Numerical Methods for Civil Engineers

Lecture 9 Numerical Integration

- Basic Ideas
- Symbolic vs. Numerical Integration
- Trapezoid Rule
- Simpson's Rule
- MATLAB quad and quad8 Functions

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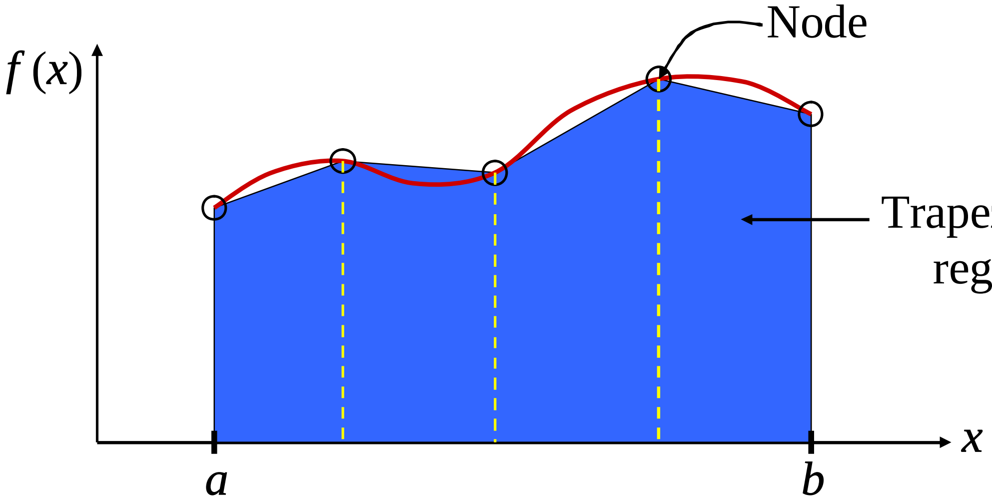
BASIC IDEAS

$$I = \int_a^b f(x)\,dx = \text{Area under curve } f(x)$$



Approximated by piecewise-linear:

Area = Sum of trapezoidal regions

Symbolic Math Toolbox

Symbolic Math Toolboxes incorporate symbolic computation into the numeric environment of MATLAB.

Symbolic Integration with MATLAB

```
>> I = int(f)            % Indefinite integral
>> I = int(f, v)         % Designating integration variable
>> I = int(f, a, b)      % Definite integral on close interval
>> I = int(f, v, a, b)
```

where **f** = symbolic expression

Variables must be define as symbolic by **sym** or **syms**

```
>> x = sym('x'), y = sym('y'), z = sym('z')
or >> syms x y z
```

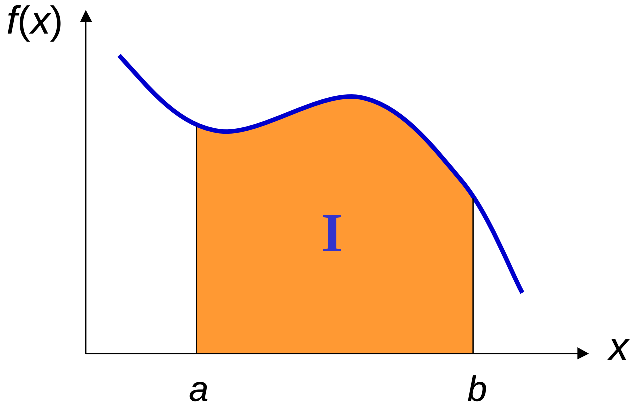
Example : $I = \int_a^b (x^3 - c)\, dx$

```
>> syms x a b c
>> I = int(x^3-c,x,a,b)

I =

1/4*b^4-c*b-1/4*a^4+c*a
```

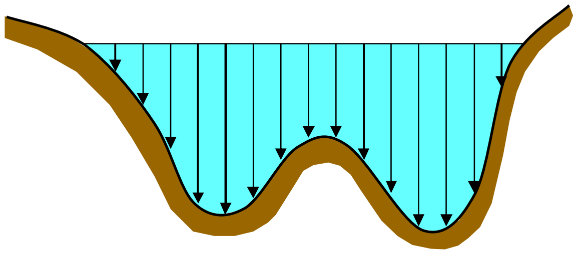
What is Integration?

The integral is equivalent to the area under the curve.

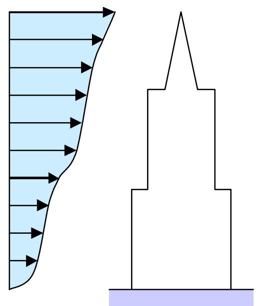
$$I = \int_a^b f(x)\,dx$$



Examples of how integration is used to evaluate areas



Cross-sectional area of a river



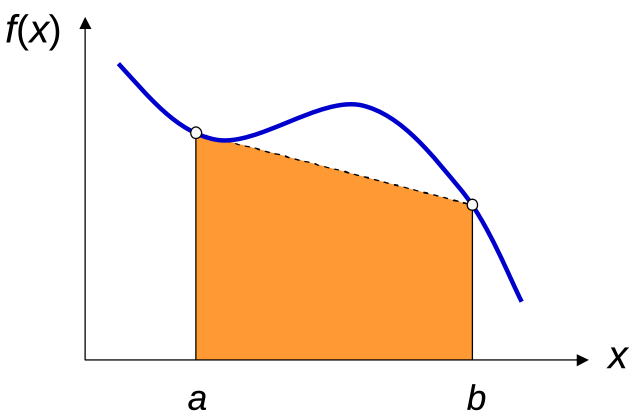
Net force due to wind blowing against the building

Newton-Cotes Formulas

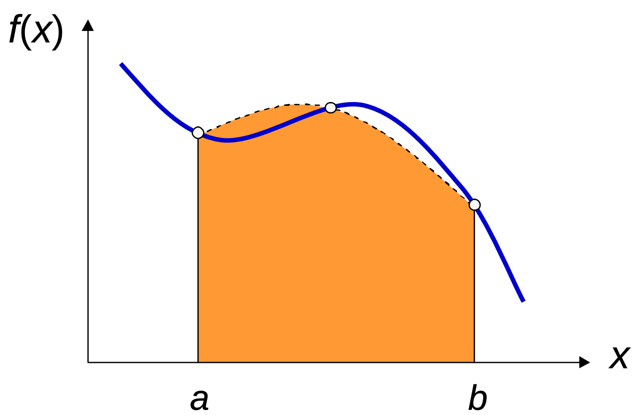
Replace a complicated function with a polynomial that is easy to integrate

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

where $f_n(x) = a_0 + a_1 x + \cdots + a_n x^{n-1} + a_{n+1} x^n$

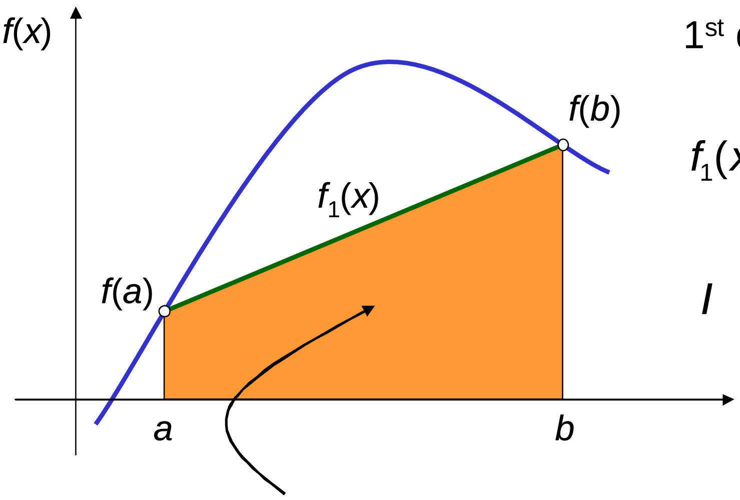


Approximate area by a straight line



Approximate area by a parabola

Trapezoidal Rule



1st degree polynomial = Linear line

$$f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$$I = \int_a^b f_1(x)\,dx = (b-a)\frac{f(a)+f(b)}{2}$$

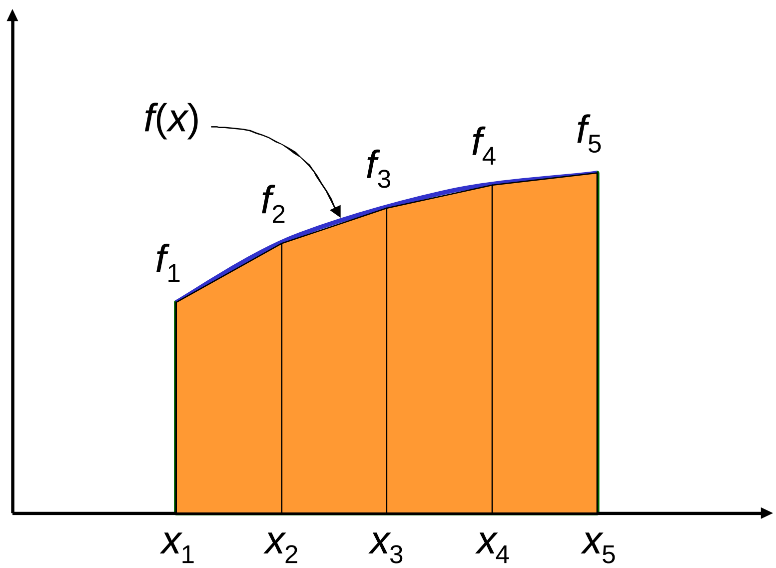
Area of trapezoid = width x average height

width = $b - a = h$

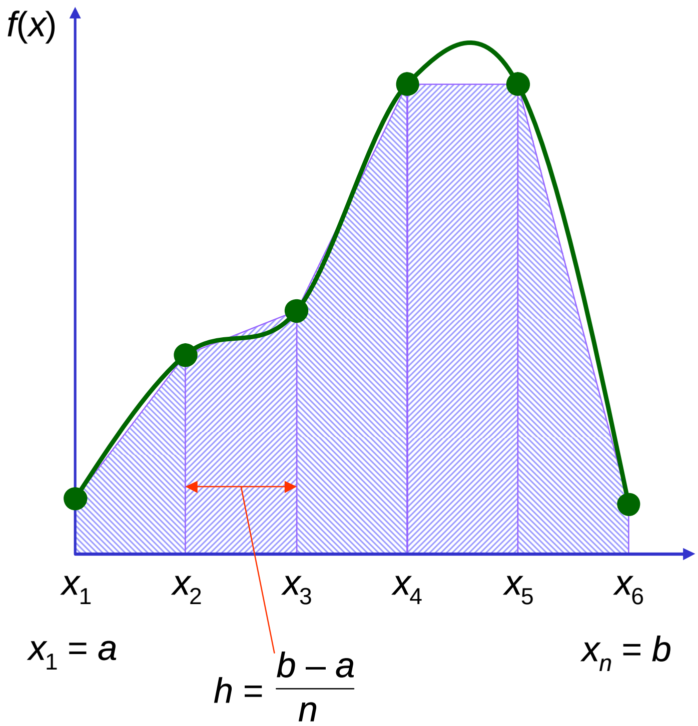
average height = $\dfrac{f(a)+f(b)}{2}$

Composite Trapezoidal Rule

Improve accuracy by dividing interval into subintervals



$$\int_{x_1}^{x_5} f(x)\,dx = \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \int_{x_3}^{x_4} f(x)\,dx + \int_{x_4}^{x_5} f(x)\,dx$$



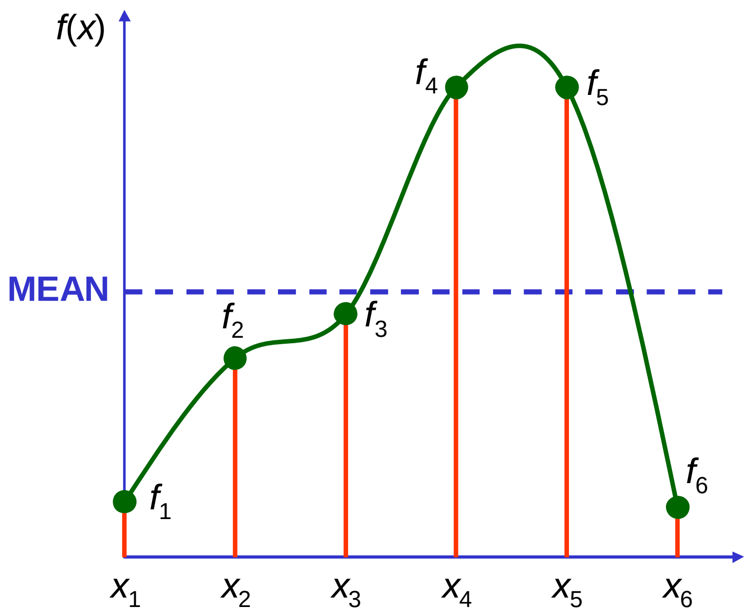
There are ***n*** segment with equal width:

$$h = \frac{b-a}{n}$$

$$I = \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \mathbf{L} + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$I = h\frac{f(x_1)+f(x_2)}{2} + h\frac{f(x_2)+f(x_3)}{2} + \mathbf{L} + h\frac{f(x_{n-1})+f(x_n)}{2}$$

$$I = h\frac{f_1+f_2}{2} + h\frac{f_2+f_3}{2} + \mathbf{L} + h\frac{f_{n-1}+f_n}{2} \Rightarrow I = \frac{h}{2}\left[f_1 + 2\sum_{i=2}^{n-1} f_i + f_n\right]$$



Substitute $h = \frac{b-a}{n}$ $\Rightarrow$ $I = \underbrace{(b-a)}_{\text{width}} \underbrace{\frac{\left[f_1 + 2\sum_{i=2}^{n-1} f_i + f_n\right]}{2n}}_{\text{average height}}$

Example 9-1 Use Trapezoidal rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a = 0 to b = 0.8. Note that the exact value of the integral can be determined analytically to be 1.640533.

or using MATLAB's Symbolic Math Toolbox :

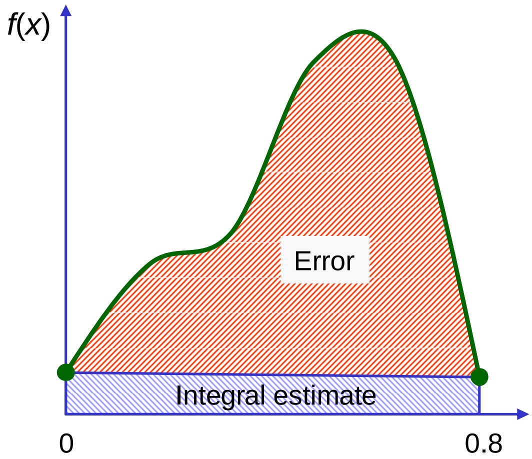
```
>> x = sym('x')
>> I=int(0.2+25*x-200*x^2+675*x^3-900*x^4+400*x^5,x,0,0.8)

I = 3076/1875 = 1.6405
```

Single Trapeziod: $f(0) = 0.2$ and $f(0.8) = 0.232$

$$I = (b-a)\frac{f(a)+f(b)}{2}$$

$$= (0.8-0)\frac{0.2+0.232}{2} = 0.1728$$

$$e_t = \left|\frac{1.6405-0.1728}{1.6405}\right| \times 100\% = 89.5\%$$



2 Trapeziods: $n = 2$ ($h = 0.4$) :

$f(0) = 0.2$ $f(0.4) = 2.456$ $f(0.8) = 0.232$

$$I = (b-a)\frac{\left[f_1 + 2\sum_{i=2}^{n-1} f_i + f_n\right]}{2n}$$

$$I = (0.8)\frac{0.2+2(2.456)+0.232}{4} = 1.0688$$

$$e_t = \left|\frac{1.6405-1.0688}{1.6405}\right| \times 100\% = 34.9\%$$

MATLAB's `trapz` function

Trapezoidal numerical integration

```
>> z = trapz(y)     % Integral of y with unit spacing
>> z = trapz(x, y)  % Integral of y with respect to x
```

Example: $\int_{0^o}^{90^o} \sin x \, dx$

```
>> angle = 0:15:90;
>> x = (pi*angle/180);
>> y = sin(x);
>> z = trapz(x,y)
>> z =
      0.9943
```

Example 9-2 Use MATLAB's `trapz` function to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a = 0 to b = 0.8. Note that the exact value of the integral can be determined analytically to be 1.640533.

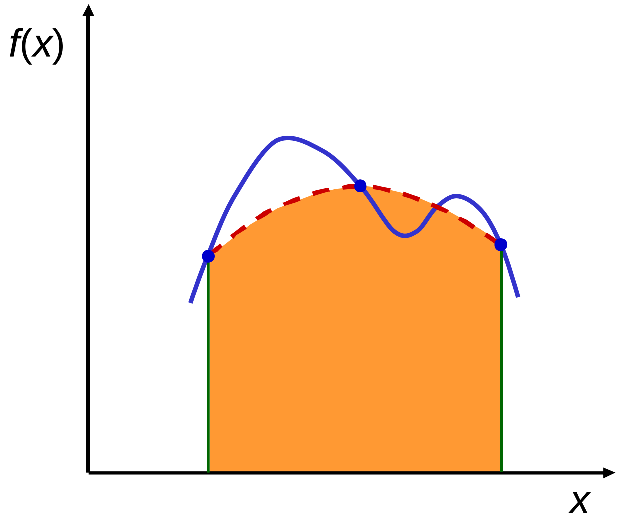
For $n = 2$:

```
>> x = linspace(0,0.8,3);
>> fx = 0.2+25*x-200*x.^2+675*x.^3-900*x.^4+400*x.^5;
>> I = trapz(x,fx)
I =
  1.0688
```

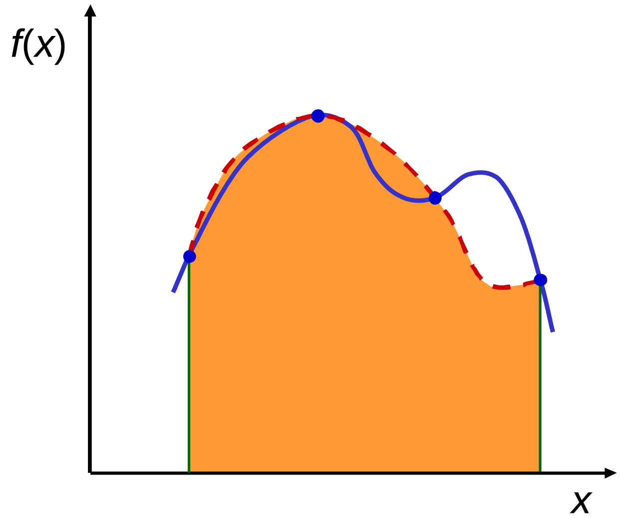
n	I	e_t
2	1.0688	34.9
3	1.3639	16.5
4	1.4848	9.49
5	1.5399	6.13
6	1.5703	4.28

Simpson's Rules

Using higher-order polynomials to connect the points



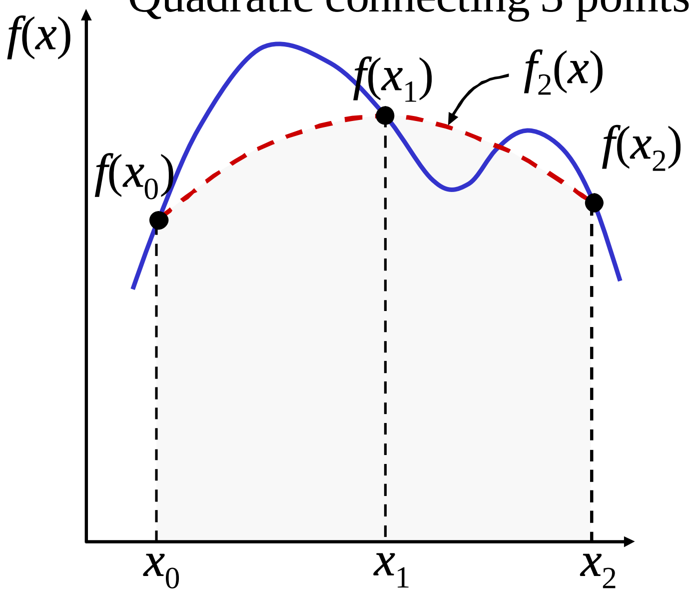
Simpson's 1/3 rule
Quadratic connecting 3 points



Simpson's 3/8 rule
Cubic connecting 4 points

Simpson's 1/3 Rules

Quadratic connecting 3 points



$$I = \int_{x_0}^{x_2} f(x)dx \cong \int_{x_0}^{x_2} f_2(x)dx$$

$$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

$$I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)], \quad h = \frac{x_2 - x_0}{2}$$

Example 9-3 Use Simpson's 1/3 rule to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a = 0 to b = 0.8. Note that the exact value of the integral can be determined analytically to be 1.640533.

Solution: n = 2 (h = 0.4) :

$f(0) = 0.2 \quad f(0.4) = 2.456 \quad f(0.8) = 0.232$

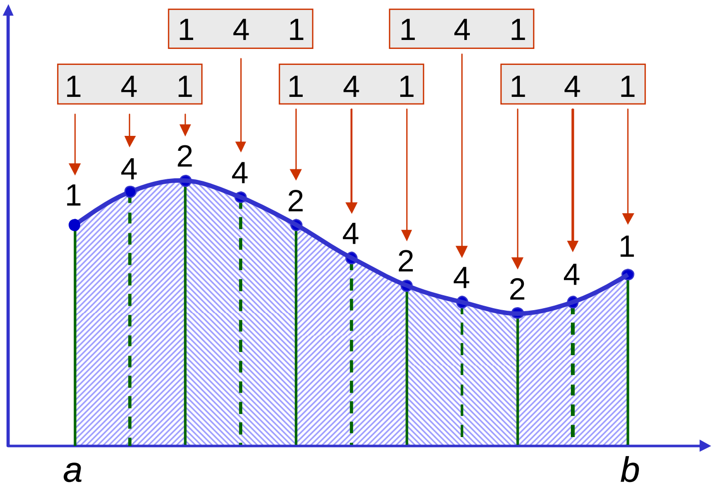
$$I = \frac{0.4}{3}(0.2 + 4(2.456) + 0.232) = 1.3675$$

$$e_t = \left|\frac{1.6405 - 1.3675}{1.6405}\right| \times 100\% = 16.6\%$$

Composite Simpson's 1/3 Rules

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)dx \quad \text{where} \quad h = \frac{b-a}{n}, \quad n = \text{even number}$$

$$I = \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \ldots + \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n]$$



$$I = \frac{(b-a)}{3n}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right]$$

Example 9-4 Use composite Simpson's 1/3 rule with n = 4 to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a = 0 to b = 0.8. Note that the exact value of the integral can be determined analytically to be 1.640533.

Solution: $n = 4$ ($h = 0.2$) :

$f(0) = 0.2$ $f(0.2) = 1.288$ $f(0.4) = 2.456$ $f(0.6) = 3.464$ $f(0.8) = 0.232$

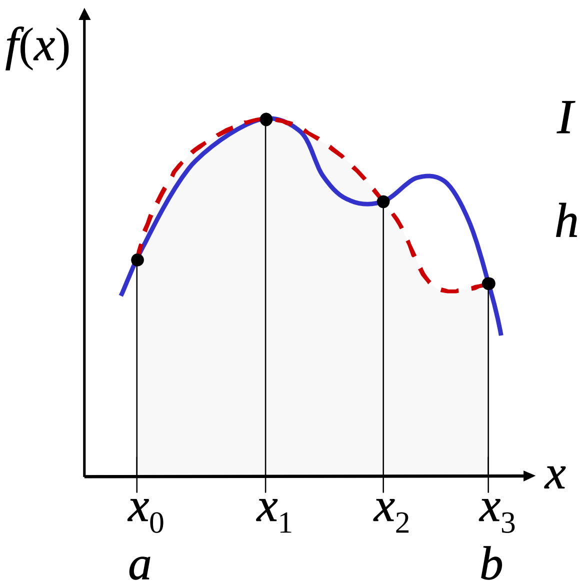
$$I = \frac{(b-a)}{3n}\left[f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)\right]$$

$$I = \frac{0.8}{3(4)}(0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232) = 1.624$$

$$e_t = \left|\frac{1.6405 - 1.624}{1.6405}\right| \times 100\% = 1.04\%$$

Simpson's 3/8 Rules

Cubic connecting 4 points



$$I = \int_a^b f(x)dx \cong \int_a^b f_2(x)dx$$

$$I \cong \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

$$h = (x_3 - x_0)/3$$

or…

$$I = \frac{(b-a)}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

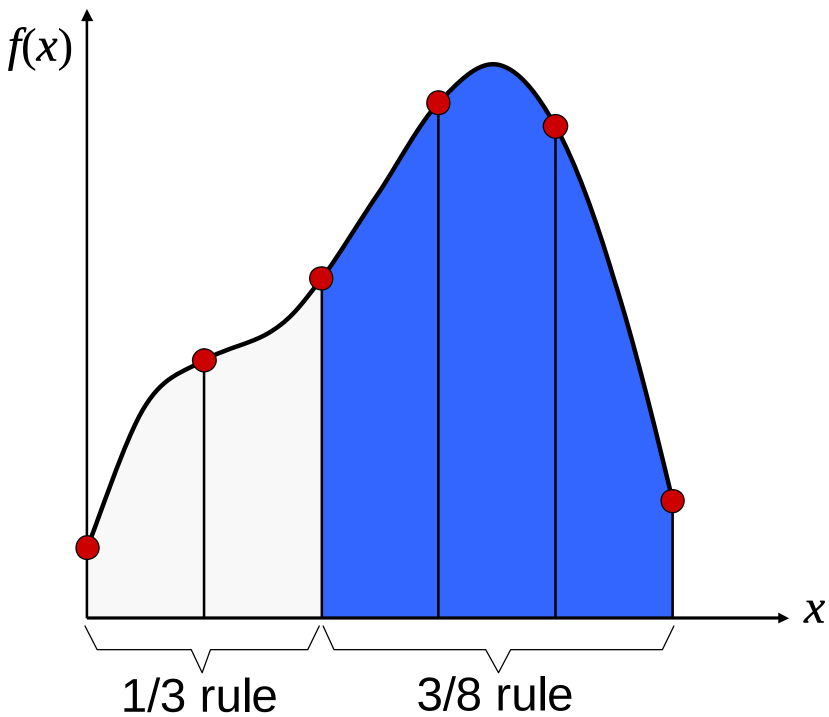
Truncation Errors

Method	E_t	Interval width	E_t
Trapezoid	$\frac{1}{12}h^3 f''(x)$	$h = b - a$	$\frac{(b-a)^3}{12} f''(x)$
Simpson's 1/3	$\frac{1}{90}h^5 f^{(4)}(x)$	$h = \frac{b-a}{2}$	$\frac{(b-a)^5}{2880} f^{(4)}(x)$
Simpson's 3/8	$\frac{3}{80}h^5 f^{(4)}(x)$	$h = \frac{b-a}{3}$	$\frac{(b-a)^5}{6480} f^{(4)}(x)$

Simpson's 3/8 rule is used when number of segments is odd.

Apply Simpson's 1/3 and 3/8 Rules

To handle multiple application with odd number of intervals



Example 9-5 Use Simpson's 1/3 + 3/8 rule with n = 5 to numerically integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from a = 0 to b = 0.8. Note that the exact value of the integral can be determined analytically to be 1.640533.

Solution: $n = 5$ ($h = 0.16$) :

$f(0) = 0.2$ $f(0.16) = 1.297$
$f(0.32) = 1.743$ $f(0.48) = 3.186$
$f(0.64) = 3.182$ $f(0.8) = 0.232$

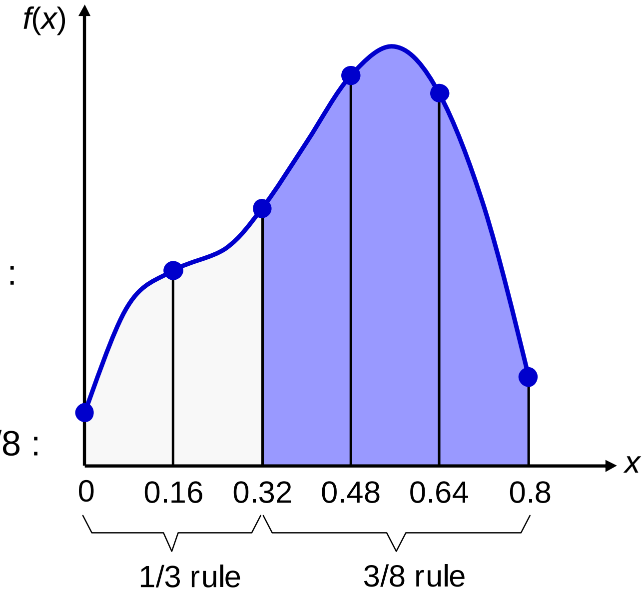
For the first two segments use Simpson's 1/3 :

$$I = \frac{0.16}{3}(0.2 + 4(1.297) + 1.743) = 0.380$$

For the last three segments use Simpson's 3/8 :

$$I = \frac{0.48}{8}(1.743 + 3(3.186 + 3.182) + 0.232)$$

$$= 1.265$$



Total integral : $I = 0.380 + 1.265 = 1.645$ $e_t = \left|\frac{1.6405 - 1.645}{1.6405}\right| \times 100\% = 0.274\%$

MATLAB's **quad** and **quad8** Functions

quad : low order method, **quad8**: high order method

```
>> q = quad('f',a,b)
```

Approximates the integral of f(x) from a to b within a relative error of 1e-3 using an adaptive recursive Simpson's rule.

`'f'` is a string containing the name of the function.

```
>> quad('sin', 0, pi/2)
```

Function `f` must return a vector of output values if given a vector of input values.

Example: $\int_0^{0.8} 0.2+25x-200x^2+675x^3-900x^4+400x^5 dx$

poly5.m

```
function p = poly5(x)
p=0.2+25*x-200*x.^2+675*x.^3-900*x.^4+400*x.^5;
```

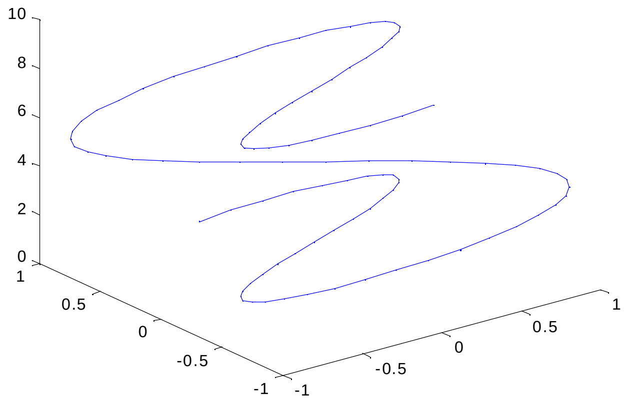
```
>> quad('poly5', 0, 0.8)

ans =

    1.6405
```

Example: Computing the Length of a Curve

$$x(t)=\sin(2t), \quad y(t)=\cos(t), \quad z(t)=t$$

where $t \in [0, 3p]$

```
>> t = 0:0.1:3*pi;
>> plot3(sin(2*t), cos(t), t)
```



Length of the curve:

Norm of derivative

$$\int_0^{3p} \sqrt{4\cos(2t)^2 + \sin(t)^2 + 1}\ dt$$

hcurve.m

```
function f = hcurve(t)
f = sqrt(4*cos(2*t).^2 + sin(t).^2 + 1);
```

```
>> len = quad('hcurve',0,3*pi)

len =
      17.2220
```

Example: Double Integration

$$\int_{y\min}^{y\max} \int_{x\min}^{x\max} f(x, y)\ dx\, dy$$

For example: $f(x, y) = y \sin(x) + x \cos(y)$

integrnd.m

```
function out = integrnd(x,y)
out = y*sin(x) + x*cos(y);
```

```
>> xmin = pi; xmax = 2*pi;
>> ymin = 0;  ymax = pi;
>> result = dblquad('integrnd',xmin,xmax,ymin,ymax);
result =
      -9.8698
```