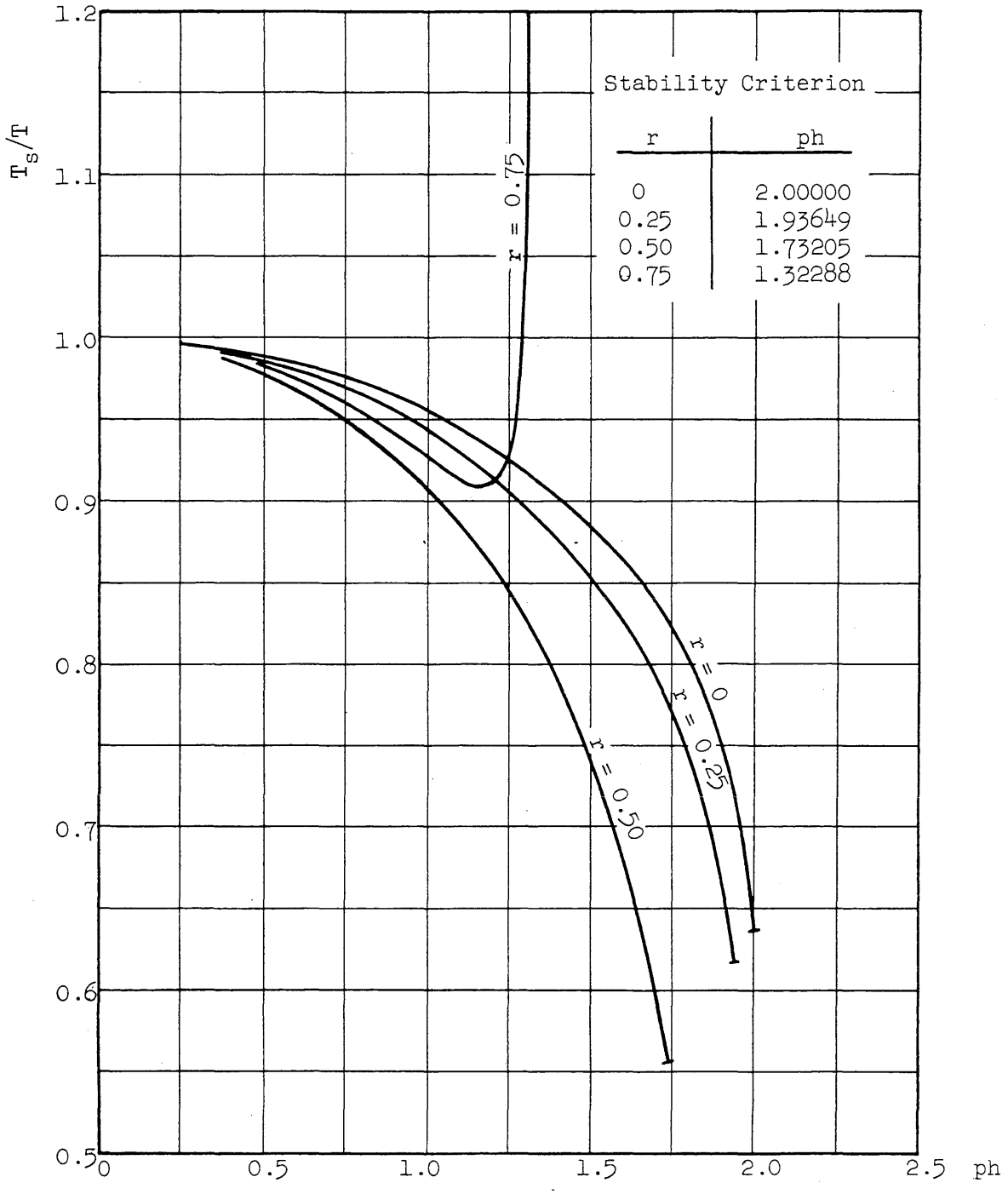

Fig. 14 - Levy's Method (or Newmark's Method for $\beta = 0$): Error in Period

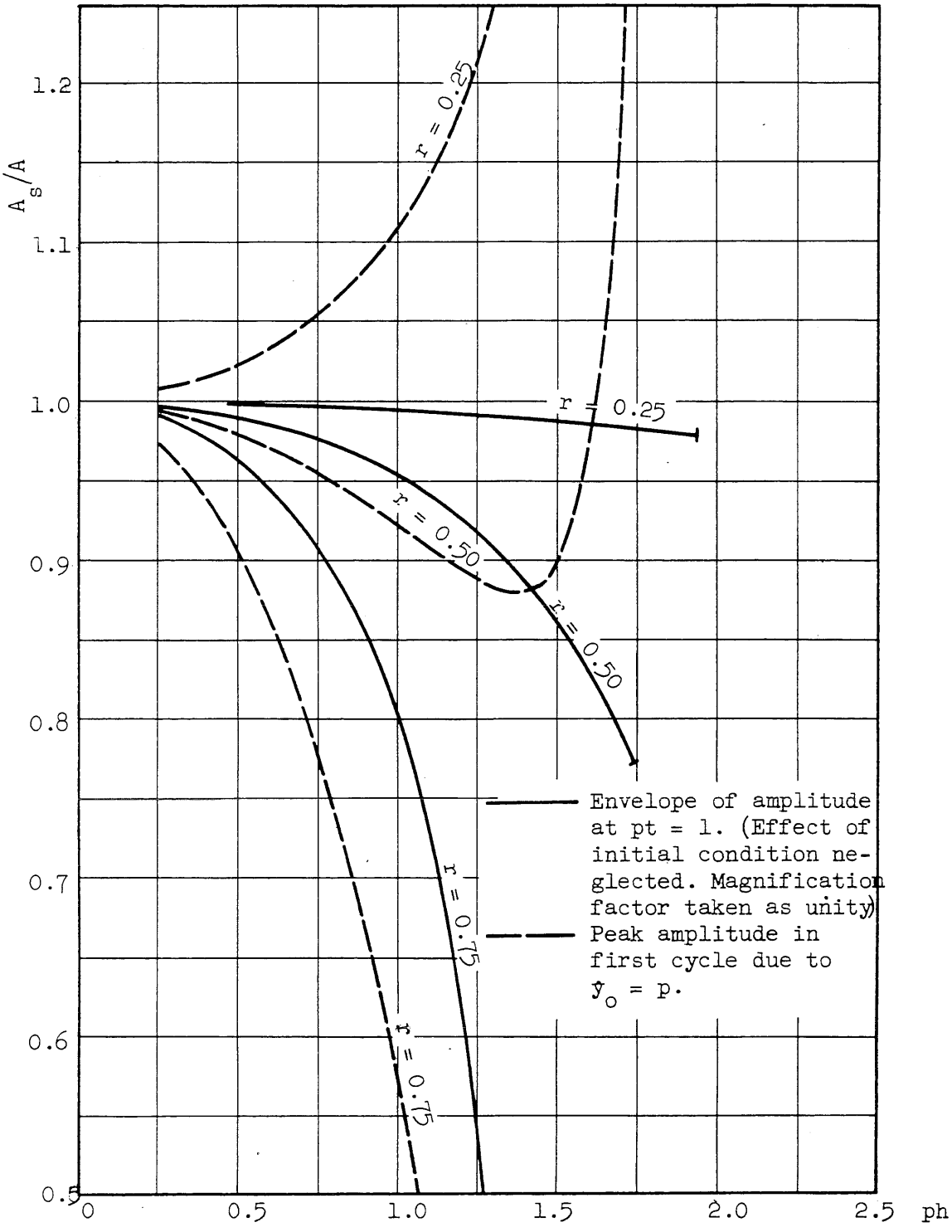

Fig. 15 - Levy's Method or Newmark's Method for $\beta = 0$: Error in Amplitude.

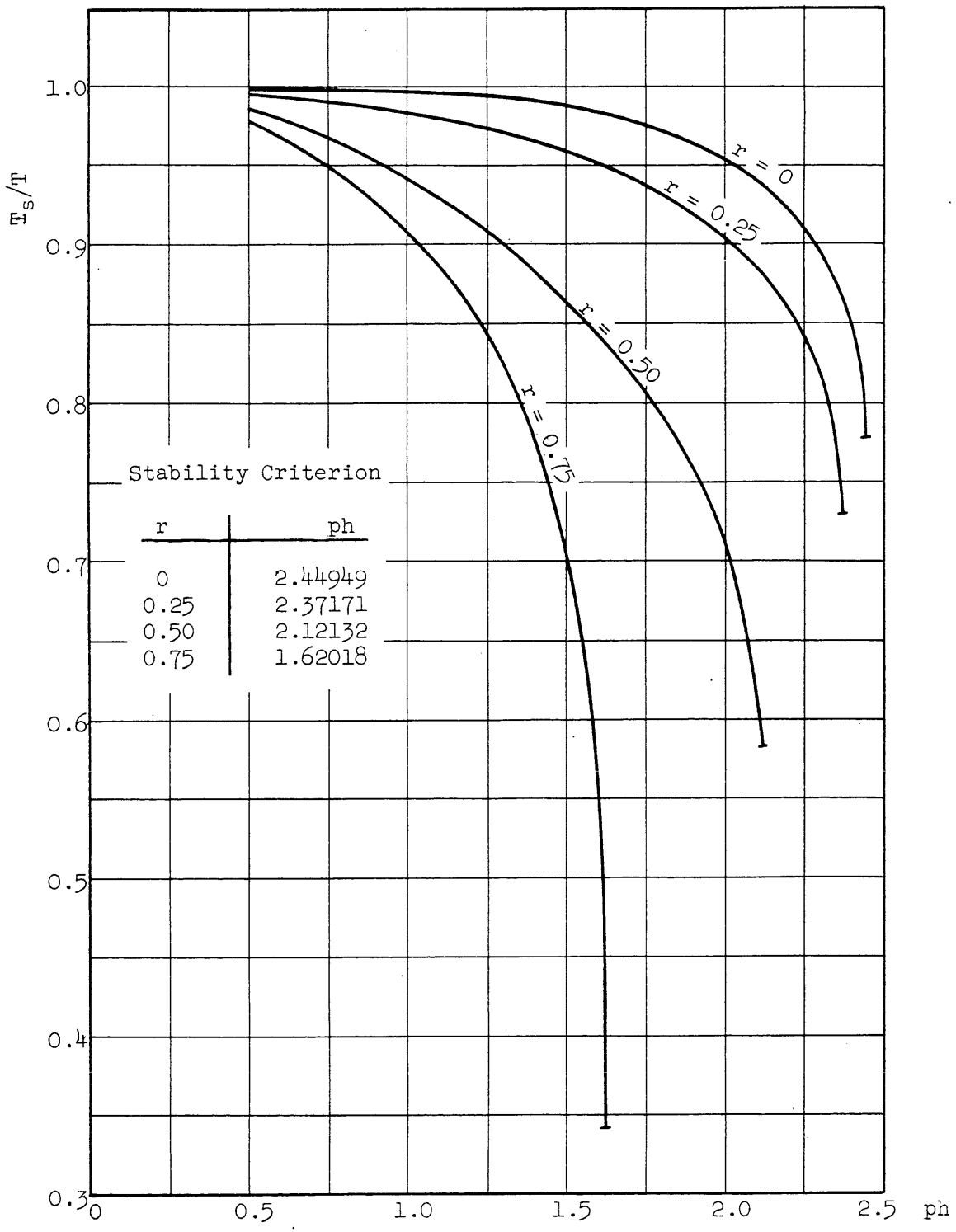

Fig. 16 - Newmark's β-Method for β = 1/12 and Salvadori's Method: Error in Period

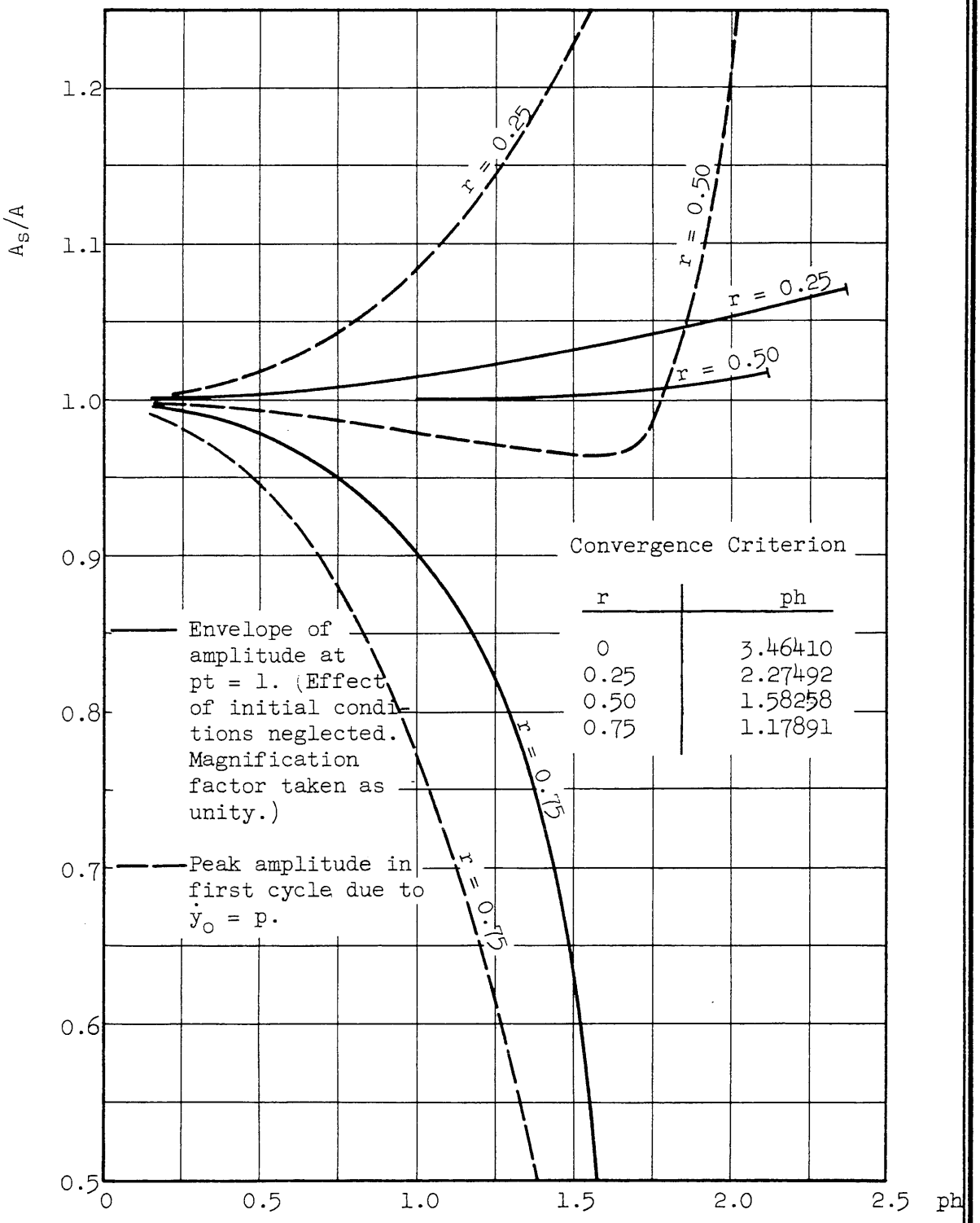

Fig. 17 - Newmark's β-Method for β = 1/12 and Salvadori's Method: Error in Amplitude

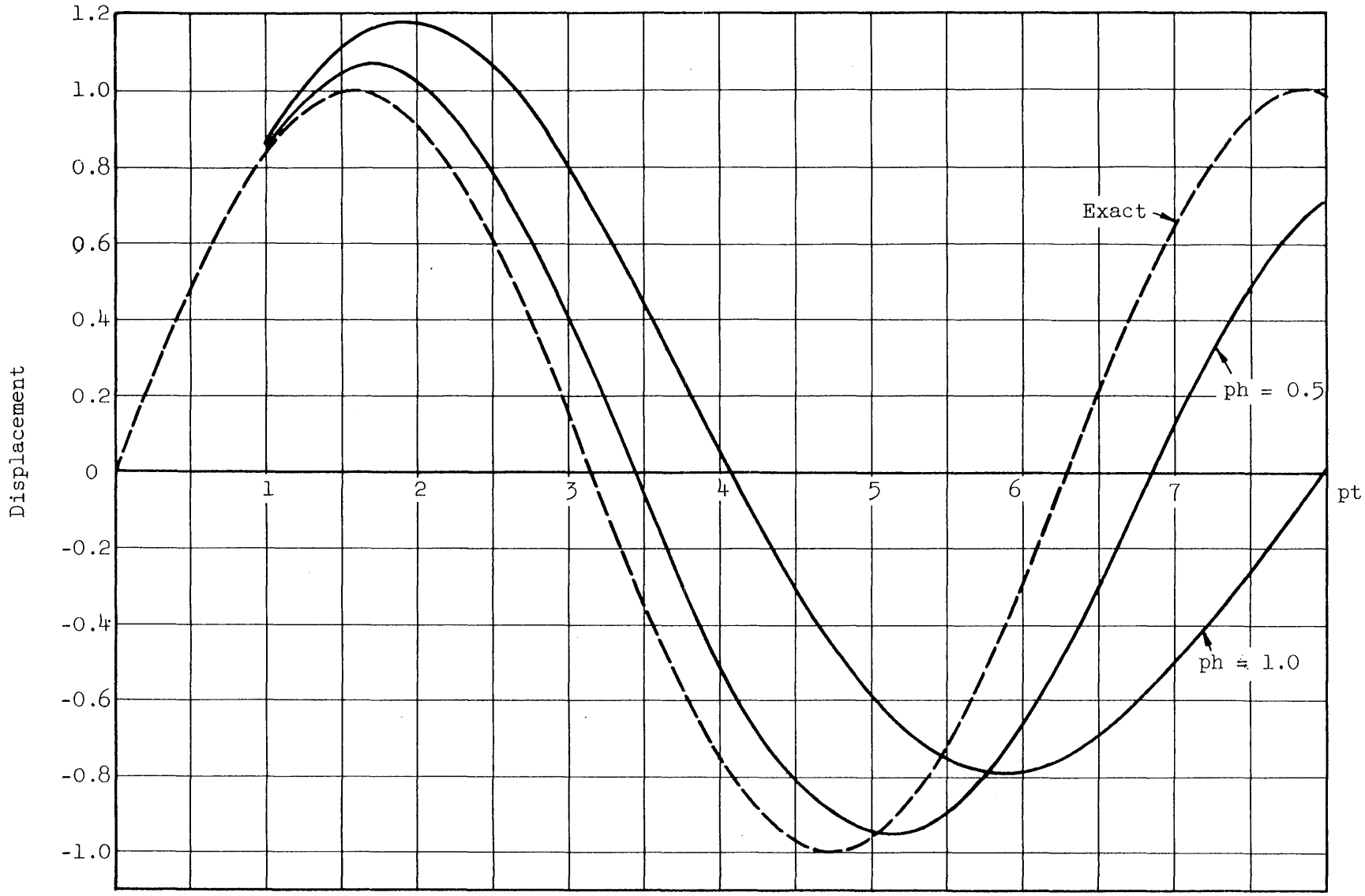

Fig. 18 - Houbolt's Method - Displacement in First Cycle due to $\dot{y}_o = p$.
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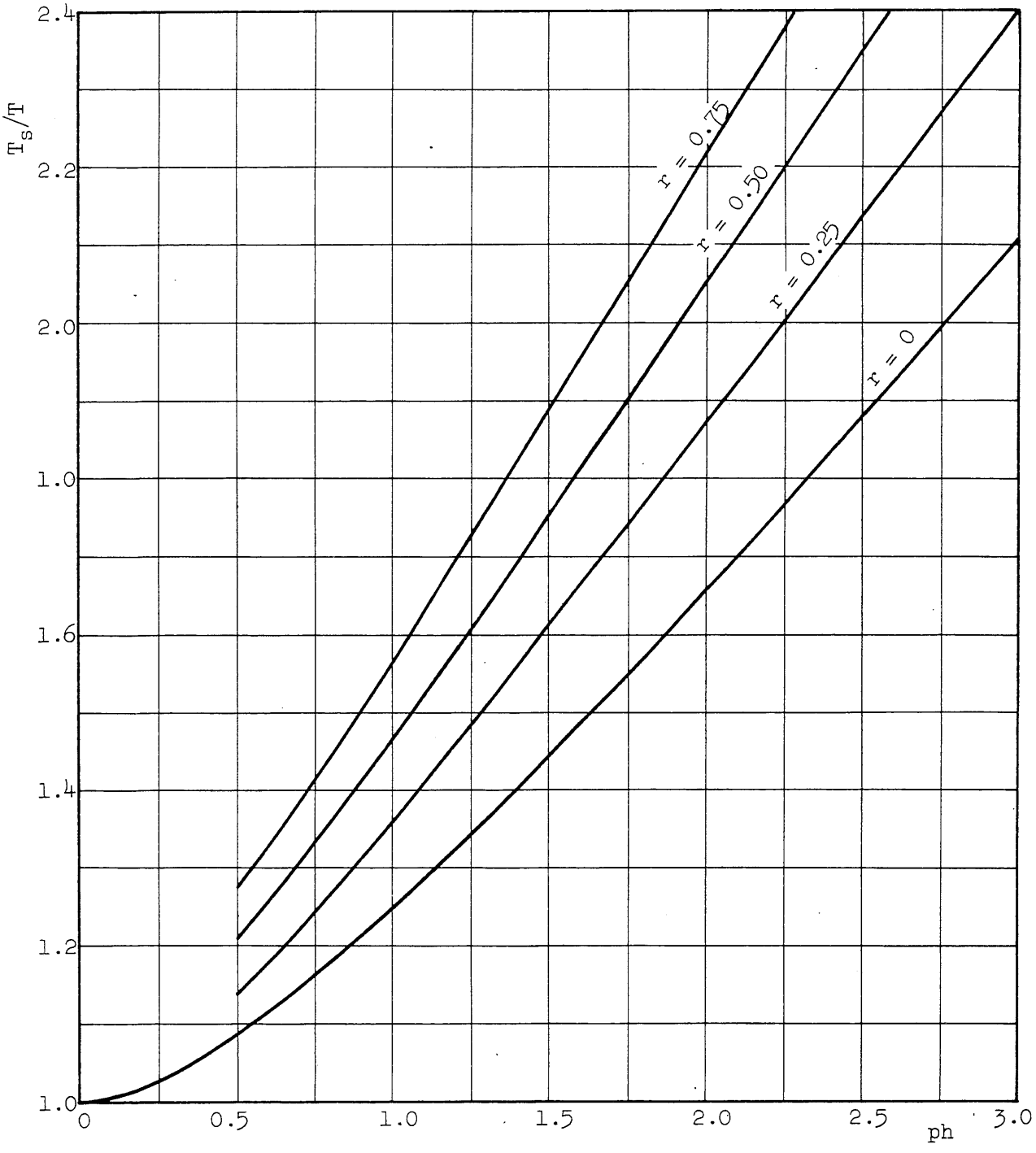


Fig. 19 - Houbolt's Method: Error in Period

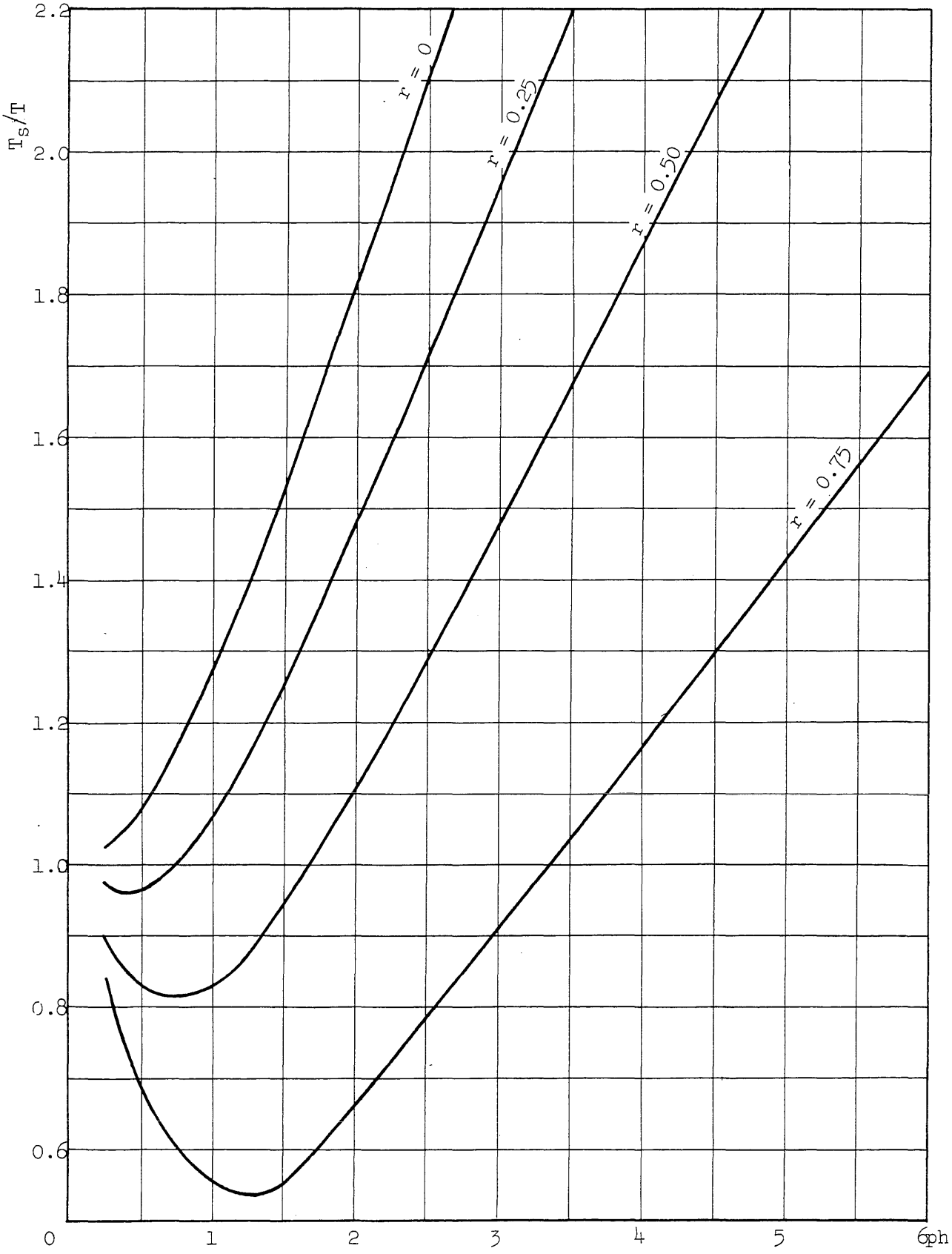
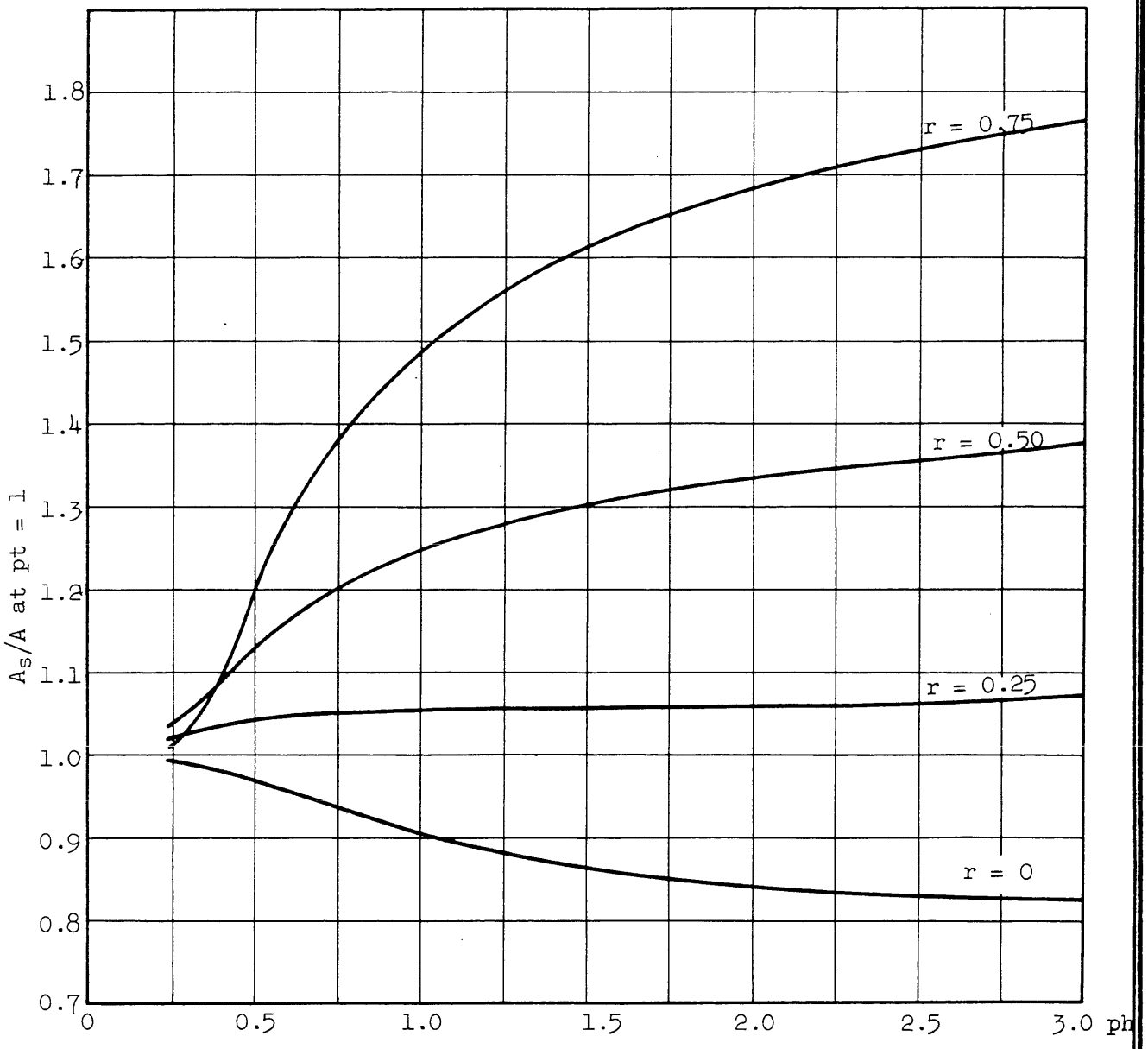


Fig. 21. - Euler's Method: Error in Period



Fog. 20 - Houbolt's Method: Error in Amplitude

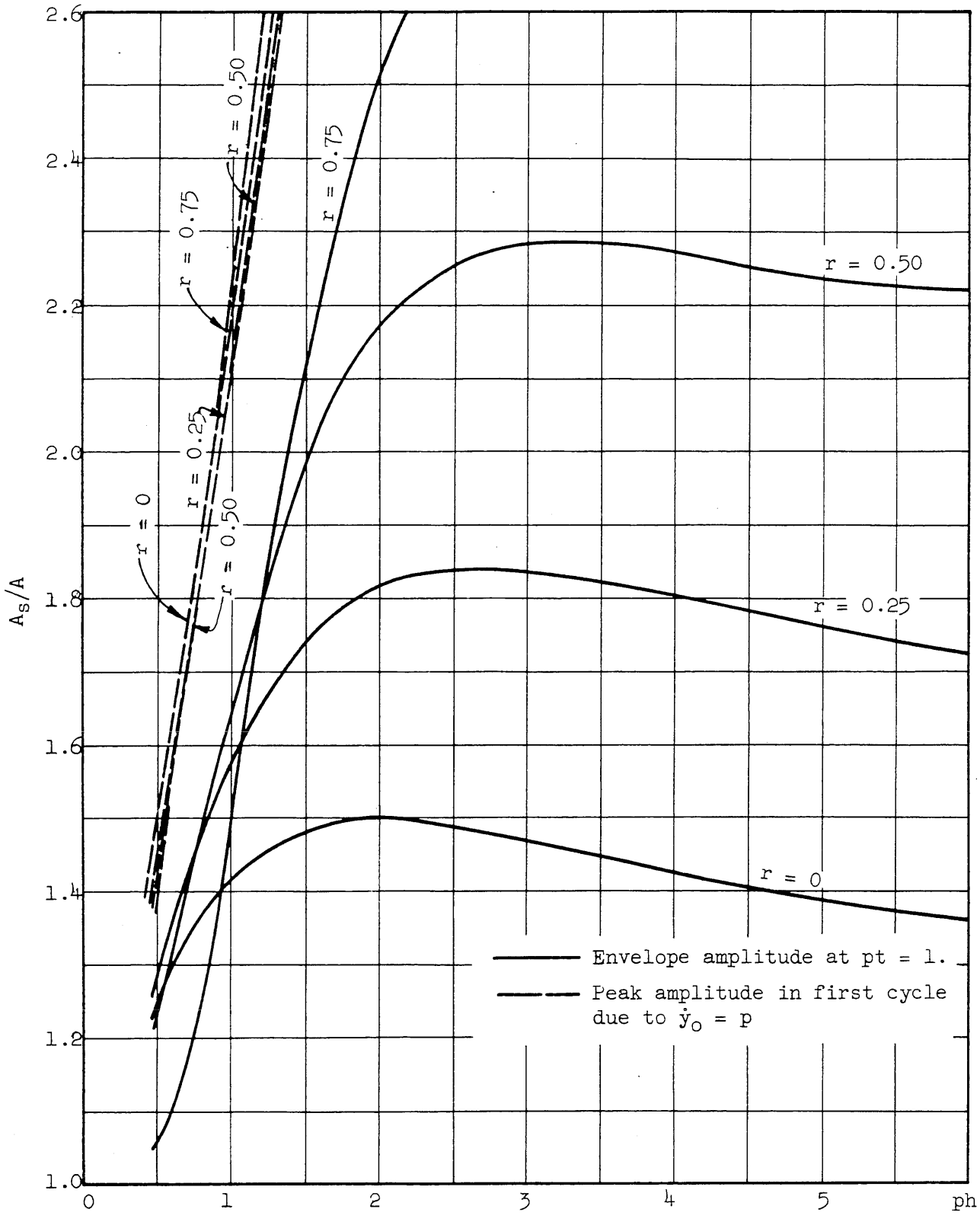


Fig. 22. - Euler's Method - Error in Amplitude

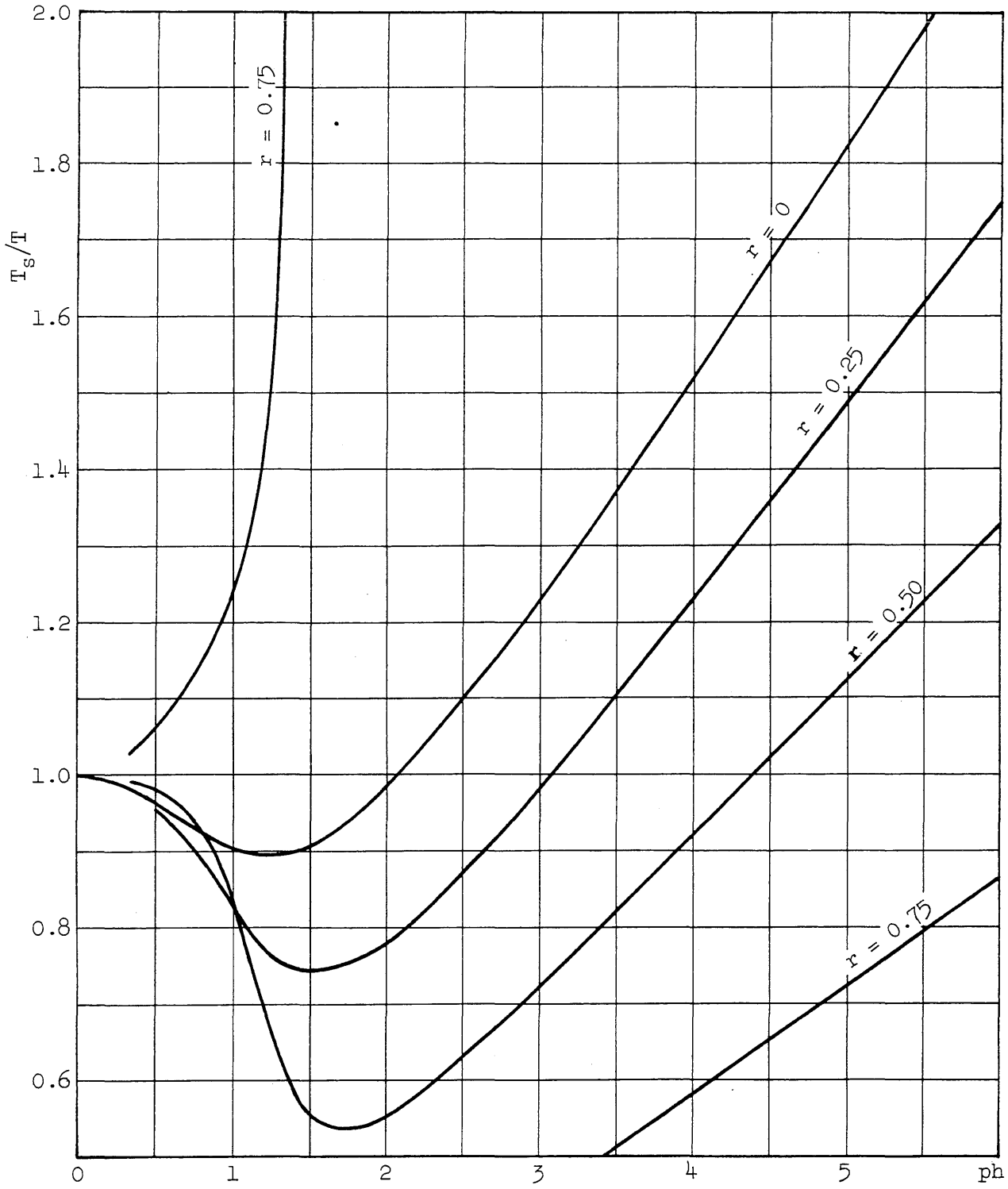


Fig. 23 - Modified Euler Method: Error in Period

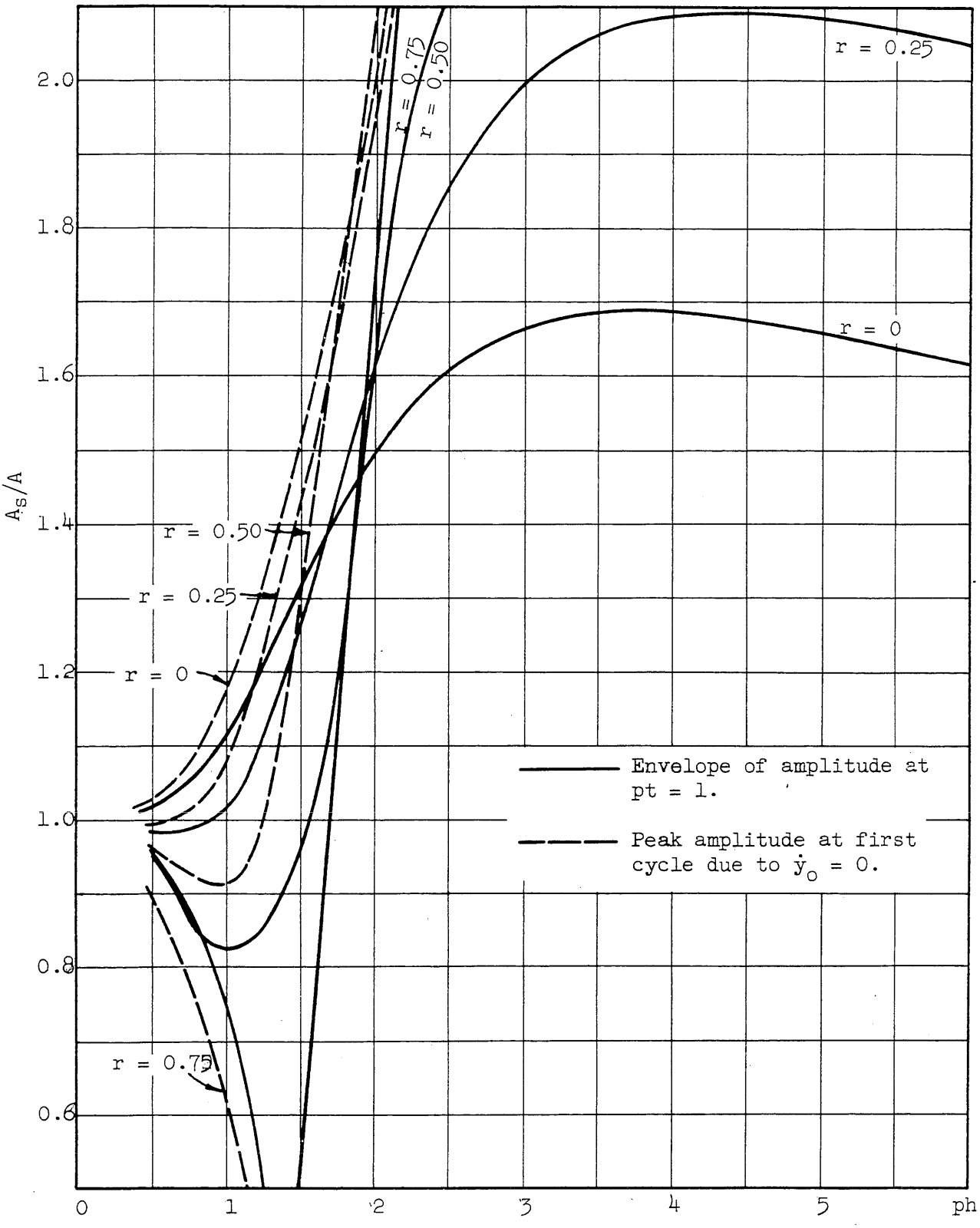


Fig. 24 - Modified Euler Method: Error in Amplitude

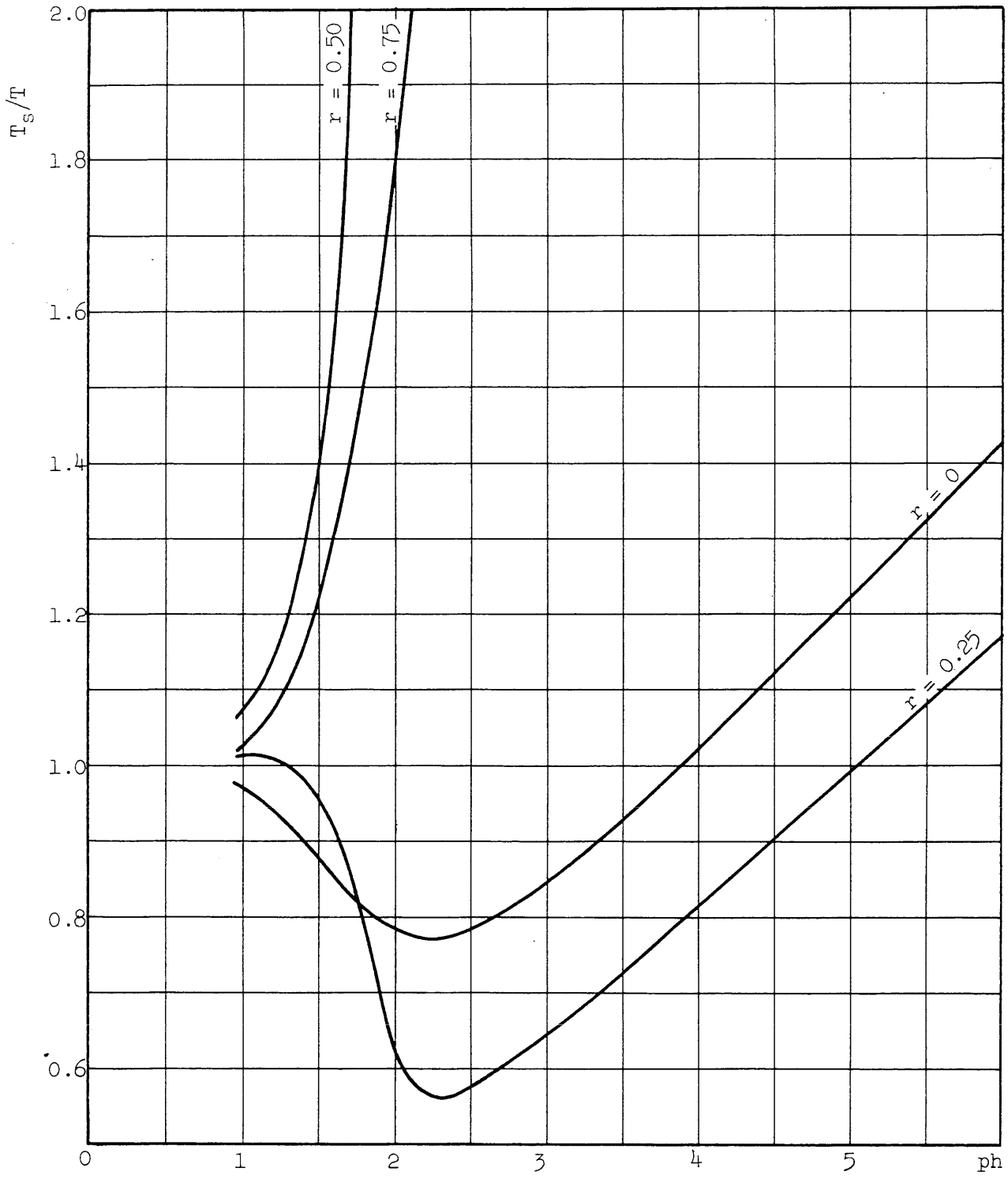


Fig. 25 - Runge, Heun and Kutta's Third Order Rule: Error in Period

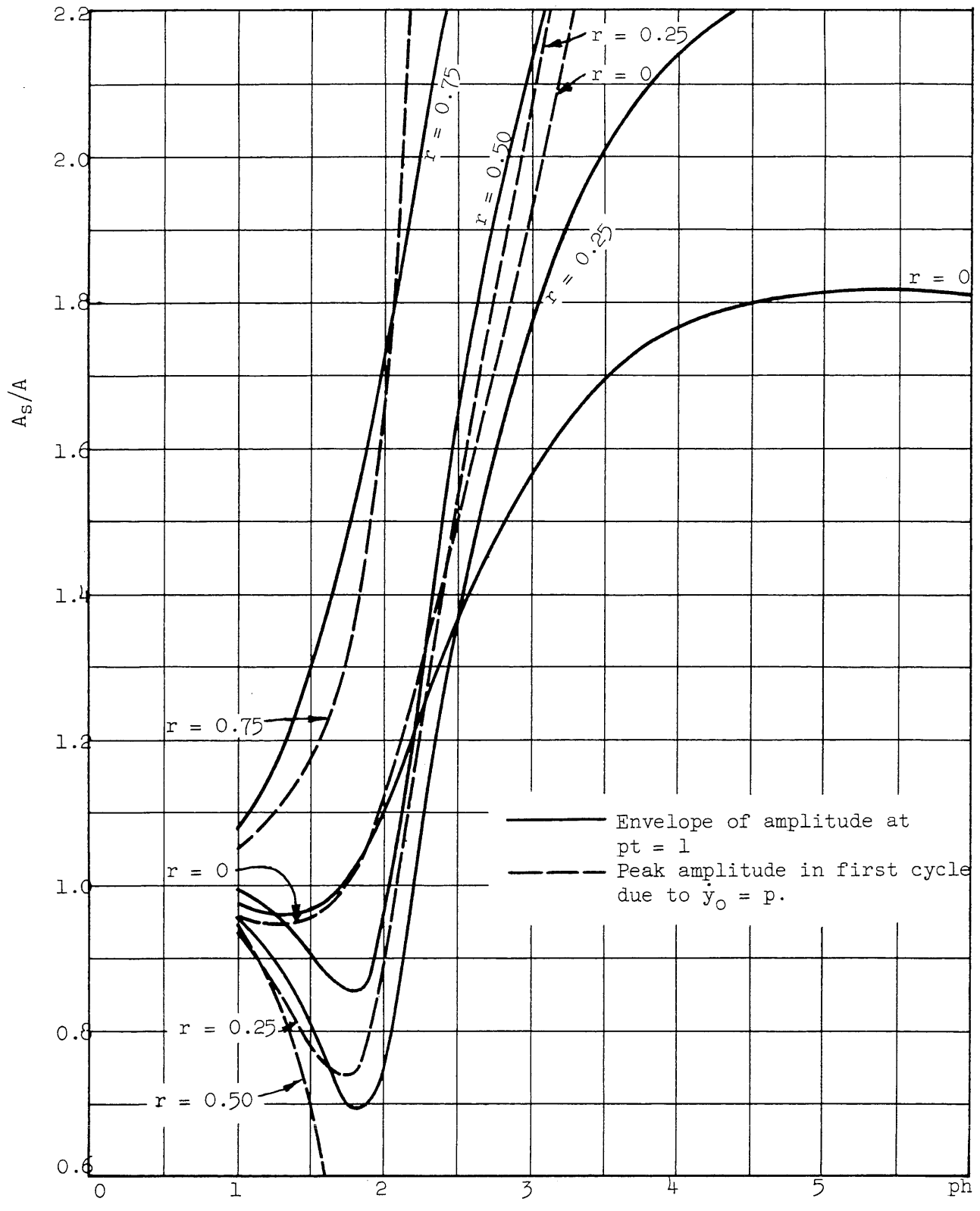


Fig. 26 - Runge, Heun and Kutta's Third Order Rule
Error in amplitude

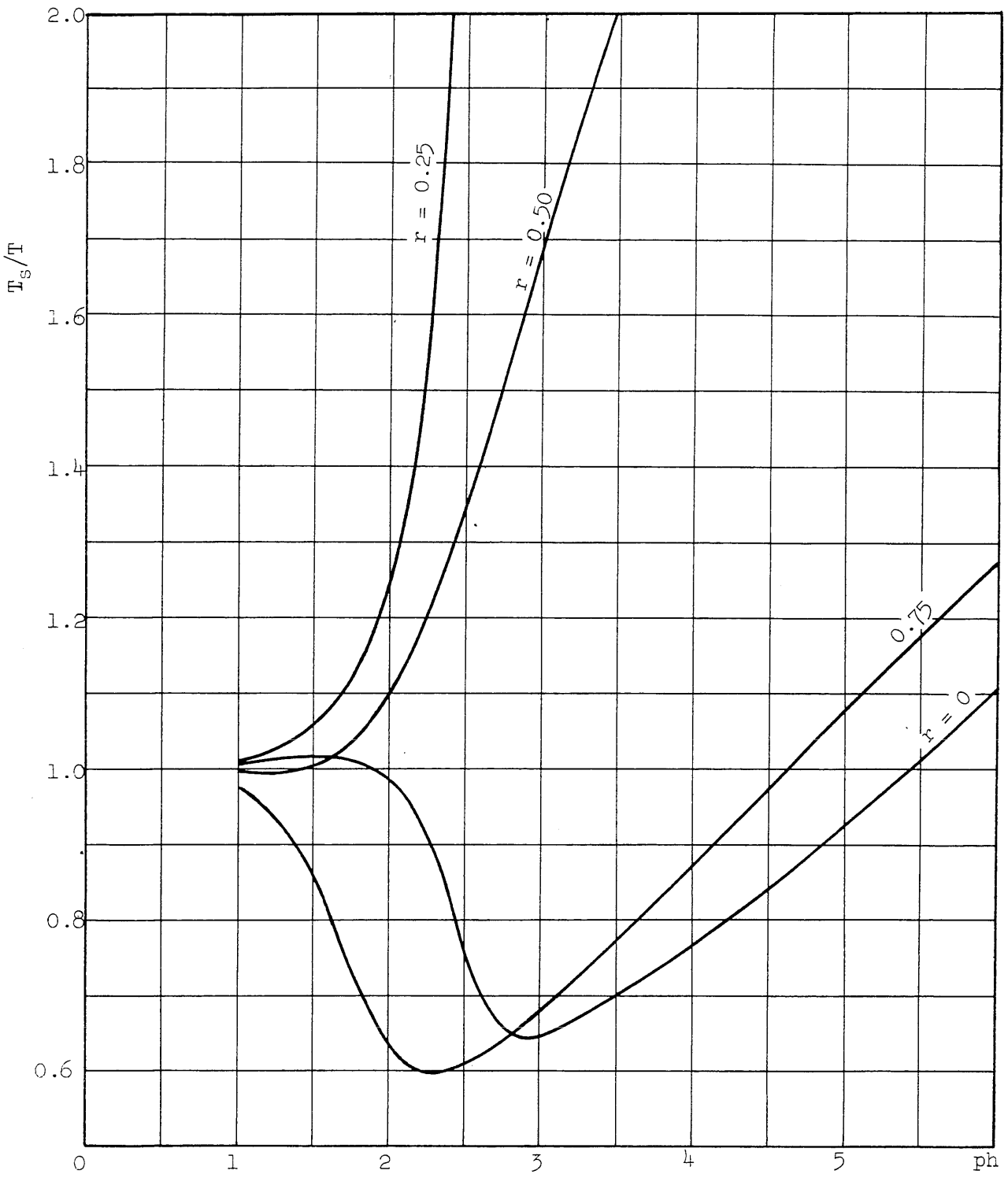


Fig. 27 - Kutta's Fourth Order Method: Error in Period

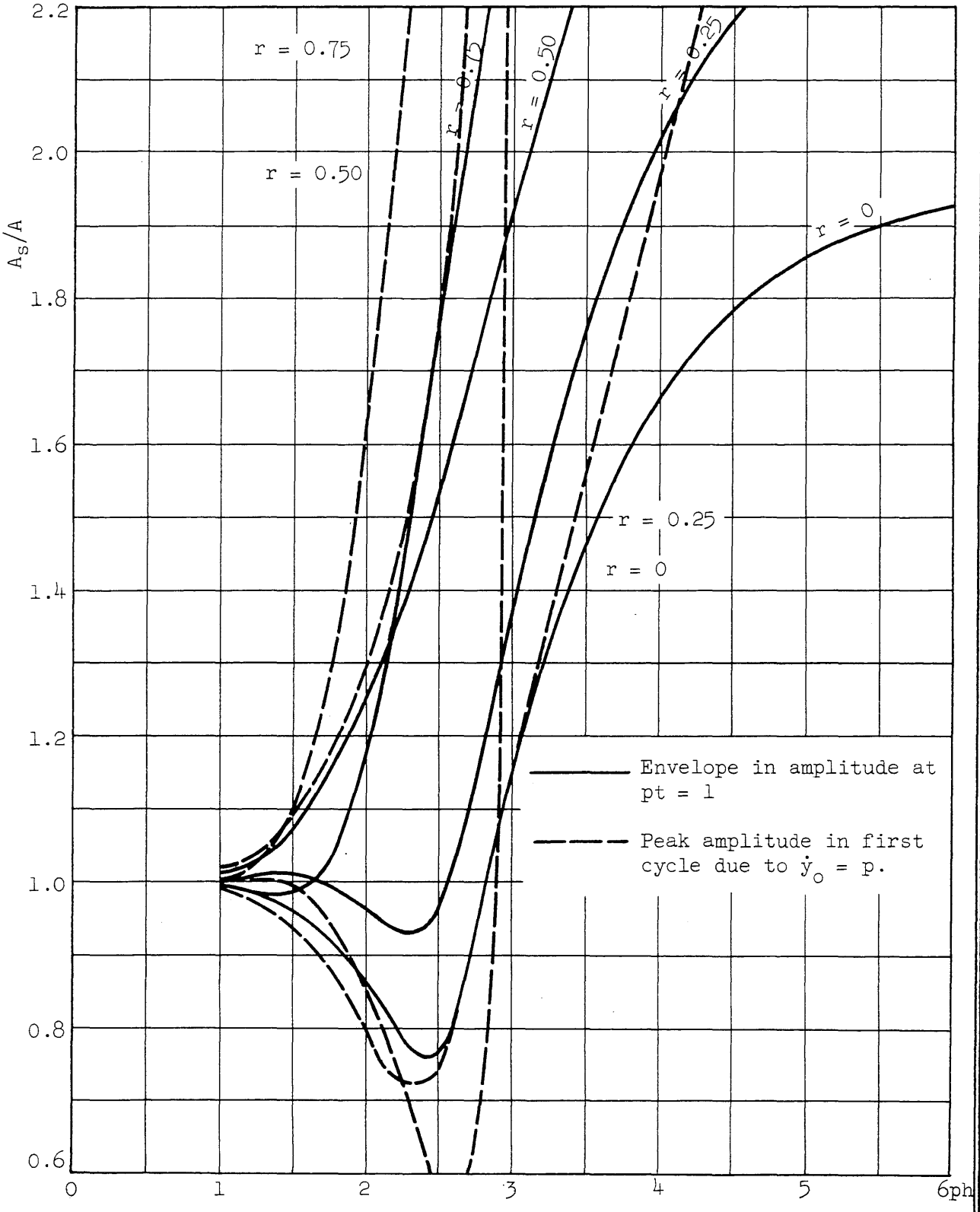


Fig. 28 - Kutta's Fourth Order Method: Error in Amplitude

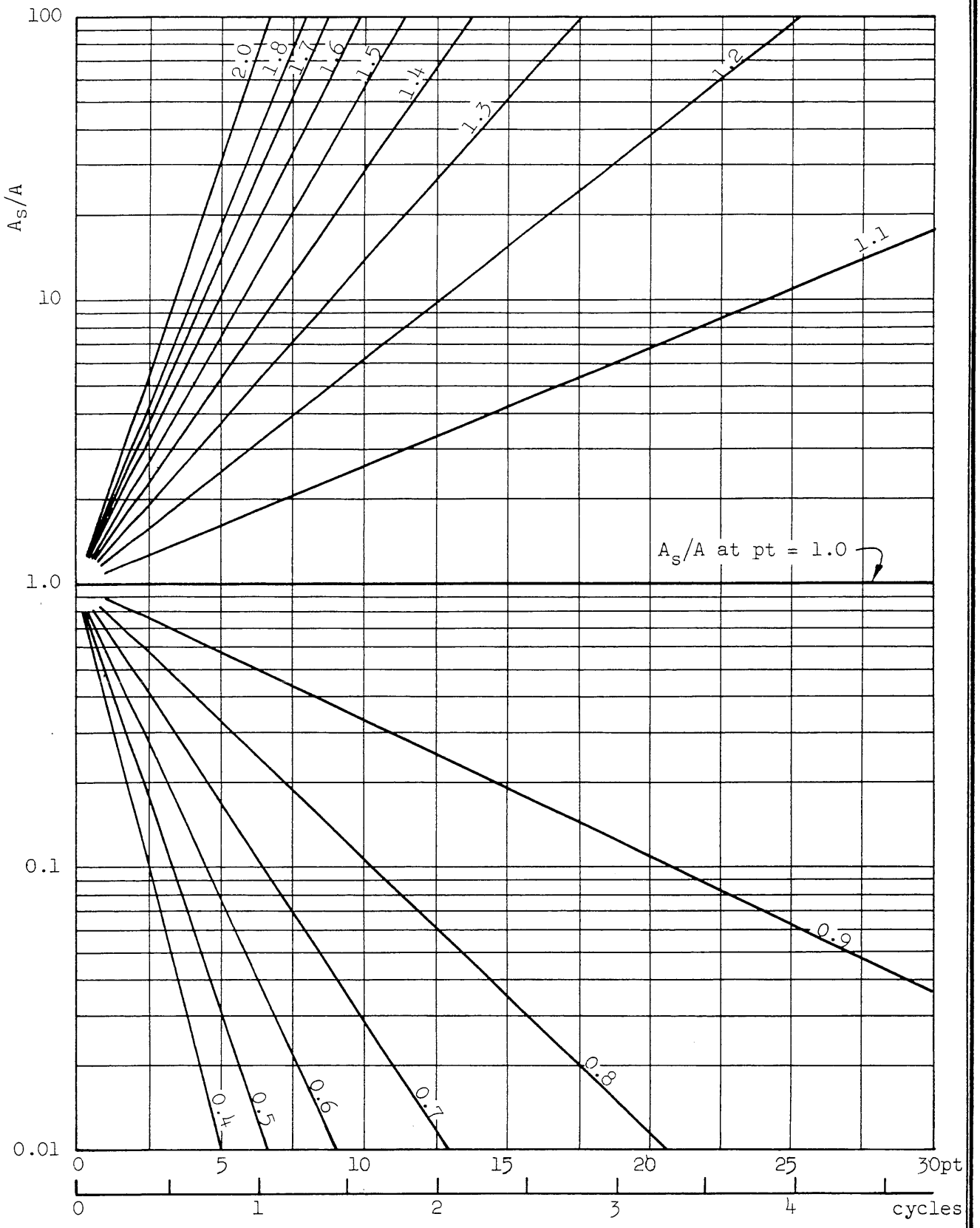


Fig. 29 - Values of A_s/A at Any Stage for Various Values of A_s/A at pt = 1.

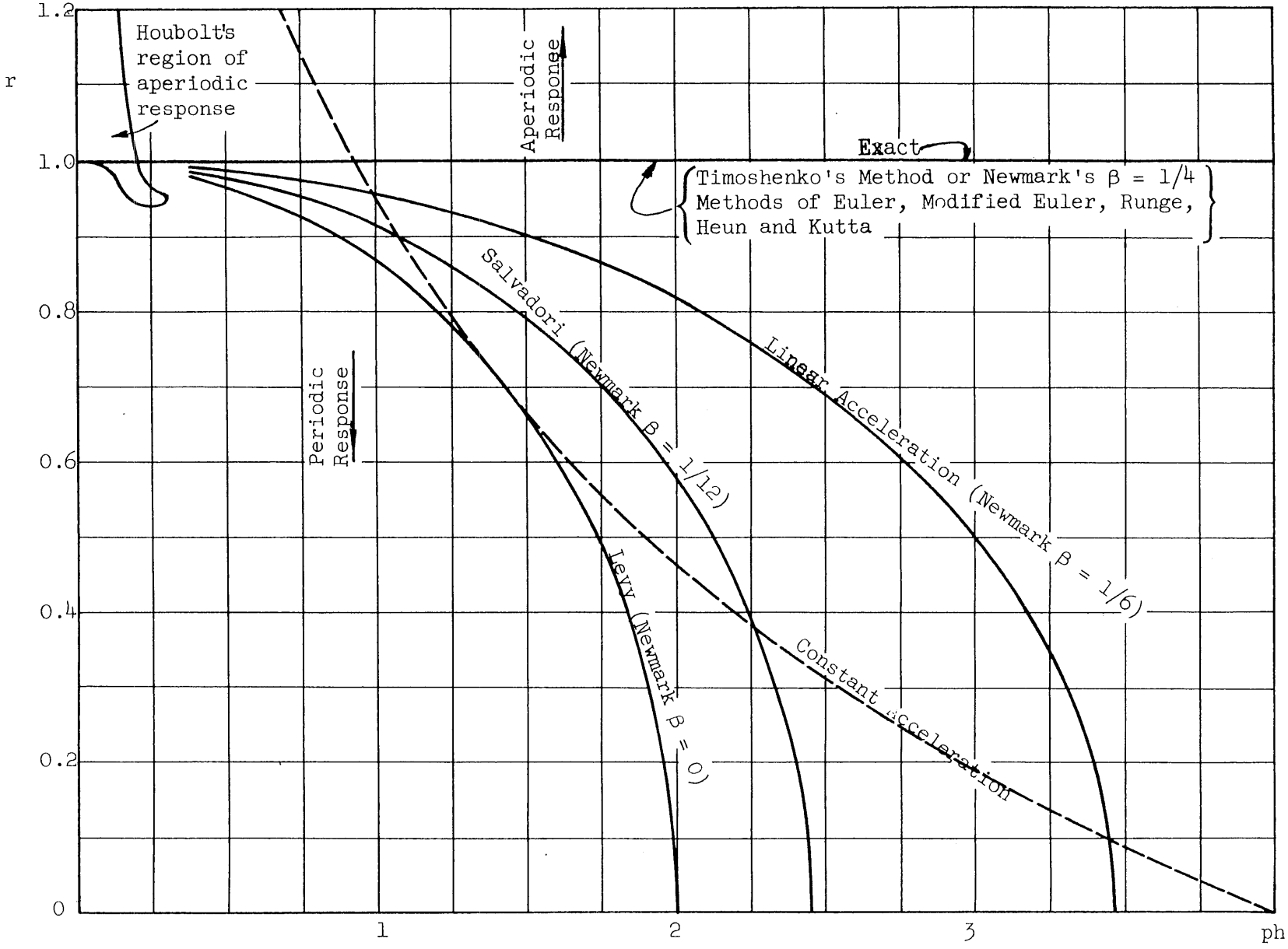


Fig. 30 - Criteria for Critical Damping

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